HILL PUBLISHING

Journal of Applied Mathematics and Computation, 2022, 6(1), 24-29
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

Large Time Behavior of Entropy Solutions to Two-Dimensional Unipolar Hydrodynamic Model for Semiconductor Devices with Variable Coefficient Damping

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How to cite this paper: Lili Chen. (2022) Large Time Behavior of Entropy Solutions to Two-Dimensional Unipolar Hydrodynamic Model for Semiconductor Devices with Variable Coefficient Damping. *Journal of Applied Mathematics and Computation*, 6(1), 24-29.
DOI: 10.26855/jamc.2022.03.005

Received: January 6, 2022
Accepted: January 30, 2022
Published: February 22, 2022

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Abstract

This paper mainly studies the large time behavior of two-dimensional isothermal spherically symmetric compressible Euler-Poisson equations with variable coefficient damping in a bounded region. This equation and its variants have been used to describe the dynamic behavior of many important physical flows including the propagation of electrons in submicron semiconductor devices, the biological transport of ions for channel proteins, the motion of stars in the theory of general relativity and so on. This paper mainly proves that the weak solution converges exponentially to the unique stationary solution in time. The main methods are entropy estimation and energy method. Here, the key step is to construct an appropriate entropy estimation to cooperate with the energy method to obtain that when $t \to \infty$, the spherically symmetric weak solution of the isothermal spherically symmetric Euler-Poisson equations with variable coefficient damping converges to the unique smooth solution of the corresponding stationary equations at an exponential rate under the condition that the corresponding initial values are satisfied.

Keywords

Euler-Poisson equations, Damping, Large time behavior, Weak entropy solution

1. Introduction

In this paper, we consider the large time behavior of two-dimensional isothermal spherically symmetric compressible Euler-Poisson equations with variable coefficient damping.

The Euler-Poisson equations for compressible isothermal fluids are of the form:

$$\begin{cases} \rho_t + \nabla \cdot \vec{m} = 0 \\ \vec{m}_t + \nabla \left(\frac{\vec{m} \otimes \vec{m}}{\rho} \right) + \nabla p(\rho) = \rho \nabla \phi + H(\vec{x}, t)\vec{m}, \\ \Delta \phi = \rho - D(\vec{x}), \end{cases} \tag{1.1}$$

where space variable $\vec{x} \in R^N$, and $L_1 \leq |\vec{x}| \leq L_2$ (L_1 and L_2 are two positive constants), time variable $t \in [0, +\infty)$. Here, $\rho \geq 0$, m, $p(\rho)$ and $\nabla \phi$ denote the electron density, electron current density, pressure, and electric filed, respectively. The pressure $p(\rho) = \rho$ for the isothermal fluid. $D(\vec{x}) \geq 0$ is the doping profile. $H(\vec{x}, t)$ represents the damping coefficient.

We are interested in the radially symmetric solutions to system (1.1) with the form

$$(\rho,\vec{m},\phi)(\vec{x},t)=(\rho(x,t),m(x,t)\frac{\vec{x}}{x},\phi(x,t)),x=|\vec{x}|. \tag{1.2}$$

Then $(\rho(x,t),m(x,t),\phi(x,t))$ in (1.2) is governed by the one-dimensional Euler-Poisson equations with geometric source terms:

$$\begin{cases}\rho_t+m_x=-\frac{N-1}{x}m,\\ m_t+(\frac{m^2}{\rho}+\rho)_x=-\frac{N-1}{x}\frac{m^2}{\rho}+H(x,t)m+\rho\phi_x,\\ \phi_{xx}=-\frac{N-1}{x}\phi_x+\rho-D(x),x\epsilon[L_1,L_2],t\epsilon[0,+\infty),\end{cases} \tag{1.3}$$

where the doping profile $D(\vec{x})=D(x)$, and damping coefficient $H(\vec{x},t)=H(x,t)$ are radial symmetric functions.

We assume that $D(x)$ satisfies

$$D(x)\epsilon C[L_1,L_2],D^*=\sup_x D(x)\geq \inf_x D(x)=D_*>0, \tag{1.4}$$

Where $D^*$and D_* are two positive constants.

First, to make full use of the law of conservation of mass, we use following transformation:

$$n=x^{N-1}\rho,J=x^{N-1}m,E=x^{N-1}\phi_x. \tag{1.5}$$

Then, n, J, E satisfies the equations

$$\begin{cases}n_t+J_x=0,\\ J_t+(\frac{J^2}{n}+n)_x=\frac{N-1}{x}n+H(x,t)\,J+\frac{nE}{x^{N-1}}\\ E_x=n-x^{N-1}D(x)\coloneqq n-b(x).\end{cases} \tag{1.6}$$

We consider the initial boundary value condition

$$\begin{aligned}&(n,J)|_{t=0}=(n_0(x),J_0(x)),L_1\leq x\leq L_2,\\ &J(L_1,t)=J(L_2,t)=0,E(L_1)=E(L_2)=-1.\end{aligned} \tag{1.7}$$

Obviously, the first two equations of system (1.6) are typical 1-D system of balance laws. We can rewrite them into the following form:

$$\begin{cases}v_t+F(v)_x=G(x,v),x\epsilon[L_1,L_2],t\epsilon[0,+\infty)\\ v|_{t=0}=v_0(x),\end{cases} \tag{1.8}$$

where $v=(n,J)^T,\nabla F=\begin{pmatrix}0 & 1\\ -\frac{J^2}{n^2}+1 & \frac{2J}{n}\end{pmatrix}$ and the source term $G(x,v)=(0,\frac{N-1}{x}n+H(x,t)J+\frac{nE}{x^{N-1}})^T$.

In recent years, a lot of research has been carried out on Euler-Poisson systems in bounded and unbounded regions. For large time behavior, Huang-Pan-Yu [1] first studied the large time behavior of weak entropy solutions of Euler-Poisson systems, and established a large time behavior framework of arbitrary uniformly bounded weak entropy solutions, assuming that the upper bound of density is independent of time *t*. Yu [2] obtained the large time behavior of two-dimensional spherically symmetric Euler-Poisson system with constant coefficient damping. Then, when $\gamma>3$, Yu [3] improved the result in [1] and extended it to that the density can increase slowly with time, i.e. $\|\rho(x,t)\|_{L^\infty}\leq Ct^2$. Other results are shown in [4]-[12].

In [2], Yu studied the large time behavior of spherically symmetric weak entropy solutions of Euler-Poisson system with dimension $N=2$ and constant damping coefficient. Here, we generalize the results and prove the large time behavior of spherically symmetric weak entropy solutions of Euler-Poisson system with variable damping coefficients when dimension $N=2$.

Before introducing the main theorems in this paper, we first give the definition of weak entropy solution and stationary solution when $N=2$.

Definition 1.1 *A bounded measurable vector function* $v(x,t)$ *is called a global entropy solution of system (1.8) in* $\prod=\{(x,t)|L_1\leq x\leq L_2,t\geq 0\}$, *if* $v(x,t)$ *satisfies the following*:

(i) System (1.8) satisfies in the sense of distributions;

(ii) The entropy inequality holds in the sense of distributions, i.e.,

$$\eta_t+q_x-\nabla\eta\cdot G(x,v)\leq 0 \qquad in\ D'$$

for any weak entropy-entropy flux pair (η, q) *with convex* $\eta(v)$, *satisfying*

$$\nabla q(v)=\nabla\eta(v)\nabla F. \tag{1.9}$$

Definition 1.2 *The stationary solution of problems (1.6) (1.7) is the smooth solution of*

$$\begin{cases}\tilde{J}_x=0,\\ (\frac{\tilde{J}^2}{\tilde{n}}+\tilde{n})_x=\frac{\tilde{n}}{x}+H(x,t)\tilde{J}+\frac{\tilde{n}\tilde{E}}{x}\\ \tilde{E}_x=\tilde{n}-b(x),\end{cases} \tag{1.10}$$

with the boundary condition

$$\tilde{J}(L_1)=\tilde{J}(L_2)=0,L_1\le x\le L_2,$$
$$\tilde{E}(L_1)=\tilde{E}(L_2)=-1. \tag{1.11}$$

Using the viscosity vanishing method and compensation compactness, we can prove the existence of global entropy weak solutions. This result will be proved in another article, and we omit it here. Throughout this paper, we assume:

(**H1**) Let $(n,J,E)\,(x,t)$ be any globally defined entropy weak solution of (1.6) (1.7) and the density $n(x,t)$ satisfies

$$0\le n(x,t)\le C_0,(x,t)\in\Pi$$

for certain positive constant C_0.

Since $\tilde{J}=0$, according to the results in [2], for the stationary problem (1.10), we have the following existence and uniqueness theorem:

Theorem 1.1 *We assume* $b(x)\epsilon L^2[L_1,L_2]$ *and satisfy*

$$0<b_*\le b(x)\le b^*,x\epsilon[L_1,L_2]. \tag{1.12}$$

Then the stationary equations (1.10) has a unique solution $(\tilde{n}(x),\tilde{E}(x))$ *on* $C^1[L_1,L_2]\times C^1[L_1,L_2]$ *and satisfies*

$$0<b_*\le\tilde{n}(x)\le b^*,x\epsilon[L_1,L_2], \tag{1.13}$$

where b∗ and b∗ are two positive constants.

Now, we introduce new variables.

$$y(x,t)=-\int_{L_1}^{x}\big(n(s,t)-\tilde{n}(s)\big)ds=-\big(E-\tilde{E}\big)(x,t),x\epsilon[L_1,L_2]. \tag{1.14}$$

Then

$$y_x=-(n-\tilde{n})(x,t),y_t=J(x,t). \tag{1.15}$$

The relative entropy flux pair is defined as

$$\eta_*(n,J)=\frac{J^2}{2n}+n\ln n-\tilde{n}\ln\tilde{n}-(\ln\tilde{n}+1)(n-\tilde{n}), \tag{1.16}$$

$$q_*(n,J)=\frac{J^3}{2n^2}+(\ln n+1)J-(\ln\tilde{n}+1)J. \tag{1.17}$$

Then we give our main result in Section 3 as follows.

Theorem 1.2 *(Large time behavior) Suppose N = 2, denote* $(n,J,E)\,(x,t)$ *is the global entropy solution of (1.6) (1.7),* $(\tilde{n}(x),\tilde{E}(x))$ *is the stationary solution,*$b(x)\epsilon L^2[L_1,L_2]$*, and*

$$0<b_*\le b(x)\le b^*,x\epsilon[L_1,L_2].$$

Assume the initial value $(n_0(x),\ J_0(x))$ *satisfied*

$$\int_{L_1}^{L_2}\big(n_0(x)-b(x)\big)dx=0,$$

$$y(x,0)\epsilon L^2[L_1,L_2],\int_{L_1}^{L_2}\eta_*(x,0)dx<\infty. \tag{1.18}$$

The damping coefficient meet $H(x,t)<-\delta,H_t(x,t)>\frac{1}{L_1^2}+\frac{2}{L_1}(\frac{L_2-1}{2L_2}b^*-\frac{7L_1+1}{6L_1}b_*)$*, where* δ *is a small positive constant. Then, it holds that*

$$\int_{L_1}^{L_2}(y^2+{y_t}^2+{y_x}^2)dx\le Ce^{-\tilde{C}t}\int_{L_1}^{L_2}(\eta_*(x,0)+y^2(x,0))dx, \tag{1.19}$$

for some positive constants C*and* $\tilde{C}$.

Remark 1.1 $H(x,t) < -\delta$ *indicates that it is a damping.* $H_t(x,t) > \frac{1}{L_1^2} + \frac{2}{L_1}\left(\frac{L_2-1}{2L_2}b^* - \frac{7L_1+1}{6L_1}b_*\right)$ *indicates that the damping cannot be too weak. For example, when* $t \to \infty, H(x,t) \to 0$, *this condition is not feasible.*

2. Preliminaries

We first consider the following homogeneous system of conservation laws:

$$\begin{cases} n_t + J_x = 0, \\ J_t + (\frac{J^2}{n} + n)_x = 0. \end{cases} \tag{2.1}$$

The eigenvalues are

$$\lambda_1 = \frac{J}{n} - 1, \lambda_2 = \frac{J}{n} + 1, \tag{2.2}$$

and the corresponding right eigenvectors are

$$\mathrm{r}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \mathrm{r}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}. \tag{2.3}$$

The Riemann invariants (w,z) are given by

$$\mathrm{w} = \frac{J}{n} + \ln n, z = \frac{J}{n} - \ln n, \tag{2.4}$$

satisfying $\nabla w \cdot r_1 = 0$ and $\nabla z \cdot r_2 = 0$, where $\nabla = (\partial_n, \partial_J)$ is the gradient with respect to (n,J).

The mechanical energy $\eta^*(n,J)$ and mechanical energy flux $q^*(n,J)$ have the following formula

$$\eta^*(n,J) = \frac{J^2}{2n} + n\ln n, q^*(n,J) = \frac{J^3}{2n^2} + J\ln n. \tag{2.5}$$

3. Large time behavior

In this section, our aim is to prove theorem 1.2. We need to prove that when $t \to \infty$, Under condition (1.18), the spherically symmetric weak entropy solution (n,J,E) of (1.6) (1.7) converges to the unique smooth solution $\left(\tilde{n},\tilde{J},\tilde{E}\right)$ of (1.10) (1.11) at an exponential rate. Combining (1.6) and (1.10), we know that y satisfies the following equation:

$$y_{tt} + (\frac{J^2}{n})_x - y_{xx} - H(x,t)y_t - \frac{yy_x - \tilde{E}y_x - \tilde{n}y}{x} + \frac{y_x}{x} = 0. \tag{3.1}$$

The corresponding boundary conditions are

$$\begin{aligned} y(L_2,t) &= -\int_{L_1}^{L_2}\big(n(x,t) - \tilde{n}(x)\big)dx = -\int_{L_1}^{L_2}\big(n_0(x) - \tilde{n}(x)\big)dx = -\int_{L_1}^{L_2}\big(n_0(x) - b(x) + b(x) - \tilde{n}(x)\big)dx \\ &= -\int_{L_1}^{L_2}\big(n_0(x) - b(x)\big)dx + \tilde{E}(L_2) - \tilde{E}(L_1) = 0 = y(L_1,t). \end{aligned} \tag{3.2}$$

Multiplyyby (3.1) and then integrate on $[L_1, L_2]$ to get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{L_1}^{L_2}\left(yy_t - \frac{H(x,t)}{2}y^2\right)dx + \int_{L_1}^{L_2}{y_x}^2dx - \int_{L_1}^{L_2}{y_t}^2dx - \int_{L_1}^{L_2}\frac{J^2}{n}y_xdx + \int_{L_1}^{L_2}\frac{H_t(x,t)}{2}y^2dx \\ &= \int_{L_1}^{L_2}\frac{y^2y_x - \tilde{E}yy_x - \tilde{n}y^2}{x}dx - \int_{L_1}^{L_2}\frac{yy_x}{x}dx, \end{aligned} \tag{3.3}$$

i.e.,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{L_1}^{L_2}\left(yy_t - \frac{H(x,t)}{2}y^2\right)dx + \int_{L_1}^{L_2}{y_x}^2dx \\ &= \int_{L_1}^{L_2}{y_t}^2dx + \int_{L_1}^{L_2}\frac{J^2}{n}y_xdx - \int_{L_1}^{L_2}\frac{H_t(x,t)}{2}y^2dx - \int_{L_1}^{L_2}\frac{y^2}{2x^2}dx \\ &+ \int_{L_1}^{L_2}\frac{y^2}{x^2}\left(\frac{y}{3x} - \tilde{n} + \frac{\tilde{E}_x}{2} - \frac{\tilde{E}}{2x}\right)dx, \end{aligned} \tag{3.4}$$

where

$$\frac{y}{3x} - \tilde{n} + \frac{\tilde{E}_x}{2} - \frac{\tilde{E}}{2x} = -\frac{\tilde{n}+b(x)}{2} - \frac{1}{3x}\int_{L_1}^{x}(n-\tilde{n})dx - \frac{1}{2x}(\int_{L_1}^{x}(\tilde{n}-b(x))dx - 1)$$
$$\leq \frac{1}{3x}\int_{L_1}^{x}(b(x)-n)dx - \frac{1}{6x}\int_{L_1}^{x}(\tilde{n}-b(x))dx - b_* + \frac{1}{2x}$$
$$\leq -b_* + \frac{1}{3x}(x-1)b^* + \frac{1}{6x}(x-1)(b^*-b_*) + \frac{1}{2x}$$
$$\leq \frac{L_2-1}{2L_2}b^* - \frac{7L_1+1}{6L_1}b_* + \frac{1}{2L_1}, \tag{3.5}$$

so,

$$-\frac{H_t(x,t)}{2} - \frac{1}{2x^2} + \frac{1}{x}\left(\frac{y}{3x} - \tilde{n} + \frac{\tilde{E}_x}{2} - \frac{\tilde{E}}{2x}\right)$$
$$\leq -\frac{H_t(x,t)}{2} + \frac{1}{2L_1^2} + \frac{1}{x}\left(\frac{L_2-1}{2L_2}b^* - \frac{7L_1+1}{6L_1}b_*\right)$$
$$\leq -\frac{H_t(x,t)}{2} + \frac{1}{2L_1^2} + \frac{1}{L_1}\left(\frac{L_2-1}{2L_2}b^* - \frac{7L_1+1}{6L_1}b_*\right). \tag{3.6}$$

And because we already know

$$H_t(x,t) > \frac{1}{L_1^2} + \frac{2}{L_1}\left(\frac{L_2-1}{2L_2}b^* - \frac{7L_1+1}{6L_1}b_*\right). \tag{3.7}$$

Then there exists $\sigma > 0$ such that

$$-\frac{H_t(x,t)}{2} - \frac{1}{2x^2} + \frac{1}{x}\left(\frac{y}{3x} - \tilde{n} + \frac{\tilde{E}_x}{2} - \frac{\tilde{E}}{2x}\right) \leq -\sigma. \tag{3.8}$$

Thus, we can rewrite (3.4) as follows:

$$\frac{d}{dt}\int_{L_1}^{L_2}(yy_t - \frac{H(x,t)}{2}y^2)dx + \int_{L_1}^{L_2}{y_x}^2dx + \sigma\int_{L_1}^{L_2}y^2dx \leq \int_{L_1}^{L_2}{y_t}^2dx + \int_{L_1}^{L_2}\frac{J^2}{n}y_x\,dx = \int_{L_1}^{L_2}\frac{\tilde{n}}{n}{y_t}^2dx, \forall x\epsilon[L_1,L_2]. \tag{3.9}$$

In addition, according to definitions (1.16) and (1.17) of $\eta^*$and q^*, we have the following inequality holds in the sense of distribution:

$$\eta_{*t} + q_{*x} + \frac{\tilde{n}_x}{\tilde{n}}J - \frac{JE}{x} - \frac{H(x,t)J^2}{n} - \frac{J}{x} \leq 0. \tag{3.10}$$

We notice that

$$\frac{JE}{x} = \frac{J\tilde{n}_x}{\tilde{n}} - \frac{J}{x} - \frac{yy_t}{x}. \tag{3.11}$$

Putting (3.11) into (3.10), we have

$$\eta_{*t} + q_{*x} - \frac{H(x,t)J^2}{n} + \frac{yy_t}{x} \leq 0. \tag{3.12}$$

Integrate (3.12) on interval $[L_1, L_2]$ to arrive at

$$\frac{d}{dt}\int_{L_1}^{L_2}(\eta_* + \frac{y^2}{2x})dx - \int_{L_1}^{L_2}\frac{H(x,t)J^2}{n}dx \leq 0. \tag{3.13}$$

Let Λ sufficiently big so that $\Lambda > \frac{b^*}{\delta_0} + \| n \|_{L^\infty} + 1$, multiply (3.13) by Λ and add the result to (3.9), we have

$$\frac{d}{dt}\int_{L_1}^{L_2}(\Lambda\eta_* + \left(\frac{\Lambda}{2x} - \frac{H(x,t)}{2}\right)y^2 + yy_t)dx + \int_{L_1}^{L_2}({y_x}^2 + \sigma y^2 + \frac{-\Lambda H - \tilde{n}}{n}{y_t}^2)dx \leq 0. \tag{3.14}$$

Since

$$-\Lambda H - \tilde{n} > \left(\frac{b^*}{\delta_0} + \|n\|L_\infty + 1\right)\delta_0 - b^* > \|n\|L_\infty\delta_0, \tag{3.15}$$

so there exists $\tilde{C}$ such that

$$\frac{d}{dt}\int_{L_1}^{L_2}(\Lambda\eta_* + \left(\frac{\Lambda}{2x} - \frac{H(x,t)}{2}\right)y^2 + yy_t)dx + \tilde{C}\int_{L_1}^{L_2}({y_x}^2 + y^2 + \frac{{y_t}^2}{n})dx \leq 0. \tag{3.16}$$

We define

$$\eta_* = Q_1 + Q_2, Q_1 = \frac{J^2}{2n}, Q_2 = n\ln n - \tilde{n}\ln\tilde{n} - (\ln\tilde{n} + 1)(n - \tilde{n}).$$

Obviously, Q_2 is the quadratic remainder of $n\ln n$ according to the Taylor expansion of $\tilde{n}$, because of the convexity of $n\ln n$, we know that Q_2 is nonnegative, so there are two positive constants $C_1$and C_1 such that

$$C_1({y_x}^2 + {y_t}^2) \le \eta_* \le C_2({y_x}^2 + \frac{{y_t}^2}{n}),$$

And$x \in [L_1, L_2]$, then there exists two positive constants C_3 and C_4 such that

$$C_3(y^2 + {y_x}^2 + {y_t}^2) \le \Lambda\eta_* + (\frac{\Lambda}{2x} - \frac{H(x,t)}{2})y^2 + yy_t \le C_4(y^2 + \eta_*).$$

We set

$$F(t) = \int_{L_1}^{L_2}(\Lambda\eta_* + (\frac{\Lambda}{2x} - \frac{H(x,t)}{2})y^2 + yy_t)dx.$$

Then we can rewrite (3.16) into the following form:

$$\frac{d}{dt}F(t) + C_5F(t) \le 0. \tag{3.17}$$

Using Gronwall inequality for (3.17), we obtain

$$C_3\int_{L_1}^{L_2}(y^2 + {y_x}^2 + {y_t}^2)(x,t)dx \le F(t) \le F(0)e^{-C_5t} \le C_4e^{-C_5t}\int_{L_1}^{L_2}(y^2 + \eta_*)(x,0)dx. \tag{3.18}$$

This completes the proof of the Theorem 1.2.

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