Existence and Uniqueness of Solution for a Class of Seven-Order Exponential Fuzzy Difference Equations

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Abstract

At present, the difference equations studied by most scholars are ordinary difference equations with real parameters and initial values. However, for the difference equation describing many natural phenomena, the parameter information is uncertain, incomplete and fuzzy. Based on the above fact, if the parameters and initial values in the ordinary difference equation model are transformed into fuzzy numbers, the research on the existence and uniqueness of its solution will have greater practical value and significance. In this paper, a class of seven-order exponential fuzzy difference equations is studied. Firstly, the fuzzy difference equation is transformed into the corresponding ordinary difference equations with parameter by using the fuzzy set theory, in which the value range of the parameter is 0 to 1. Then, by using iterative method, inequality technique and mathematical induction, the existence and uniqueness of solutions of ordinary differential equations are obtained. Thus, the existence and uniqueness for the solution of the exponential fuzzy difference equation is proved.

Keywords

Fuzzy Difference Equation, Positive Solution, Existence, Uniqueness

1. Introduction and Preliminaries

Difference equations are usually expressed in the form of recursive sequences, which are widely used in various practical problems. It is worth noting that the difference equation model has been important developed and applied in the fields of infectious disease dynamics [1], computer [2], engineering [3], biology [4], physics [5]. A large number of studies show that the dynamic behavior of difference equations with order greater than 1 is relatively complex, but can be better applied to practical problems. Therefore, it has attracted the interest of many experts and scholars, and also produced a large number of meaningful results [6-9].

In this paper, we study the following seven-order fuzzy difference equation

\[ x_{n+1} = \frac{A x_{n-1}}{B + C x_{n-2}^p x_{n-4}^q x_{n-6}^r}, \quad n = 0,1,2,\cdots, \]

where the parameters \( p, q, r \) are positive real number, \( A, B, C \) and the initial conditions \( x_{-6}, \cdots, x_{-1}, x_0 \) are positive fuzzy numbers. When the parameters \( p = q = r = 1 \), \( A, B, C \) and initial conditions are positive real numbers, Abo-Zeid
[10] studied the global asymptotic stability of solutions to the above ordinary difference equation (1).

Next, we give some definitions and preliminary results, which can be found in [11-13].

**Definition 1** Let \( X \) be a non-empty set, assume \( T \) is a mapping from \( X \) to \([0,1]\), a.e. \( T : X \rightarrow [0,1] \), \( x \rightarrow T(x), x \in X \), then we say \( T \) is a fuzzy set on \( X \), \( T(x) \) be called a membership function on a fuzzy set \( T \).

**Definition 2** For a set \( X \), we denote by \( \overline{X} \) the closure of \( X \). Assume \( T \) is a fuzzy set and \( \alpha \in (0,1] \), the \( \alpha \)-cuts of \( T \) on \( R \) is defined as \( [T]_\alpha = \{x \in R : T(x) \geq \alpha \} \) and \( \overline{[T]}_\alpha = \{x \in R | T(x) > 0 \} \). It is clear that the \( [T]_\alpha \) is a bounded closed interval in \( R \) for any specific \( \alpha \in [0,1] \).

**Definition 3** We say that a fuzzy set \( T \) is a fuzzy number if it satisfies the following properties:

1. \( T \) is a normal fuzzy set, i.e., there exists \( x \in R \) such that \( T(x) = 1 \);
2. \( T \) is a fuzzy convex set, i.e., \( T(ax + (1-a)y) \geq \min \{T(x), T(y)\}, \forall a \in (0,1), x, y \in R \);
3. \( T \) is upper semicontinuous on \( R \);
4. \( T \) is compactly supported, i.e., \( \text{supp}(T) = \bigcup_{\alpha \in [0,1]} [T]_\alpha \).

Let us denote by \( R_f \) the set of all fuzzy numbers. A fuzzy number \( T \) is positive if \( \text{supp}(u) \subset (0, +\infty) \), we denote by \( R^+_f \) the space of all positive fuzzy numbers. Similarly, a fuzzy number \( T \) is negative if \( \text{supp}(u) \subset (-\infty, 0) \), we denote by \( R^-_f \) the space of all negative fuzzy numbers. If \( T \) is a positive real number, then \( T \) is also a positive fuzzy number with \( [T]_\alpha = [T,T], \alpha \in [0,1] \), and we say that \( T \) is a trivial fuzzy number.

**Definition 4** For any \( u, v \in R_f, [u]_\alpha = [u_{l,a}, u_{r,a}], [v]_\alpha = [v_{l,a}, v_{r,a}] \), and \( \lambda \in R \), the sum \( u + v \), the scalar product \( \lambda u \), multiplication \( uv \) and division \( \frac{u}{v} \) in the standard interval arithmetic (SIA) setting are defined by

\[
[u + v]_\alpha = [u]_\alpha + [v]_\alpha, \quad [\lambda u]_\alpha = \lambda [u]_\alpha, \quad [uv]_\alpha = \min \{u_{l,a}v_{l,a}, u_{l,a}v_{r,a}, u_{r,a}v_{l,a}, u_{r,a}v_{r,a}\}, \quad \max \{u_{l,a}v_{l,a}, u_{l,a}v_{r,a}, u_{r,a}v_{l,a}, u_{r,a}v_{r,a}\}.
\]

**Lemma 1** Let \( I_x, I_y \) be some intervals of real numbers and let \( f : I_x^{k+1} \times I_y^{l+1} \rightarrow I_x, g : I_x^{k+1} \times I_y^{l+1} \rightarrow I_y \) be continuously differentiable functions. Then for every set of initial conditions \( (x_i, y_j) \in I_x \times I_y, (i = -k, -k+1, \ldots, 0, j = -l, -l+1, \ldots, 0) \), the following system of difference equations

\[
\begin{align*}
x_{n+1} &= f(x_n, x_{n+1}, \ldots, x_{n+k}, y_n, y_{n+1}, \ldots, y_{n+l}), \\
y_{n+1} &= g(x_n, x_{n+1}, \ldots, x_{n+k}, y_n, y_{n+1}, \ldots, y_{n+l}),
\end{align*}
\]

has a unique solution \( \{(x_i, y_j)\}_{i, j = -k, -l}^{k, l} \). Here, \( I_x^{k+1} \) is the product of \( k+1 \) intervals \( I_x \), and \( I_y^{l+1} \) is the product of \( l+1 \) intervals \( I_y \).

**Lemma 2** [14] Let \( u \in R_f \), write \( [u]_\alpha = [u_{l,a}, u_{r,a}], \alpha \in (0,1] \), then \( u_{l,a} : u_{r,a} \) are functions on \((0,1]\), which satisfy the following conditions: (a) \( u_{l,a} \) is non-decreasing and left continuous; (b) \( u_{r,a} \) is non-increasing and left continuous; (c) \( u_{l,a} \leq u_{r,a} \). Conversely for any functions \( a(\alpha) \) and \( b(\alpha) \) defined on \((0,1]\) which satisfy (a)-(c) in the above, there exists a unique \( u \in R_f \) such that \( u(\alpha) = [a(\alpha), b(\alpha)] \) for any \( \alpha \in (0,1] \).

2. **Main results and proofs**

In this section, by using fuzzy set theory, iterative method, inequality technology and mathematical induction as well as the above lemmas, the existence and uniqueness of positive solutions of the seven-order exponential fuzzy difference
equation (1) are studied.

**Theorem 1** For the fuzzy difference equation (1), if the parameters \( p, q, r \) are positive real number, \( A, B, C \) and initial conditions \( x_{a}, \ldots, x_{a}, x_{b} \) are positive fuzzy numbers, then for any \( x_{a}, \ldots, x_{a}, x_{b} \) there exists a unique positive fuzzy solution \( \{x_{a}\} \) to the fuzzy difference equation (1).

**Proof.** Assume for initial values \( x_{a}, \ldots, x_{a}, x_{b} \), there exists a sequence of positive fuzzy numbers \( \{x_{a}\} \) that satisfies the difference equation (1).

Considering \( \alpha - cuts \) of the parameters \( A, B, C \) and the initial values \( x_{a}, \ldots, x_{a}, x_{b} \), we have

\[
\begin{align*}
A_{\alpha} &= \left[ A_{l,\alpha}, A_{r,\alpha} \right], B_{\alpha} = \left[ B_{l,\alpha}, B_{r,\alpha} \right], C_{\alpha} = \left[ C_{l,\alpha}, C_{r,\alpha} \right],
\end{align*}
\]

\[
\begin{align*}
E_{\alpha} &= \left[ E_{l,\alpha}, E_{r,\alpha} \right], F_{\alpha} = \left[ F_{l,\alpha}, F_{r,\alpha} \right], G_{\alpha} = \left[ G_{l,\alpha}, G_{r,\alpha} \right],
\end{align*}
\]

\[
\begin{align*}
x_{\alpha} &= \left[ x_{l,\alpha}, x_{r,\alpha} \right],
\end{align*}
\]

Then from (1), (2) and **Definition 3**, we have

\[
\begin{align*}
L_{\alpha+1,\alpha} &= \frac{A_{l,\alpha} L_{\alpha+1,\alpha} + B_{l,\alpha} + C_{l,\alpha} R_{\alpha+1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} R_{\alpha+1,\alpha}},
\end{align*}
\]

Moreover, according to **Lemma 1**, for any initial condition \( (L_{0,\alpha}, R_{0,\alpha}) \) \( (i = 0, 1, \ldots, 6), \alpha \in (0, 1) \), there exists a unique positive solution \( (L_{n,\alpha}, R_{n,\alpha}), n = 1, 2, \ldots \) to system (3).

Conversely, we need to prove that \( (L_{n,\alpha}, R_{n,\alpha}), \alpha \in (0, 1) \) determines the solution \( \{x_{a}\} \) to the equation (1) with initial conditions \( x_{a}, \ldots, x_{a}, x_{b} \) such that

\[
\begin{align*}
x_{\alpha} &= \left[ L_{n,\alpha}, R_{n,\alpha} \right], \alpha \in (0, 1), \text{ } n = -6, -5, -4, -3, -1, 0, \ldots.
\end{align*}
\]

In view of the parameters and initial conditions \( A, B, C, x_{a}, x_{a}, x_{b} \) are positive fuzzy numbers, for arbitrarily \( \alpha, \alpha \in (0, 1), \alpha \leq \alpha \), according to **Lemma 2**, it follows that

\[
\begin{align*}
0 < A_{l,\alpha} &\leq A_{r,\alpha}, 0 < B_{l,\alpha} \leq B_{r,\alpha}, 0 < C_{l,\alpha} \leq C_{r,\alpha},
\end{align*}
\]

Next, using the mathematical induction it hold that

\[
\begin{align*}
0 < L_{n,\alpha} \leq R_{n,\alpha}, \alpha \in (0, 1), \text{ } n = 1, 2, \ldots.
\end{align*}
\]

From (5), it easy to see that (6) hold for \( n = -6, \ldots, -1, 0 \). Suppose that (6) are true for \( n \leq k, \text{ } k \in \{0, 1, 2, \ldots\} \), then from (4)-(6), we have that for \( n = k + 1 \)

\[
\begin{align*}
L_{k+1,\alpha} &= \frac{A_{l,\alpha} L_{k+1,\alpha} + B_{l,\alpha} + C_{l,\alpha} R_{k+1,\alpha}}{B_{l,\alpha} + C_{l,\alpha} R_{k+1,\alpha}},
\end{align*}
\]

Next, we will prove that the support set \( \bigcup_{\alpha \in (0, 1)}(L_{n,\alpha}, R_{n,\alpha}) \) of \( x_{a} \) is compact. It is easy to know that we just need to
prove that \( \bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \) is bounded.

When \( n = 1 \), because of \( A, B, C, x_i, (i = 0, 1, \ldots, 6) \) are positive fuzzy numbers, so there exists positive constants \( M_A, N_A, M_B, N_B, M_C, N_C, M_j, N_j \) \((j = -6, -5, -2, -1, 0)\), so that for all \( \alpha \in (0,1] \), it follows that
\[
\begin{align*}
[A_{i,\alpha}, A_{i,\alpha}] & \subseteq [M_A, N_A], \\
[B_{i,\alpha}, B_{i,\alpha}] & \subseteq [M_B, N_B], \\
[C_{i,\alpha}, C_{i,\alpha}] & \subseteq [M_C, N_C], \\
[L_{j,\alpha}, R_{j,\alpha}] & \subseteq [M_j, N_j](j = -6, -5, -2, -1, 0).
\end{align*}
\]
(8)

So, from (7) and (8), it holds that
\[
\begin{align*}
[L_{1,\alpha}, R_{1,\alpha}] & \subseteq \left[\frac{M_1 M_{-1}}{N_2 + N_3 N_5 N_6}, \frac{N_1 N_{-1}}{N_2 + M_2 M_4 M_5 M_6}\right], \alpha \in (0,1].
\end{align*}
\]
Thus, we have
\[
\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subseteq \left[\frac{M_1 M_{-1}}{N_2 + N_3 N_5 N_6}, \frac{N_1 N_{-1}}{N_2 + M_2 M_4 M_5 M_6}\right], \alpha \in (0,1]
\]
(9)

According to (9), we can obtain that \( \bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \) is compact and \( \bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \subset (0, +\infty) \). Moreover, from mathematical induction it can be concluded that \( \bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \) is compact. Thus, it holds that
\[
\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}] \subset (0, +\infty), n = 1, 2, \ldots.
\]
(10)

Therefore, from Lemma 2, relations (6) and (10), and \( L_{n,\alpha}, R_{n,\alpha} \) are left continuous it is proved that \( [L_{n,\alpha}, R_{n,\alpha}] \) determines a sequence of positive fuzzy numbers \( \{x_n\} \), which satisfies equation (1).

Now, we prove that \( \{x_n\} \) is the solution to equation (1) with initial values \( x_{-6}, \ldots, x_{-1}, x_0 \). Because for all \( \alpha \in (0,1] \), we have
\[
\begin{align*}
[L_{n+1,\alpha}, R_{n+1,\alpha}] & = \left[\frac{A_{i,\alpha} L_{n-1,\alpha} + C_{i,\alpha} R_{n-1,\alpha}}{B_{i,\alpha} + C_{i,\alpha} R_{n-1,\alpha}}, \frac{A_{i,\alpha} R_{n-1,\alpha} + C_{i,\alpha} L_{n-1,\alpha}}{B_{i,\alpha} + C_{i,\alpha} R_{n-1,\alpha}}\right] \\
& = \left[\frac{A_{i,\alpha} x_{n-1} + C_{i,\alpha} x_{n-1}}{B_{i,\alpha} + C_{i,\alpha} x_{n-1}}\right].
\end{align*}
\]
So \( \{x_n\} \) is the solution to exponential fuzzy difference equation (1) with \( x_{-6}, \ldots, x_{-1}, x_0 \) as initial conditions.

Finally, we prove that the solution of equation (1) is unique. Suppose \( \{\overline{x}_n\} \) is another solution to equation (1) with initial values \( x_{-6}, \ldots, x_{-1}, x_0 \). In the light of the above method, it holds that
\[
\begin{align*}
[\overline{x}_{n+1}] & = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0,1], n = -6, \ldots, -1, 0.
\end{align*}
\]
(11)

According to (4) and (11), it follows that
\[
[x_n] = [\overline{x}_n], \alpha \in (0,1], n = -6, \ldots, -1, 0,
\]
from which it holds that \( x_n = \overline{x}_n, n = -2, -1, 0, \ldots \), and then the proof is completed.

3. Acknowledgments

This work is supported by the Scientific Research Fund of Chengdu University of Information Technology (Grant no. KYTZ201820) of China, the State Key Laboratory of Robotics (Grant no. 2019-O13) of China, the Sichuan Science and Technology Program (Grant no. 2021ZYD0009) of China.
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