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Weighted Approximation Properties of New (*p*, *q*)—Analogue of Balazs Szabados Operators

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Abstract

Korovkin-type theorems provide simple and useful tools for finding out whether a given sequence of positive linear operators, acting on some function space is an approximation processor, equivalently, converges strongly to the identity operator. These theorems exhibit a variety of test subsets of functions which guarantee that the approximation property holds on the whole space provided it holds on them. These kinds of results are called "Korovkin-type theorems" which refers to P.P. Korovkin who in 1953 discovered such a property for the functions 1, x and x^2 in the space $C([0,1])$. After this discovery, several mathematicians have undertaken the program of extending Korovkin's theorems in many ways and to several settings. Such developments delineated a theory which is nowadays referred to as Korovkin-type approximation theory. In this paper, we study weighted approximation properties of new (p, q) - analogue of the Balázs-Szabados operators by using the weighted modulus of continuity and we give a Korovkin type theorem for weighted approximation.

Keywords

(p, q)- analysis, moments, Bernstein operators, Balázs-Szabados operators, (p, q)-Balázs-Szabados operators, weighted modulus of continuity

1. Introduction

In the year 1975, Catherine Balázs defined and studied Bernstein type rational functions as follows (see [1]),

$$R_n(f;x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_nx)^k \qquad (n=1,\ 2,\ \ldots),$$

where f is a real valued and single valued function which is defined on the unbounded interval $[0,\infty)$, a_n and b_n are real numbers which are selected suitably and do not depend on x. Seven years later in 1982, Catherine Balázs and J. Szabados studied together to improve the estimation in [2] by selecting suitable a_n and b_n under some restrictions for $f(x)$.

Recently, generalizations of Balázs-Szabados operators based on the q-integers are studied by Hayatem Hamal and Pembe Sabancigil ([3]), Ogün Doğru ([4]) and Esma Yıldız Özkan ([5]). Approximation properties of the q-Balázs-Szabados complex operators are studied by Nazım I. Mahmudov in [6] and by Nurhayat Ispir and Esma Yıldız Özkan in [7].

Moreover, the fast rise of (p,q)-analysis has encouraged many authors in this subject to discover different generalizations and examine their approximation properties. In the last seven years, Mohammad Mursaleen et al. introduced and studied (p,q)-analogue of Bernstein operators, (p,q)-analogue of Bernstein-Stancu operators, Bernstein-Kantorovich operators based on (p,q)-calculus, (p,q)-Lorentz polynomials on a compact disc, Bleimann-Butzer-Hahn operators defined by (p,q)-integers and (p,q)-analogue of two parametric Stancu-Beta operators (see [8]-[14]). (p,q)-generalization of Szász-Mirakyan operators is studied by Tuncer Acar (see [15]), Kantorovich modification of (p,q)-Bernstein operators is studied by Tuncer Acar and Ali Aral (see [16]). A generalization of q-Balázs-Szabados operators based on (p,q)-integers is studied by Esma Yıldız Özkan and Nurhayat Ispir in [17]. Hayatem Hamal and Pembe Sabancigil introduce a new (p,q)-generalization of q-Balázs-Szabados operators as follows (see [18]),

$$R_{n,p,q}(f,x)=\frac{1}{p^{n(n-1)/2}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2} f\left(\frac{p^{n-k}[k]_{p,q}}{b_n}\right)\left(\frac{a_n x}{1+a_n x}\right)^k \prod_{j=0}^{n-k-1}\left(p^j-q^j\frac{a_n x}{1+a_n x}\right),$$

where $a_n=[n]_{p,q}^{\beta-1}$, $b_n=[n]_{p,q}^{\beta}$, $0<\beta\le\frac{2}{3}$, $n\in\mathbb{N}$, $x\ge 0$, f is a real-valued function defined on the unbounded interval $[0,\infty)$.

In this paper, we study weighted approximation properties of new (p,q)-analogue of the Balázs-Szabados operators by using the weighted modulus of continuity and we give a Korovkin type theorem for weighted approximation.

Before giving the main results for the mentioned operators above, we present some important notations and some basic definitions of (p,q)-analysis. For any two non-negative numbers p, q and a non-negative integer n, the (p,q)-integer of the number n is defined as follows:

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2+\ldots+pq^{n-2}+q^{n-1}=\begin{cases}\dfrac{p^n-q^n}{p-q} & if\ p\ne q\ne 1\\ np^{n-1} & if\ p=q\ne 1\\ [n]_q & if\ p=1\\ n & if\ p=q=1\end{cases}.$$

The (p,q)-factorial is defined by $[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q}$, for $n\ge 1$ and particularly we have $[0]_{p,q}!=1.$

(p,q)-binomial cofficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},\quad 0\le k\le n,$$

and the formula of (p,q)-binomial expantion is defined by

$$\begin{aligned}(ax+by)_{p,q}^n&=\sum_{k=0}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}a^{n-k}b^k x^{n-k}y^k\\ &=(ax+by)(pax+qby)(p^2ax+q^2by)\ldots(p^{n-1}ax+q^{n-1}by).\end{aligned}$$

2. Maın Results

Firstly we consider the following three spaces:

$B_2[0,\infty)=\left\{f:[0,\infty)\to R:|f(x)|\le M_f(1+x^2)\right\}$, where M_f is a constant depending on f.

$C_2[0,\infty)=B_2[0,\infty)\cap C[0,\infty)$ and $C_2^*[0,\infty)=\left\{f\in C_2[0,\infty):\lim\limits_{x\to\infty}\frac{f(x)}{1+x^2}<\infty\right\}$.

The norm on the space $C_2^*[0,\infty)$ is shown as $\|f(x)\|_2=\sup\limits_{x\in[0,\infty)}\frac{f(x)}{1+x^2}$.

The modulus of continuity of f on a closed and bounded interval $[0,b]$, $b>0$ is defined as follows: $\omega_b(f,\delta)=\sup\limits_{|t-x|\le\delta}\sup\limits_{x\in[0,b]}|f(t)-f(x)|$. It is obvious that for a function $f\in C_2[0,\infty)$, the modulus of continuity $\omega_b(f,\delta)$ tends to zero as $\delta\to 0$.

Definition 1 [18] Let $0<q<p\le 1$, we introduce a new (p,q)-analogue of Balázs-Szabados operators by

$$R_{n,p,q}(f,x)=\frac{1}{p^{n(n-1)/2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{k(k-1)/2}f\left(\frac{p^{n-k}[k]_{p,q}}{b_n}\right)\left(\frac{a_nx}{1+a_nx}\right)^k\prod_{j=0}^{n-k-1}\left(p^j-q^j\frac{a_nx}{1+a_nx}\right),$$

where $a_n=[n]_{p,q}^{\beta-1}$, $b_n=[n]_{p,q}^{\beta}$, $0<\beta\le\frac{2}{3}$, $n\in\mathbb{N}$, $x\ge 0$, f is a real-valued function which is defined on the unbounded interval $[0,\infty)$.

In the following theorem we give the rate of convergence for the new (p,q)-analogue of Balázs-Szabados operators, $R_{n,p,q}(f,x)$.

Theorem 1 ([19], [20]). Let $f\in C_2[0,\infty), 0<q<p\le 1$ and $\omega_{b+1}(f,\delta)$ be the modulus of continuity on $[0,b+1]\subset[0,\infty)$, where $b>0$. Then for every $n\in\mathbb{N}$, we have

$$\left\|R_{n,p,q}(f,x)-f(x)\right\|_{[0,b]}\le L+2\omega_{b+1}(f,\delta)$$

where L is a positive constant.

Proof. For $x\in[0,b]$ and $t>b+1$, since $t-x>1$ we have

$$|f(t)-f(x)|\le M_f\left(2+x^2+t^2\right)\le M_f\left(2(t-x)^2+3x^2(t-x)^2+2(t-x)^2\right)$$
$$\le M_f\left(4+3x^2\right)(t-x)^2\le 4M_f\left(1+b^2\right)(t-x)^2. \quad (1)$$

For $x\in[0,b]$, $t<b+1$, we have

$$|f(t)-f(x)|\le\omega_{b+1}(f,|t-x|)\le\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta). \quad (2)$$

So, with $\delta>0$, $x\in[0,b]$, $t\ge 0$ and by the inequalities (1) and (2) we may write

$$|f(t)-f(x)|\le 4M_f\left(1+b^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta),$$

by applying $R_{n,p,q}(f,x)$ to the above inequality and by the well known Cauchy-Schwarz Inequality, we obtain

$$\left|R_{n,p,q}(f,x)-f(x)\right|\le 4M_f\left(1+b^2\right)R_{n,p,q}\left((t-x)^2,x\right)+\left(1+R_{n,p,q}\left(\frac{|t-x|}{\delta},x\right)\right)\omega_{b+1}(f,\delta),$$

$$\left|R_{n,p,q}(f,x)-f(x)\right|\le 4M_f\left(1+b^2\right)R_{n,p,q}\left((t-x)^2,x\right)+\left(1+\frac{1}{\delta}\left(R_{n,p,q}\left((t-x)^2,x\right)\right)^{\frac{1}{2}}\right)\omega_{b+1}(f,\delta).$$

Now by using Lemma 3 in [18], we may write

$\left|R_{n,p,q}\left(f,x\right)-f\left(x\right)\right|\leq 4M_f\left(1+b^2\right)D_1\left(1+x\right)^2+\left(1+\frac{1}{\delta}\sqrt{D_1}\left(1+x\right)\right)\omega_{b+1}\left(f,\delta\right)$, where D_1 is a positive constant.

For $x\in\left[0,b\right]$, we have the following explicit formula:

$$\left|R_{n,p,q}\left(f,x\right)-f\left(x\right)\right|\leq 4M_f\left(1+b^2\right)D_1\left(1+b\right)^2+\left(1+\frac{1}{\delta}\sqrt{D_1}\left(1+b\right)\right)\omega_{b+1}\left(f,\delta\right).$$

Then, by taking $\delta=\sqrt{D_1}\left(1+b\right)$, $L=\left(4M_f+D_1\right)$ we get the desired result.

In the following theorem, we give Korovkin's approximation property for the new $\left(p,q\right)$-analogue of Balázs-Szabados operators.

Theorem 2. Assume that $q=q_n$, $p=p_n$ are sequences such that $0<q_n<p_n\leq 1$ and $q_n\to 1$ as $n\to\infty$. Then for each $f\in C_2^*\left[0,\infty\right)$ we have $\lim_{n\to\infty}\left\|R_{n,p_n,q_n}\left(f,x\right)-f\left(x\right)\right\|_2=0$.

Proof. By using the Korovkin theorem for weighted approximation (see [21], [22], [23]), it is sufficient to show that

$$\lim_{n\to\infty}\left\|R_{n,p_n,q_n}\left(t^m;x\right)-x^m\right\|_2=0\text{, for } m=0,\ 1,\ 2. \tag{3}$$

Since $R_{n,p_n,q_n}\left(1;x\right)=1$, (3) holds for $m=0$. Now by Lemma 2 in [18], we have

$R_{n,p_n,q_n}\left(t;x\right)-x=\frac{x}{1+a_{n,p_n,q_n}x}-x=-\frac{a_{n,p_n,q_n}x^2}{\left(1+a_{n,p_n,q_n}x\right)}$. By using triangle inequality, we get

$$\left|R_{n,p_n,q_n}\left(t;x\right)-x\right|\leq\frac{a_{n,p_n,q_n}x^2}{\left(1+a_{n,p_n,q_n}x\right)}.$$

Now we may write

$$\left\|R_{n,p_n,q_n}\left(t;x\right)-x\right\|_2\leq\sup_{0\leq x<\infty}\frac{1}{1+x^2}\left\{\frac{a_{n,p_n,q_n}x^2}{\left(1+a_{n,p_n,q_n}x\right)}\right\}\leq a_{n,p_n,q_n}\cdot\sup_{0\leq x<\infty}\frac{x^2}{1+x^2\left(1+a_{n,p_n,q_n}x\right)}.$$

By taking the limit overall the last inequality, we have

$$\lim_{n\to\infty}\left\|R_{n,p_n,q_n}\left(t;x\right)-x\right\|_2\leq\lim_{n\to\infty}a_{n,p_n,q_n}\cdot 1=0.$$

Again by using Lemma 2 in [18], we have

$$R_{n,p_n,q_n}\left(t^2;x\right)-x^2=\frac{p_n^{n-1}}{a_{n,p_n,q_n}b_{n,p_n,q_n}}\left(\frac{a_{n,p_n,q_n}x}{1+a_{n,p_n,q_n}x}\right)+\frac{q_n\left[n-1\right]_{p_n,q_n}}{a_{n,p_n,q_n}b_{n,p_n,q_n}}\left(\frac{a_{n,p_n,q_n}x}{1+a_{n,p_n,q_n}x}\right)^2-x^2,$$

$$=\frac{p_n^{n-1}}{a_{n,p_n,q_n}b_{n,p_n,q_n}}\left(\frac{a_{n,p_n,q_n}x}{1+a_{n,p_n,q_n}x}\right)+\left\{\frac{q_n\left[n-1\right]_{p_n,q_n}}{\left[n\right]_{p_n,q_n}}\frac{1}{\left(1+a_{n,p_n,q_n}x\right)^2}-1\right\}x^2.$$

Therefore,

$$\left|R_{n,p_n,q_n}\left(t^2;x\right)-x^2\right|\leq\frac{p_n^{n-1}}{a_{n,p_n,q_n}b_{n,p_n,q_n}}\left(\frac{a_{n,p_n,q_n}x}{1+a_{n,p_n,q_n}x}\right)+\left\{1-\frac{q_n\left[n-1\right]_{p_n,q_n}}{\left[n\right]_{p_n,q_n}}\frac{1}{\left(1+a_{n,p_n,q_n}x\right)^2}\right\}x^2$$

Then, we have

$$\left\|R_{n,p_n,q_n}\left(t^2;x\right)-x^2\right\|_2 \le \frac{p_n^{n-1}}{b_{n,p_n,q_n}}\sup_{0\le x<\infty}\frac{x}{\left(1+x^2\right)\left(1+a_{n,p_n,q_n}x\right)}+\sup_{0\le x<\infty}\frac{x^2}{\left(1+x^2\right)}-\sup_{0\le x<\infty}\frac{x^2}{\left(1+x^2\right)\left(1+a_{n,p_n,q_n}x\right)^2}$$
$$-\frac{P_n^{n-1}}{[n]_{p_n,q_n}}\sup_{0\le x<\infty}\frac{x^2}{\left(1+x^2\right)\left(1+a_{n,p_n,q_n}x\right)^2}.$$

Now by taking the limit overall the last inequality, we have

$$\lim_{n\to\infty}\left\|R_{n,p_n,q_n}\left(t^2;x\right)-x^2\right\|_2 \le \lim_{n\to\infty}\frac{p_n^{n-1}}{b_{n,p_n,q_n}}\sup_{0\le x<\infty}\frac{x}{\left(1+x^2\right)\left(1+a_{n,p_n,q_n}x\right)}+\lim_{n\to\infty}\sup_{0\le x<\infty}\frac{x^2}{\left(1+x^2\right)}$$
$$-\lim_{n\to\infty}\sup_{0\le x<\infty}\frac{x^2}{\left(1+x^2\right)\left(1+a_{n,p_n,q_n}x\right)^2}-\lim_{n\to\infty}\frac{P_n^{n-1}}{[n]_{p_n,q_n}}\sup_{0\le x<\infty}\frac{x^2}{\left(1+x^2\right)\left(1+a_{n,p_n,q_n}x\right)^2}.$$

Hence,

$$\lim_{n\to\infty}\left\|R^*_{n,q_n}\left(t^2;x\right)-x^2\right\|_2=0.$$

Now, we present the next theorem to approximate all functions in the space $C_2^*[0,\infty)$. These types of results are given in [24] for locally integrable functions.

Theorem 3. Let $0<q_n<p_n<1$, $q_n\to1$ as $n\to\infty$. Then for each $f\in C_2^*[0,\infty)$ and all $\upsilon>0$, we have $\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|R_{n,p_n,q_n}(f,x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}=0$.

Proof. Let $x_0>0$. Then

$$\sup_{x\in[0,\infty)}\frac{\left|R_{n,p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}=\sup_{x\le x_0}\frac{\left|R_{n,p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}+\sup_{x>x_0}\frac{\left|R_{n,p_n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}$$

$$\le\left\|R_{n,p_n,q_n}(f;x)-f(x)\right\|_{C[0,x_0]}+\sup_{x\in[0,\infty)}\frac{\left|R_{n,p_n,q_n}\left(\left(1+t^2\right)f;x\right)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}$$

$$\le\left\|R_{n,p_n,q_n}(f;x)-f(x)\right\|_{C[0,x_0]}+\|f\|_2\sup_{x>x_0}\frac{\left|R_{n,p_n,q_n}\left(\left(1+t^2\right);x\right)\right|}{\left(1+x^2\right)^{1+\upsilon}}+\sup_{x>x_0}\frac{|f(x)|}{\left(1+x^2\right)^{1+\upsilon}}. \tag{4}$$

Now, by definition of the norm of each function belonging to $C_2^*[0,\infty)$, we have $|f(x)|\le\|f\|_2\left(1+x^2\right)$, also we have $\sup_{x>x_0}\frac{|f(x)|}{\left(1+x^2\right)^{1+\upsilon}}\le\frac{\|f\|_2}{\left(1+x^2\right)^{\upsilon}}\le\frac{\|f\|_2}{\left(1+x_0^2\right)^{\upsilon}}$.

Let $\varepsilon>0$. We can choose x_0 in such a way that

$$\frac{\|f\|_2}{\left(1+x_0^2\right)^{\upsilon}}<\frac{\varepsilon}{3}. \tag{5}$$

By Theorem 2, we get

$$\|f\|_2 \lim_{n\to\infty} \frac{\left|R_{n,p_n,q_n}\left(\left(1+t^2\right);x\right)\right|}{\left(1+x^2\right)^{1+\upsilon}} = \frac{1+x^2}{\left(1+x^2\right)^{1+\upsilon}}\|f\|_2 \le \frac{\|f\|_2}{\left(1+x^2\right)^{\upsilon}} \le \frac{\|f\|_2}{\left(1+x_0^2\right)^{\upsilon}} < \frac{\varepsilon}{3}.$$

By using Theorem 1, we can see that the first term of the inequality (4) implies that

$$\left\|R_{n,p_n,q_n}\left(f;x\right)-f\left(x\right)\right\|_{C[0,x_0]} < \frac{\varepsilon}{3}, \qquad \text{as } n\to\infty \tag{6}$$

By taking the limit over inequality (4) and combining inequalities (5) and (6), we get the desired result.

Next, we discuss the order of approximation of the functions $f \in C_2^*$ by the operators $R_{n,p,q}$ with the help of the following weighted modulus of continuity (see [25] and [26]).

The weighted modulus of continuity is defined by

$$\Omega_2\left(f;\delta\right) = \sup_{0<h<\delta,\, x\in[0,\infty)} \frac{\left|f\left(x+h\right)-f\left(x\right)\right|}{1+\left(x+h\right)^2}, \qquad \forall f \in C_2^*[0,\infty).$$

The weighted modulus of continuity and usual modulus of continuity have similar properties.

Lemma 1 ([26]). Let $f \in C_2^*[0,\infty)$. Then, we have the following:

1) $\Omega_2\left(f;\delta\right)$ is a monotonic increasing function of δ.
2) For each $f \in C_2^*$, $\lim_{\delta\to 0^+} \Omega_2\left(f;\delta\right)$.
3) For each $\lambda > 0$, $\Omega_2\left(f;\lambda\delta\right) \le \left(1+\lambda\right)\Omega_2\left(f;\delta\right)$.

In the following theorem we give the main convergence result which gives an expression of the approximation error with the operators $R_{n,p,q}$ using Ω_2.

Theorem 4. Let $f \in C_2^*$, $0 < q_n < p_n < 1$ such that $q_n \to 1$ as $n\to\infty$. Then we have the following inequality

$$\sup_{x\in[0,\infty)} \frac{\left|R_{n,p_n,q_n}\left(f,x\right)-f\left(x\right)\right|}{\left(1+x^2\right)^{\frac{5}{2}}} \le A_3\Omega_2\left(f,\sqrt{D_1}\right),$$

where, $A_3 = 2\left(1+A_1+A_2\right) > 0$ and $D_1 > 0$.

Proof. It is known that $R_{n,p_n,q_n}\left(1,x\right)=1$, by monotonicity of R_{n,p_n,q_n} , we may write

$$\left|R_{n,p_n,q_n}\left(f,x\right)-f\left(x\right)\right| \le R_{n,p_n,q_n}\left(\left|f\left(t\right)-f\left(x\right)\right|,x\right),$$

now, by using the definition of $\Omega_2\left(f;\delta\right)$ and (3) in the previous lemma we have

$$\left|f\left(t\right)-f\left(x\right)\right| \le \left(1+\left(x+\left|t-x\right|\right)^2\right)\Omega_2\left(f,\left|t-x\right|\right)$$

$$\left|f\left(t\right)-f\left(x\right)\right| \le 2\left(1+x^2\right)\left(1+\left(t-x\right)^2\right)\left(1+\frac{\left|t-x\right|}{\delta}\right)\Omega_2\left(f,\delta\right).$$

By using linearity and positivity properties of the operators R_{n,p_n,q_n} , we obtain

$$\left|R_{n,p_n,q_n}\left(f,x\right)-f\left(x\right)\right| \le 2\left(1+x^2\right)\Big\{\left(1+R_{n,p_n,q_n}\left(\left(t-x\right)^2,x\right)\right)$$
$$+R_{n,p_n,q_n}\left(\left(1+\left(t-x\right)^2\right)\frac{\left|t-x\right|}{\delta},x\right)\Big\}\Omega_2\left(f,\delta\right).$$

Applying the Cauchy-Schwarz inequality on the second term of the last inequality, we get

$$R_{n,p_n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta},x\right)\le\left(R_{n,p_n,q_n}\left(\left(\frac{|t-x|}{\delta}\right)^2,x\right)\right)^{\frac{1}{2}}+\left(R_{n,p_n,q_n}(t-x)^4,x\right)^{\frac{1}{2}}\left(R_{n,p_n,q_n}\left(\left(\frac{|t-x|}{\delta}\right)^2,x\right)\right)^{\frac{1}{2}},$$

$$\left|R_{n,p_n,q_n}(f,x)-f(x)\right|\le 2\left(1+x^2\right)\left\{\left(1+R_{n,p_n,q_n}\left((t-x)^2,x\right)\right)+\left(R_{n,p_n,q_n}\left(\left(\frac{|t-x|}{\delta}\right)^2,x\right)\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(R_{n,p_n,q_n}(t-x)^4,x\right)^{\frac{1}{2}}\left(R_{n,p_n,q_n}\left(\left(\frac{|t-x|}{\delta}\right)^2,x\right)\right)^{\frac{1}{2}}\right\}\Omega_2(f,\delta), \tag{7}$$

On the other hand, in [18], by using (12) and (13) in Lemma 4, we have the following inequalities

$$\left(1+R_{n,p_n,q_n}\left((t-x)^2,x\right)\right)\le A_1\left(1+x^2\right),\quad A_1>0. \tag{8}$$

$$\left(R_{n,p_n,q_n}\left((t-x)^4,x\right)\right)^{\frac{1}{2}}\le\left(\frac{1}{b_n^2}D_2\left(1+x^2\right)\right)^{\frac{1}{2}},\ \text{ where } D_2>0.$$

Since $\lim\limits_{n\to\infty}\frac{1}{b_n}=0$, there exists a positive constant A_2 such that

$$\left(R_{n,p_n,q_n}\left((t-x)^4,x\right)\right)^{\frac{1}{2}}\le A_2\left(1+x^2\right). \tag{9}$$

$$\left(R_{n,p_n,q_n}\left(\left(\frac{|t-x|}{\delta}\right)^2,x\right)\right)^{\frac{1}{2}}\le\left(D_1\frac{1}{\delta^2}\left(1+x^2\right)\right)^{\frac{1}{2}}\le\sqrt{D_1}\,\frac{1}{\delta}\left(1+x^2\right)^{\frac{1}{2}},\quad D_1>0. \tag{10}$$

For $0<q_n<p_n<1$, by substituting (8), (9) and (10) into the inequality (7), we can write

$$\left|R_{n,p_n,q_n}(f,x)-f(x)\right|\le 2\left(1+x^2\right)\left\{A_1\left(1+x^2\right)+\sqrt{D_1}\,\frac{1}{\delta}\left(1+x^2\right)^{\frac{1}{2}}\right.$$

$$\left.+A_2\left(1+x^2\right)\sqrt{D_1}\,\frac{1}{\delta}\left(1+x^2\right)^{\frac{1}{2}}\right\}\Omega_2(f,\delta).$$

Now, by taking $\delta=\frac{1}{\sqrt{D_1}}$, $A_3=2\left(1+A_1+A_2\right)$, and we get the desired result.

3. Conclusion

In this paper, by using the notion of (p,q)-calculus and weighted modulus of continuity we study weighted approximation properties of new (p,q)-analogue of the Balázs-Szabados operators. We give the rate of convergence for these operators and we give a Korovkin type theorem for weighted approximation.

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