Journal of Applied Mathematics and Computation, 2021, 5(4), 237-246
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

New Algorithm of the Optimal Homotopy Asymptotic Method for Solving Lane-Emden Equations

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How to cite this paper: Haider A. Mkharrib, Tahrier N. Salem. (2021) New Algorithm of the Optimal Homotopy Asymptotic Method for Solving Lane-Emden Equations. *Journal of Applied Mathematics and Computation*, 5(4), 237-246.
DOI: 10.26855/jamc.2021.12.001

Received: August 30, 2021
Accepted: September 24, 2021
Published: October 13, 2021

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Abstract

The objective of this paper is to get an approximate solution for Lane-emden and Emden-Fowler initial and boundary value problems. For this, we used the Optimal Homotopy Asymptotic Method (OHAM) which is a semi-analytical method. OHAM Results show the eÂ˘gectiveness and reliability of OHAM for Lane-Emden and Emden-Fowler initial and boundary value problems. The results we obtained are compared to the exact solutions in addition to we presented a new modification of the optimal homotopy asymptotic method (NOHAM) and applied upon singular initial value Lane-Emden type equations and results are compared with the available exact solutions. The modified algorithm gives the exact solution for diffierential equations by using one iteration only.

Keywords

Optimal homotopy asymptotic method, Lane-Emden equations, singular initial value problems

1. Introduction

Mathematical modeling of many physical systems and engineering are generally described by differential equations. These equations are often solved by many methods such as Adomian's decomposition method (ADM) [1], Variational iteration method (VIM) [2-5], Homotopy analysis method (HAM) [6-10], and Homotopy perturbation method (HPM) [2, 7-8, 11]. In this work, we present a new modification of the optimal homotopy asymptotic method (NOHAM) and applied upon singular initial value Lane-Emden type equations, that first, this equations were published by Jonathan Homer Lane in 1870 [12], and further explored in detail by Emden [13].

The Lane-Emden equations have the following form

$$\mu''(x) + \frac{m}{x}\mu'(x) + f(\mu) = \lambda(x), \quad 0 < x < 1, m \geq 1$$

subject to following initial conditions

$$\mu(0) = \theta, \qquad \mu'(0) = \rho$$

where θ, ρ and m are constants and $f(\mu)$ is a real valued continuous function.

This modification demonstrates a rapid convergence of the series solution if compared with standard OHAM. In addition, the modified algorithm gives the exact solution for diffierential equations by using one iteration only. These results reveal that the NOHAM is very effective, simple and has closed agreement with exact solution.

2. Basic Ideas of Optimal Homotopy Asymptotic OHAM

Consider the following equation,

$$\ell(\mu(x)) + \lambda(x) + \eta(\mu(x)) = 0,\ B(\mu, d\mu/dx) = 0 \tag{1}$$

where ℓ is a linear operator, x denotes independent variable, $\mu(x)$ is an unknown function, $\lambda(x)$ is a known function, η is a nonlinear operator and B is a boundary operator.

According to OHAM, we construct a homotopy:

$\mathcal{H}(v(x,q),q): \mathcal{R} \times [0,1] \to \mathcal{R}$ which satisfies

$$(1-q)\ell(v(x,q)) + \lambda(x)) = H(q)[\ell(v(x,q)) + \lambda(x) + \eta(v(x,q))], \tag{2}$$
$$B(v(x,q), \partial v(x,q)/\partial x) = 0$$

where $x \in \mathcal{R}$ and $q \in [0,1]$ is an embedding parameter, $H(q)$ is a nonzero auxiliary function for $q \neq 0$, $H(0) = 0$ and $v(x,q)$ is an unknown function.

Obviously, when $q = 0$ and $q = 1$, it holds that $v(x,0) = \mu_0(x)$ and $v(x,1) = \mu(x)$, respectively.

Thus, as q varies from $q = 0$ to $q = 1$ the solution $v(x,q)$ approaches from $\mu_0(x)$ to $\mu(x)$, where $\mu_0(x)$ is obtained from Eq. (2) for $q = 0$ and we have

$$\ell(\mu_0(x)) + \lambda(x)) = 0,\ B(\mu_0, d\mu_0/dx) = 0 \tag{3}$$

Next, we choose auxiliary function $H(q)$ in the form

$$H(q) = \sum_{i=1}^{\infty} \propto_i q^i \tag{4}$$

Where $\propto_i, i = 1,2,3,\dots$ are constants to be determined. $H(q)$ can be expressed in many forms as reported by V. Marinca et al. [13-15].

To get an approximate solution, we expand $v(x,q,\propto_i)$ in Taylor's series about q in the following manner

$$v(x,q,\propto_i) = \mu_0(x) + \sum_{i=1}^{\infty} \mu_i(x, \propto_1 \propto_2 \propto_3 \dots \propto_i) q^i \tag{5}$$

Substituting Eq. (5) into Eq. (2) and equating the coefficient of the same power of q, we obtain the following linear equations. The zeroth and the first order are given by Eq. (3) and Eq. (6) respectively,

$$\ell(\mu_1(x)) + \lambda(x)) = \propto_1 \eta_0(\mu_0(x)),\ B(\mu_1, d\mu_1/dx) = 0 \tag{6}$$

The general governing equations for $\mu_i(x)$ are given by

$$\ell(\mu_i(x)) - \ell(\mu_{i-1}(x)) = \propto_i \eta_0(\mu_0(x)) + \sum_{k=1}^{i-1} \propto_i [\ell(\mu_{i-1}(x)) + \eta_{i-k}(\mu_0(x), \mu_1(x),$$
$$\dots, \mu_{i-1}(x)],\ i = 2,3,\dots \quad , B(\mu_i, d\mu_i/dx) = 0, \tag{7}$$

Where $\eta_m(\mu_0(x), \mu_1(x), \dots, \mu_m(x))$ is the coefficient of q^m in the expansion of $\eta(v(x,q))$ about the embedding parameter q.

$$\eta(v(x,q,\propto_i)) = \eta_0(\mu_0(x)) + \sum_{m=1}^{\infty} \eta_0(\mu_0(x), \mu_1(x), \dots, \mu_m(x)) q^m \tag{8}$$

It has been observed that the convergence of the series (5) depends upon the auxiliary constants $\propto_1 \propto_2 \propto_3 \dots$

If it is convergent at $q = 1$, one has

$$v(x,\propto_i) = \mu_0(x) + \sum_{i=1}^{\infty} \mu_i(x, \propto_1, \propto_2, \dots, \propto_i) \tag{9}$$

The result of the m^{th} order approximations are given by

$$\tilde{\mu}(x, \propto_1, \propto_2, \dots, \propto_m) = \mu_0(x) + \sum_{k=1}^{m} \mu_k(x, \propto_1, \propto_2, \dots, \propto_k) \tag{10}$$

Substituting Eq. (10) into Eq. (1), it results the following residual

$$\Re(x, \propto_1, \propto_2, \dots, \propto_m) = \ell(\tilde{\mu}(x, \propto_1, \propto_2, \dots, \propto_m)) + \lambda(x) + \eta(\tilde{\mu}(x, \propto_1, \propto_2, \dots, \propto_m)).$$

If $\Re = 0$, then $\tilde{\mu}$ will be the exact solution. Generally it does not happen, especially in nonlinear problems.

In order to find the optimal values of $\propto_k, k = 1,2,3, \dots$, we first construct the functional

$$\Im(\propto_1, \propto_2, \dots, \propto_m) = \int_a^b \Re^2(x, \propto_1, \propto_2, \dots, \propto_m)\, dx \tag{11}$$

and then minimizing it, we have

$$\frac{\partial \Im}{\partial \propto_1} = \frac{\partial \Im}{\partial \propto_2} = \dots = \frac{\partial \Im}{\partial \propto_m} = 0$$

where a and b are in the domain of the problem. With these constants known, the approximate solution (of the order m) is well determined.

3. Basic Ideas of New Algorithm of the Optimal Homotopy Asymptotic (NOHAM)

The new algorithm of the optimal homotopy asymptotic (NOHAM) is established. It is assumed that the nonhomogeneous terms $\lambda(x)$ in (1) can be expressed in Taylor series based on a kind of a continuous homotopy mapping with respect to q, $\lambda(x) \to \varphi(x;q)$ as,

$$\varphi(x;q) = \sum_{m=0}^{\infty} \lambda_m^z(x)\, q^m, \tag{12}$$

where $\lambda_m^z(x) = \frac{1}{mz!}[\frac{d^{mz}}{dx^{mz}}\lambda(x)]_{x=0}x^{mz} + \frac{1}{(mz+1)!}[\frac{d^{(mz+1)}}{dx^{(mz+1)}}\lambda(x)]_{x=0}x^{(mz+1)} + \cdots$

$$+\frac{1}{(mz+z-1)!}[\frac{d^{(mz+z-1)}}{dx^{(mz+z-1)}}\lambda(x)]_{x=0}x^{(mz+z-1)} \tag{13}$$

We note that $\lambda_m^z(x)$ depend on the order of the differential equation z.

And to this assumption the Eq. (2) becomes

$$(1-q)\ell(v(x,q)) + \lambda_0^z(x)) = H(q)[\ell(v(x,q)) + \varphi(x;q) + \eta(v(x,q))], \tag{14}$$

$$B(v(x,q), \partial v(x,q)/\partial x) = 0$$

For to communicate the reliability of NOHAM, we deal with different examples.

4. Numerical experiments

In this section, we solve some examples by OHAM and NOHAM.

Example 1. Consider the linear Lane-Emden equation [14]

$$\mu'' + \frac{2}{x}\mu' + \mu - \lambda(x) = 0,\ 0 < x < 1 \tag{15}$$

$$\mu(0) = 0\ ,\quad \mu'(0) = 0 \tag{16}$$

The exact solution of (15) subject to (16) in the case $\lambda(x) = 30x^3 + x^5$ is $\mu(x) = x^5$.

OHAM solution.

$$(1-q)\left[\mu'' + \frac{2}{x}\mu' + \mu - 30x^3 - x^5\right] = H(q)[\mu'' + \frac{2}{x}\mu' + \mu - 30x^3 - x^5]$$

The zeroth order problem is

$$\mu_0'' + \frac{2}{x}\mu_0' + \mu_0 - 30x^3 - x^5 = 0$$

$$\mu_0(0) = 0\ ,\qquad \mu_0'(0) = 0$$

$$\mu_0(x) = \frac{x^7}{56} + x^5$$

The first order problem is

$$\mu_1'' + \frac{2}{x}\mu_1' = (1+\alpha_1)\mu_0'' + (1+\alpha_1)\frac{2}{x}\mu_0' + \alpha_1\,\mu_0 - \alpha_1\,x^5 - 30\,\alpha_1\,x^3 - 30x^3 - x^5,$$

$$\mu_1(0) = 0\ ,\qquad \mu_1'(0) = 0$$

$$\mu_1(x) = \frac{\alpha_1\,x^9}{5050} + \frac{\alpha_1\,x^7}{56}$$

The second order problem is

$$\mu_2'' + \frac{2}{x}\mu_2' = (1+\alpha_1)\mu_1'' + (1+\alpha_1)\frac{2}{x}\mu_1' + \alpha_1\,\mu_1 + \alpha_2\,\mu_0'' + \alpha_2\frac{2}{x}\mu_0' + \alpha_2\,\mu_0 - \alpha_2\,\mu_0$$

$$-\alpha_2\,x^5 - 30\,\alpha_2\,x^3$$

$$\mu_2(0) = 0\ ,\qquad \mu_2'(0) = 0$$

$$\mu_2(x) = \frac{\propto_1^2 x^{11}}{665280} + \frac{\propto_1^2 x^9}{2520} + \frac{\propto_1^2 x^7}{56} + \frac{\propto_1 x^9}{5040} + \frac{\propto_1 x^7}{56} + \frac{\propto_2 x^9}{5040} + \frac{\propto_2 x^7}{56}$$

Now, $\mu(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solutions if necessary as:

$$\mu(x) = \mu_0(x) + \mu_1(x) + \cdots$$

By using the procedure mentioned in Section 2, we can calculate the constant $\propto_1$ and $\propto_2$, as follows:

$$\propto_1 = -0.9915643704541924 \text{ and } \propto_2 = 0.000037809003255378 55:$$

By using these values of $\propto_1$ and $\propto_2$, the approximate solution becomes

$$\mu(x) \approx x^5 + 1.9458723052 \times 10^{-6} x^7 - 3.3117322217 \times 10^{-6} x^9$$
$$+1.4778738287 \times 10^{-6} x^{11}$$

Table 1 shows the approximate solutions obtained from the solution using OHAM.

Table 1. The exact and OHAM of order 2 solutions

x	*Exact solution*	*OHAM solution*	*Error*
0	0	0	0
0.1	0.0000100	0.0000100	1.912903×10^{-13}
0.2	0.0003200	0.0003200	2.324183×10^{-11}
0.3	0.0024300	0.0024300	3.629955×10^{-10}
0.4	0.0102400	0.0102400	2.381953×10^{-9}
0.5	0.0312500	0.0312500	9.455518×10^{-9}
0.6	0.0777600	0.0777600	2.645902×10^{-8}
0.7	0.1680700	0.1680701	5.583301×10^{-8}
0.8	0.3276800	0.3276801	9.053422×10^{-8}
0.9	0.5904900	0.5904901	1.114442×10^{-7}
1	1.0000000	1.0000001	1.120139×10^{-7}

NOHAM solution.

We shall now apply the NOHAM to (15). First, we expand the homotopy $\varphi(x;q)$ in powers of the embedding parameter q with $z = 2$ as follows:

$$\varphi(x;q) = \sum_{m=0}^{\infty} \lambda_m^2(x)\, q^m$$

$$\lambda_m^2(x) = \frac{1}{m2!}\left[\frac{d^{2m}}{dx^{2m}}\lambda(x)\right]_{x=0} x^{2m} + \frac{1}{(2m+1)!}\left[\frac{d^{(2m+1)}}{dx^{(2m+1)}}\lambda(x)\right]_{x=0} x^{(2m+1)}$$

$$\lambda_0^2(x) = 30x^3,\ \lambda_1^2(x) = x^5\ ,\ \lambda_m^2(x) = 0,\ m = 2,3,\ldots$$

For the modification NOHAM, we construct homotopy in the following form

$$(1-q)\left[\mu'' + \frac{2}{x}\mu' - 30x^3\right] = H(q)\left[\mu'' + \frac{2}{x}\mu' + \mu - 30x^3 - x^5\right]$$

$$\mu_0(0) = 0\ , \qquad {\mu_0}'(0) = 0$$

We get

$$\mu_0(x) = x^5$$
$$\mu_1(x) = 0$$
$$\mu_2(x) = 0$$
$$\mu_m(x) = 0\ , m = 3,4,\ldots$$

Then, the series solution expression by NOHAM can be written in the following form:

$\mu(x) = \sum_{m=0}^{\infty} \mu_m(x) = x^5$. Thus, we obtain an exact solution at level $m = 0$.

Example 2. Consider the linear Lane-Emden equation [14]

$$\mu'' + \frac{8}{x}\mu' + x\mu - \lambda(x) = 0,\ 0 < x < 1 \tag{17}$$

$$\mu(0) = 0\ ,\ \mu'(0) = 0 \tag{18}$$

The exact solution of (17) subject to (18) in the case $\lambda(x) = -30x + 44x^2 - x^4 + x^5$ is $\mu(x) = x^4 - x^3$.

OHAM solution.

$$(1-q)\left[\mu'' + \frac{8}{x}\mu' + 30x - 44x^2 + x^4 - x^5\right] = H(q)[\mu'' + \frac{8}{x}\mu' + x\mu + 30x - 44x^2 + x^4 - x^5]$$

The zeroth order problem is

$$\mu_0'' + \frac{8}{x}\mu_0' + x\mu_0 + 30x - 44x^2 + x^4 - x^5 = 0$$

$$\mu_0(0) = 0\ ,\quad \mu_0'(0) = 0$$

$$\mu_0(x) = \frac{x^7}{98} - \frac{x^6}{78} + x^4 - x^3$$

The first order problem is

$$\mu_1'' + \frac{8}{x}\mu_1' = (1 + \propto_1)\mu_0'' + (1 + \propto_1)\frac{8}{x}\mu_0' + \propto_1 x\mu_0 - \propto_1 30x - \propto_1 44x^2 + \propto_1 x^4 - \propto_1 x^5 + 30x - 44x^2 + x^4 - x^5,$$

$$\mu_1(0) = 0\ ,\quad \mu_1'(0) = 0$$

$$\mu_1(x) = \frac{\propto_1 x^{10}}{16660} - \frac{\propto_1 x^9}{11232} + \frac{\propto_1 x^7}{98} - \frac{\propto_1 x^6}{78}$$

The second order problem is

$$\mu_2'' + \frac{8}{x}\mu_2' = (1 + \propto_1)\mu_1'' + (1 + \propto_1)\frac{8}{x}\mu_1' + \propto_1 x\mu_1 + \propto_2 \mu_0'' + \propto_2 \frac{8}{x}\mu_0' + \propto_2 x\mu_0 - \propto_2 x\mu_0$$
$$- \propto_2 30x - \propto_2 44x^2 + \propto_2 x^4 - \propto_2 x^5$$

$$\mu_2(0) = 0\ ,\quad \mu_2'(0) = 0$$

$$\mu_2(x) = \frac{\propto_1^2 x^{13}}{4331600} - \frac{\propto_1^2 x^{12}}{2560896} + \frac{\propto_1^2 x^{10}}{8330} - \frac{\propto_1^2 x^9}{5616} + \frac{\propto_1^2 x^7}{98} - \frac{\propto_1^2 x^6}{78} + \frac{\propto_1 x^{10}}{16660} - \frac{\propto_1 x^9}{11232} + \frac{\propto_1 x^7}{98} - \frac{\propto_1 x^6}{78}$$
$$+ \frac{\propto_2 x^{10}}{16660} - \frac{\propto_2 x^9}{11232} + \frac{\propto_2 x^7}{98} - \frac{\propto_2 x^6}{78}$$

Now, $\mu(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solution if necessary as:

$$\mu(x) = \mu_0(x) + \mu_1(x) + \cdots$$

By using the procedure mentioned in Section 2, we can calculate the constant $\propto_1$ and $\propto_2$, as follows:
$\propto_1 = -0.9951916227117169$ and $\propto_2 = 0.000026392708974673 15$:
By using these values of $\propto_1$ and $\propto_2$, the approximate solution become

$$\mu(x) \approx x^4 - x^3 + 2.286467739208561 \times 10^{-7}x^{13} - 3.8674212694134408 \times 10^{-7}x^{12}$$
$$-5.7287640355933747 \times 10^{-7}x^{10} + 8.4972586211703724 \times 10^{-7}x^9 - 5.0523674613498198 \times 10^{-7}x^7$$
$$-6.347846297583312 \times 10^{-7}x^6$$

Table 2 shows the approximate solutions obtained from the solution using OHAM.

Table 2. The exact and OHAM of order 2 solutions

x	*Exact solution*	*OHAM solution*	*Error*
0	0	0	0
0.1	-0.00090000	-0.00090000	5.834689×10^{-13}
0.2	-0.00640000	-0.00640000	3.378419×10^{-11}

0.3	-0.01890000	-0.01890000	3.390894×10^{-10}
0.4	-0.03840000	-0.03840000	1.614572×10^{-9}
0.5	-0.06250000	-0.06250000	4.937685×10^{-9}
0.6	-0.08640000	-0.08640001	1.091703×10^{-8}
0.7	-0.10290000	-0.10290002	1.810387×10^{-8}
0.8	-0.10240000	-0.10240002	2.191974×10^{-8}
0.9	-0.07290000	-0.07290002	1.735431×10^{-7}
1	0.00000000	-0.00000001	1.879378×10^{-7}

NOHAM solution.

We shall now apply the NOHAM to (17). First, we expand the homotopy $\varphi(x;q)$ in powers of the embedding parameter q with $z = 2$ as follows:

$$\varphi(x;q)=\sum_{m=0}^{\infty}\lambda_m^2(x)\,q^m$$

$$\lambda_m^2(x)=\frac{1}{m2!}[\frac{d^{2m}}{dx^{2m}}\lambda(x)]_{x=0}x^{2m}+\frac{1}{(2m+1)!}[\frac{d^{(2m+1)}}{dx^{(2m+1)}}\lambda(x)]_{x=0}x^{(2m+1)}$$

$$\lambda_0^2(x)=-30x+44x^2,\ \lambda_1^2(x)=-x^4+x^5,\ \lambda_m^2(x)=0,\ m=2,3,\dots$$

For the modification NOHAM, we construct homotopy in the following form

$$(1-q)\left[\mu''+\frac{8}{x}\mu'+30x-44x^2\right]=H(q)[\,\mu''+\frac{8}{x}\mu'+x\mu+30x-44x^2+x^4-x^5\,]$$

$$\mu_0(0)=0\ ,\qquad \mu_0{}'(0)=0$$

We get

$$\mu_0(x)=x^4-x^3$$

$$\mu_1(x)=0$$

$$\mu_2(x)=0$$

$$\mu_m(x)=0\ ,m=3,4,\dots$$

Then, the series solution expression by NOHAM can be written in the following form:
$\mu(x)=\sum_{m=0}^{\infty}\mu_m(x)=x^4-x^3$. Thus, we obtain an exact solution at level $m = 0$.

Example 3. Consider nonlinear Lane-Emden equation [14]

$$\mu''+\frac{1}{x}\mu'+\mu\,\mu'-\lambda(x)=0,\ 0<x\le 1 \tag{19}$$

$$\mu(0)=1\ ,\ \mu'(0)=0 \tag{20}$$

The exact solution of (19) subject to (20) in the case $\lambda(x)=2x^3+2x+4$ is $\mu(x)=1+x^2$.

OHAM solution.

$$(1-q)\left[\mu''+\frac{1}{x}\mu'-2x^3-2x-4\right]=H(q)[\,\mu''+\frac{1}{x}\mu'+\mu\,\mu'-2x^3-2x-4\,]$$

The zeroth order problem is

$$\mu_0{}''+\frac{1}{x}\mu_0{}'+\mu_0{}'\mu_0-2x^3-2x-4=0$$

$$\mu_0(0)=1\ ,\qquad \mu_0{}'(0)=0$$

$$\mu_0(x)=\frac{2x^5}{25}+\frac{2x^3}{9}+x^2+1$$

The first order problem is

$$\mu_1{}'' + \frac{1}{x}\mu_1{}' = (1+\alpha_1)\mu_0{}'' + (1+\alpha_1)\frac{1}{x}\mu_0{}' + \alpha_1\,\mu_0\mu_0{}' - 2\,\alpha_1\,x^3 - 2\,\alpha_1\,x - 4\,\alpha_1 - 2x^3$$
$$-2x-4$$
$$\mu_1(0)=1\ ,\qquad \mu_1{}'(0)=0$$

$$\mu_1(x) = \frac{4\alpha_1 x^3}{27} - \frac{\alpha_1 x^2}{2} + \frac{\alpha_1 x^4}{24} - \frac{2\alpha_1 x^5}{125} + \frac{\alpha_1 x^6}{90} + \frac{10x^3}{27} - \frac{x^2}{2} + \frac{x^4}{12} + \frac{8x^5}{125} + \frac{43x^6}{810} + \frac{4x^7}{1323} + \frac{7x^8}{800} + \frac{32x^9}{18225} - \frac{4x^{11}}{15125}$$

The second order problem is

$$\mu_2{}'' + \frac{1}{x}\mu_2{}' = (1+\alpha_1)\mu_1{}'' + (1+\alpha_1)\frac{1}{x}\mu_1{}' + \alpha_1\,\mu_1\mu_1{}' + \alpha_2\,\mu_0{}'' + \alpha_2\,\frac{1}{x}\mu_0{}' + \alpha_2\,\mu_0\mu_1{}' - \alpha_2\,\mu_0\mu_1{}'$$
$$-2\,\alpha_2\,x^3 - 2\,\alpha_2\,x + 4\,\alpha_2$$
$$\mu_2(0)=1\ ,\qquad \mu_2{}'(0)=0$$
$$\mu_2(x) = \frac{11\,\alpha_1\,x^3}{27} - \alpha_1\,x^2 + \frac{7\,\alpha_1\,x^4}{36} - \frac{7\,\alpha_1\,x^5}{375} + \frac{2651\,\alpha_1\,x^6}{24300} + \frac{169\,\alpha_1\,x^7}{5670} + \frac{13841\,\alpha_1\,x^8}{1008000} + \cdots$$

Now, $\mu(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solutions if necessary as:

$$\mu(x) = \mu_0(x) + \mu_1(x) + \cdots$$

By using the procedure mentioned in Section 2, we can calculate the constant α_1 and α_2, as follows:
$\alpha_1 = -0.7262870183266369669595247882906 4$ and $\alpha_2 = -1.5666846630398228174101179583022$, by using these values of α_1 and α_2, the approximate solution becomes

$$\begin{aligned}\mu(x) \approx & -8.5071681405413514\times 10^{-7}x^{17} - 9.0037081148592845\times 10^{-6}x^{15}\\ & -4.6460197145527956\times 10^{-5}x^{14} + 3.2595707226913386 7\times 10^{-5}x^{13}\\ & -4.196708982501607\times 10^{-4}x^{12} - 1.0061383709219621\times 10^{-3}x^{11}\\ & -1.2438236844093688\times 10^{-3}x^{10} - 7.0845567198262452\times 10^{-3}x^{9}\\ & -8.8779323735504041\times 10^{-3}x^{8} + 1.4802190408287729\times 10^{-2}x^{7}\\ & -3.949085161037252\times 10^{-2}x^{6} + 6.0720483845049083\times 10^{-2}x^{5}\\ & -3.3464849107268245\times 10^{-2}x^{4} + 0.23085479925375204\,x^{3}\\ & +0.82868411099505713x^{2} + 1.0.\end{aligned}$$

Table 3 shows the approximate solutions obtained from the solution using OHAM.

Table 3. The exact and OHAM of order 2 solutions

x	*Exact solution*	*OHAM solution*	*Error*
0	1.0000000	1.0000000	0.0000000
0.1	1.0100000	1.0113646	1.515084×10^{-3}
0.2	1.0400000	1.0460277	5.162654×10^{-3}
0.3	1.0900000	1.1047812	9.611622×10^{-3}
0.4	1.1600000	1.1882937	1.354462×10^{-2}
0.5	1.2500000	1.2969916	1.569918×10^{-2}
0.6	1.3600000	1.4308165	1.499403×10^{-2}
0.7	1.4900000	1.5887828	1.078549×10^{-2}
0.8	1.6400000	1.7682158	3.324701×10^{-2}
0.9	1.8100000	1.9634760	5.473674×10^{-2}
1	2.0000000	2.0137805	1.378047×10^{-2}

NOHAM solution.

We shall now apply the NOHAM to (19). First, we expand the homotopy $\varphi(x;q)$ in powers of the embedding parameter q with $z = 2$ as follows:

$$\varphi(x;q) = \sum_{m=0}^{\infty} \lambda_m^2(x)\, q^m$$

$$\lambda_m^2(x) = \frac{1}{m2!}[\frac{d^{2m}}{dx^{2m}}\lambda(x)]_{x=0}x^{2m} + \frac{1}{(2m+1)!}[\frac{d^{(2m+1)}}{dx^{(2m+1)}}\lambda(x)]_{x=0}x^{(2m+1)}$$

$$\lambda_0^2(x) = 4 + 2x,\ \lambda_1^2(x) = 2x^3,\ \lambda_m^2(x) = 0,\ m = 2,3, \dots$$

For the modification NOHAM, we construct homotopy in the following form

$$(1-q)\left[\mu'' + \frac{1}{x}\mu' - 4 - 2x\right] = H(q)[\,\mu'' + \frac{1}{x}\mu' + \mu\mu' - 4 - 2x - 2x^3]$$

$$\mu_0(0) = 1\ , \qquad \mu_0{}'(0) = 0$$

We get

$$\mu_0(x) = 1 + x^2$$

$$\mu_1(x) = 0$$

$$\mu_2(x) = 0$$

$$\mu_m(x) = 0\ \ , m = 3,4, \dots$$

Then, the series solution expression by NOHAM can be written in the following form:
$\mu(x) = \sum_{m=0}^{\infty} \mu_m(x) = 1 + x^2$. Thus, we obtain an exact solution at level $m = 0$.

Example 4. Consider nonlinear Lane-Emden equation [14]

$$\mu'' + \tfrac{2}{x}\mu' + \mu^3 - \lambda(x) = 0,\ 0 < x \le 1 \qquad (21)$$

$$\mu(0) = 0\ ,\ \mu'(0) = 0 \qquad (22)$$

The exact solution of (21) subject to (22) in the case $\lambda(x) = 6 + x^6$ is $\mu(x) = x^2$.

OHAM solution.

$$(1-q)\left[\mu'' + \frac{2}{x}\mu' - 6 - x^6\right] = H(q)[\,\mu'' + \frac{2}{x}\mu' + \mu^3 - 6 - x^6\,]$$

The zeroth order problem is

$$\mu_0{}'' + \frac{2}{x}\mu_0{}' + \mu_0{}^3 - 6 - x^6 = 0$$

$$\mu_0(0) = 0\ , \qquad \mu_0{}'(0) = 0$$

$$\mu_0(x) = \frac{x^8}{72} + x^2$$

The first order problem is

$$\mu_1{}'' + \frac{2}{x}\mu_1{}' = (1 + \alpha_1)\mu_0{}'' + (1 + \alpha_1)\frac{2}{x}\mu_0{}' + \alpha_1\, \mu_0{}^3 - \alpha_1\, x^6 - 6\,\alpha_1 - 6 - x^6$$

$$\mu_1(0) = 0\ , \qquad \mu_1{}'(0) = 0$$

$$\mu_1(x) = \frac{\alpha_1\, x^{26}}{262020096} + \frac{\alpha_1\, x^{20}}{725760} + \frac{\alpha_1\, x^{14}}{5040} + \frac{\alpha_1\, x^8}{72}$$

The second order problem is

$$\mu_2{}'' + \frac{2}{x}\mu_2{}' = (1 + \alpha_1)\mu_1{}'' + (1 + \alpha_1)\frac{2}{x}\mu_1{}' + \alpha_1\, \mu_1{}^3 + \alpha_2\, \mu_0{}'' + \alpha_2\frac{2}{x}\mu_0{}' + \alpha_2\, \mu_1{}^3 - \alpha_2\, \mu_1{}^3$$
$$- \alpha_2\, x^6 - 6\,\alpha_2$$

$$\mu_2(0) = 0 \ , \qquad \mu_2{}'(0) = 0$$

$$\mu_2(x) = \frac{\propto_1{}^2 \, x^{44}}{896486037258240} + \frac{491 \ \propto_1{}^2 \, x^{38}}{652367154216960} + \frac{\propto_1{}^2 \, x^{14}}{2520} + \cdots$$

Now, $\mu(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solutions if necessary as:

$$\mu(x) = \mu_0(x) + \mu_1(x) + \cdots$$

By using the procedure mentioned in Section 2, we can calculate the constant $\propto_1$ and $\propto_2$, as follows:
$\propto_1 = -0.9828444161739907$ and $\propto_2 = 0.00007151252608689145$, by using these values of $\propto_1$ and $\propto_2$, the approximate solution becomes

$$\mu(x) \approx 1.077521686069644 \times 10^{-15} x^{44} + 7.2704108693801268 \times 10^{-13} x^{38}$$
$$+2.2054221289131272 \times 10^{-10} x^{32} + 3.568499285010508 \times 10^{-8} x^{26}$$
$$+2.6536524254083618 \times 10^{-6} x^{20} - 6.676910740296036 \times 10^{-6} x^{14}$$
$$+5.0809247569184018 \times 10^{-6}$$

Table 4 shows the approximate solutions obtained from the solution using OHAM.

Table 4. The exact and OHAM of order 2 solutions

x	*Exact solution*	*OHAM solution*	*Error*
0	0.0000000	0.0000000	0.0000000
0.1	0.0100000	0.0100000	5.080918×10^{-14}
0.2	0.04000000	0.04000000	1.300607×10^{-11}
0.3	0.09000000	0.09000000	3.330402×10^{-10}
0.4	0.16000000	0.16000000	3.311941×10^{-9}
0.5	0.25000000	0.25000002	1.944237×10^{-8}
0.6	0.36000000	0.36000008	8.020490×10^{-8}
0.7	0.49000000	0.49000025	2.497424×10^{-7}
0.8	0.64000000	0.64000059	5.894919×10^{-7}
0.9	0.81000000	0.81000098	9.846713×10^{-7}
1	1.00000000	1.00000109	1.093692×10^{-6}

NOHAM solution.

We shall now apply the NOHAM to (21). First we expand the homotopy $\varphi(x;q)$ in powers of the embedding parameter q with $z = 2$ as follows:

$$\varphi(x;q) = \sum_{m=0}^{\infty} \lambda_m^2(x)\, q^m$$

$$\lambda_m^2(x) = \frac{1}{m2!}\left[\frac{d^{2m}}{dx^{2m}}\lambda(x)\right]_{x=0} x^{2m} + \frac{1}{(2m+1)!}\left[\frac{d^{(2m+1)}}{dx^{(2m+1)}}\lambda(x)\right]_{x=0} x^{(2m+1)}$$

$$\lambda_0^2(x) = 6,\ \lambda_1^2(x) = 0\ ,\ \lambda_2^2(x) = 0,\ \ \lambda_3^2(x) = x^6,\ \lambda_m^2(x) = 0,\ m = 4,5,\ldots$$

For the modification NOHAM, we construct homotopy in the following form

$$(1-q)\left[\mu'' + \frac{2}{x}\mu' - 6\right] = H(q)\left[\mu'' + \frac{2}{x}\mu' + \mu^3 - 6 - x^6\right]$$

$$\mu_0(0) = 0 \ , \qquad \mu_0{}'(0) = 0$$

We get

$$\mu_0(x) = x^2$$

$$\mu_1(x) = 0$$

$$\mu_2(x) = 0$$

$$\mu_m(x) = 0 \ , m = 3,4, \dots$$

Then, the series solution expression by NOHAM can be written in the following form: $\mu(x) = \sum_{m=0}^{\infty} \mu_m(x) = x^2$. Thus, we obtain an exact solution at level $m = 0$.

5. Conclusion

In this paper, the modified NOHAM is applied to approximate solutions of linear and non-linear Lane-Emden equations. The results show us that this method can obtain the exact solution by only one iteration. So it is concluded that NOHAM is reliable and efficient technique for finding the solutions of Lane-Emden equations.

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