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On Modular Happy Numbers II

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Abstract

In this paper, we investigate the asymptotic behavior of the sequences generated by iterating the process of summing the powers modulo $b-1$ in base-b system where b is a power of prime. In particular, we identify *modular happy numbers*. Following the spirit of happy number [1, p. 374], a number is called b-modular happy if the sequence obtained by iterating the process of summing the powers modulo $(b-1)$ in base-b system ends with 1.

Keywords

Happy Numbers, Sequences, Recurrence Relations, Difference Equations, Modular Arithmetic

1. Motivation

Let a be an integer and $\mathbb{N}a = \{a, a+1, a+2, \cdots\}$ be the set of integers bigger than or equal a. Furthermore, define the integer-valued family of function $f_{n,b}: \mathbb{N}_0 \to \mathbb{N}_0$ by

$$f_{n,b}(d) = d^n \, mod \, (b-1) \tag{1.1}$$

where $b \in \mathbb{N}_3$ and $n \in \mathbb{N}_1 = \mathbb{N}$. In particular, if $b = 10$, then $f_{n,10}$ will be denoted by f_n.

In this paper, for a given $n \in \mathbb{N}$ and $b \in \mathbb{N}_3$, we go a step further and iterate the following process.

$$F_{n,b}(x) = \sum_{j=0}^{k} f_{n,b}(d_j) = \sum_{j=0}^{k} d_j^n \bmod (b-1) \;\; where \;\; x = (d_k \cdots d_1 d_0)_b = \sum\nolimits_{j=1}^{k} d_j b^j \tag{1.2}$$

More precisely, we consider the following iteratively defined sequence:

$$\begin{array}{ccc} x_0 & = & x \\ x_1 & = & F_{n,b}(x_0) \\ x_2 & = & F_{n,b}(x_1) \\ \vdots & \vdots & \vdots \end{array}$$

The decimal case, i.e., $b = 10$, was recently investigated in Abu-Saris and Bayyati [2]. In this paper, we continue our investigation and study other bases.

2. Preliminary Results

To be on the same page, we recall the following standard definitions; see, for example, Rosen (2019, p. 252), Weisstein

(2020), and Keef and Guichard (2018, p. 71).

Definition 2.1. *Let q be a positive integer. A sequence* a_n *is said to be* periodic of period q *if*

$$a_{n+q} = a_n \text{ for } n = 1,2,3,\ldots$$

If q is the smallest such integer, it is called minimal period.

Definition 2.2. *Let a and b be integers, and let m be a positive integer. Then $a \equiv b \,(mod\, m)$ if and only if $a\, mod\, m = b\, mod\, m$, where $a\, mod\, m = a - m\lfloor a/m \rfloor$ where $\lfloor\cdot\rfloor$ denotes the floor function, i.e., the greatest integer function.*

Definition 2.3. *The* Euler Phi function *(or* Euler's Totient Function*), written $\phi(n)$, for positive integer n is the number of non-negative integers less than n that are relatively prime to n.*

For the proof of our main results in the next section, we shall need the following well-known lemmas; see Rosen (2019, pp. 256, 297) and Keef and Guichard (2018, p. 79).

Lemma 2.1. *Let m be a positive integer and let a and b be integers. Then*

$$(a + b)\, mod\, m = ((a\, mod\, m) + (b\, mod\, m))\, mod\, m$$

and

$$ab\, mod\, m = ((a\, mod\, m)(b\, mod\, m))\, mod\, m.$$

Lemma 2.2. *(***Fermat's Little Theorem***) If p is prime and a is an integer not divisible by p, then*

$$a^{p-1} \equiv 1(mod\, p).$$

Furthermore, for every integer a, we have

$$a^p \equiv a(mod\, p).$$

Lemma 2.3. *If $n > 0$, and u is relatively prime to n, then $u^{\phi(n)} \equiv 1(mod\, n)$.*

3. Main Results

Recall the iterative sequence defined in Section 1. Our first main result in this section reads as follows:

Theorem 3.1. *Suppose $b \in N_3$. Then, for each $n = 1,2,3,\ldots$, x_i is eventually less than $b - 1$.*

Proof. Observe that if x_0 is a power of b, then $x_i = 1$ for all $i \geq 1$, and hence all powers of b are modular happy. Furthermore, if $x_0 = b - 1$, then $x_i = 0$ for all $i \geq 1$, and so multiples of $(b - 1)$ are not modular happy.

If $n = 1$, then

$$F_{1,b}(x) = x \quad \text{if} \quad k = 0, \text{ i.e., } x = d_0$$

$$F_{1,b}(x) = \sum_{j=0}^{k} d_j < \sum_{j=0}^{k} d_j\, b^j = x \quad \text{if} \quad k > 0$$

Hence, by the Monotonic Convergence Theorem, the result follows.

To this end, assume that $n \geq 2$ and $x = \sum_{j=1}^{k} d_j b^j$. Then

$$\begin{aligned} F_{n,b}(x) &= \sum_{j=0}^{k} f_{n,b-1}(d_j) \leq (k+1)(b-1) \\ &\leq x \quad \text{if} \quad x \geq (k+1)(b \geq -1). \end{aligned}$$

In particular, if $k = 1$, then $f(x) < x$ if $x \geq 1(b - 1)$. But, if $x = 1d_0$, then, by Theorem 3.1, $f(x) < x$ if $x \geq b$.

Moreover, if $k \geq 2$, then

$$\begin{aligned} F_{n,b}(d_k d_{k-1} \cdots d_0) &= f_{n,b} + F_{n,b}(d_{k-1} \cdots d_0) \\ &\leq (b-1) + d_{k-1} \cdots d_0 \quad \text{assuming the inequality holds for } k-1 \\ &= (b-1) + \left(x - d_k b^k\right) < x \end{aligned}$$

Therefore, by the Principle of Mathematical Induction, $f(x) < x$ for all $k \geq 1$. This completes the proof. □

Remark 3.1. *Observe that by Theorem 3.1, it suffices to consider the asymptotic behavior of initial conditions in* $[0,b-2] \cap \mathbb{N}_0$.

Next, we state and prove our second main result in this section.

Theorem 3.2. *Let* $b \in \mathbb{N}_3$ *and* $n \in \mathbb{N}$. *Then*

1. $f_{n,b}(x)$ *is periodic of period* $(b-1)$ in x, *i.e.,* $f_{n,b}(x+b-1) = fn{,}b(x)$ *for all* x
2. *If* $b-1$ *is prime, then* $f_{n+b-2,b} = f_{n,b}$ *for all* $n \geq 1$. *In other words,* $f_{n,b}$ *is periodic of period* $b-2$ *in* n.
3. *If* $b-1 = p^m$ *for some prime* p, *then* $f_{n+\phi(b-1),b} = f_{n,b}$ *for all* $n \geq m$. *In other words,* $f_{n,b}$ *is eventually periodic of period* $\phi(b-1)$ *in* n.

Proof.

1. By Lemma 2.1,

$$f_{n,b}(x+b-1) = (x+b-1)^n \, mod\, (b-1) = (x+(b-1)\, mod\, (b-1))^n \, mod\, (b-1) = f_{n,b}(x)$$

2. By Part (1), it is enough to consider $0 \leq x \leq b-2$.

With this in mind, if $x = 0$, then the result holds true. Furthermore, if $0 < x < b-1$, then by Lemma 2.2,

$$f_{n+b-2,b}(x) = (x)^{n+b-2} \, mod\, (b-1) = x^n x^{b-2} \, mod\, (b-1) = x^n \, mod\, (b-1) = f_{n,b}(x)$$

3. By Part (1), it is enough to consider $0 \leq x \leq b-2$.

That been said, if $x = 0$, then the result holds true. Furthermore, if $0 < x < b-1$ and $n \geq m$, then by Lemma 2.3,

$$\begin{aligned} f_{n+\phi(b-1),b}(x) &= x^{n+\phi(b-1)}\, (mod\, b-1) \\ &= \begin{cases} 0\, (mod\, b-1) & if \quad \gcd(x,b-1) > 1 \\ x^n\, (mod\, b-1) & if \quad \gcd(x,b-1) = 1 \end{cases} \\ &= f_{n,b}(x) \end{aligned}$$

□

Remark 3.2.

1. *Theorem 3.2 implies that given* $b-1$ *is a power of prime, the asymptotic behavior of the sequence obtained by iterating the process of summing the powers modulo* $(b-1)$ *in base-b for* $n+\phi(b-1)$ *matches that of* n.
2. *It is worth mentioning that Part (2) of Theorem 3.2 follows from Part (3). However, when* $m = 1$, *then* $b-2$ *is the minimal period. This, as can be seen from the table below, is not the case when* $m > 1$.

b	$b-1$	$n \geq n0$	*minimal period*	$\phi(b-1)$
3	2	1	1	1
4	3	1	2	2
5	4	2	2	2
6	5	1	4	4
7	6	1	4	2
8	7	1	6	6
9	8	3	2	4
10	9	2	6	6
11	10	1	4	4
12	11	1	10	10
13	12	2	2	4

4. Conclusion

The asymptotic behavior of the sequences obtained by iterating the process of summing the powers modulo ($b-1$) in base-b system where $b-1$ is a power of prime were investigated in this paper. Those systems include octal and decimal to mention a few. However, there is room for improvement. In particular, what if $b-1$ is a product of primes such as $6, 10, 14, 15,$ etc.? More generally, what if $b-1$ is a product of powers of primes? Furthermore, similar to happy numbers, it is our hope that the modular happy numbers will attract the attention of researchers all over the world; see [6], [7], [8].

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