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Galerkin Weighted Residual Method for Solving Fourth Order Fractional Differential and Integral Boundary Value Problems

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Abstract

In this research work, the Galerkin weighted residual method is used to find the numerical solution of fourth-order fractional value problems with homogeneous and non-homogeneous boundary conditions. The same approach is applied also to compute the approximate solutions for the two-point fourth-order integro-differential problem in fractional order. The matrix formulation of both cases is enunciated explicitly using piecewise polynomials. The operator expressing the Caputo fractional derivatives is used in this procedure. We experiment various cases from the literature in order to calculate the accuracy and efficacy of the current technique using Legendre and Bernoulli polynomials as bases. We find that the present solutions converge to the exact solutions. The absolute errors are tabulated, and we believe that absolute reliability has been achieved. The proposed method may be implemented to partial differential equations of fractional order.

Keywords

Galerkin Method, Fractional Derivatives, Riemann-Liouville Derivative, Caputo Derivative, Fractional Order BVP, Integro-differential Equations

1. Introduction

Fractional orders of differentiation are more perplexing because, unlike the traditional introduction to derivatives and integrals as slopes and areas, they have no obvious geometric meaning. The book of Podlubny [1] contains the most well-known works on fractional calculus. The numerical solutions of linear and non-linear boundary value problems of fractional-order, as well as their applications, were established by Anatoly et al. [2]. Diethelm [3] pioneered the analytical analysis to differential equations of fractional order, and the theory using the Riemann-Liouville and Caputo operators. Oldham and Spanier [4] used diffusive transport in semi-infinite media to illustrate how the differ-integral may be used to solve diffusion issues. Herrmann [5] discussed the practical implications of fractional calculus in numerous disciplines of physics, including quantum mechanics. Li and Zeng [6] provided finite difference techniques for fractional ordinary differential equations such as the Euler method, fractional Adams method, and fractional linear multistep approach. The finite element technique is a sophisticated computational tool for approximate solutions to boundary value issues. Accordingly, the weighted residual approach solves boundary value problems accurately in [7].

Islam and Shirin [8] provided the numerical solution of linear and non-linear 2nd order boundary value problems using Bernoulli polynomials. Al-Refai et al. [9] established the series solution of the fractional order differential equations with the starting condition. Many authors investigated the solution of second order fractional differential equations using various methods, including Legendre collocation method [10], Adomian Decomposition method [11], Galerkin Residual Method [12], Sinc-Galerkin Method [13], the spline collocation approach for demonstrating the correctness of the results of second order two-point fractional boundary value problems [14-18].

Kasi and Murali [19] provided Galerkin's technique using quintic B-splines as the basis function for solving fourth-order boundary value problems. Islam and Bellal used Legendre polynomials to solve linear and non-linear fourth-order two-point boundary value problems using the Galerkin method [20].

Many scholars afterward used alternative approaches to solve fourth-order fractional boundary value problems that arise in many domains of science and engineering, including physics, fluid mechanics, modeling of viscoelastic and inelastic flows, control theory, statistics, and neurology. Zahra and Elkholy [21] performed converging analysis and solve fourth-order fractional differential equations using quintic polynomial spline functions. The quintic spline collocation technique was used to find the solution to fourth-order fractional boundary value problems in Caputo's fractional derivatives [22]. Ali and Esra [23] introduced the kernel Hilbert space approach for addressing fourth-order fractional boundary value issues. The quintic spline approach was utilized by Khalid et al. [24] to discuss fourth-order fractional boundary value problems.

The Galerkin approach [31] was utilized by Rahman and Islam [25] to solve the Volterra Integral equations using Laguerre polynomials as a trial function. Shirin and Islam [26] used the Galerkin technique with piecewise Bernstein polynomials to solve non-singular Fredholm Integral equations. Rajagopal and Balaji [27] used the Bernoulli wavelet operational matrix of derivatives and integration with the Gaussian quadrature method to solve fractional Volterra integrodifferential equations. Mohamed [28] used the Taylor series expansion method to solve fractional singular integro-differential equations with the Cauchy Kernel. Jani et al. [29] presented the solutions of fractional integro-differential equations with Bernstein polynomials. The Adomian decomposition approach was used by Momani and Noor [30] to present the linear and non-linear fourth order fractional integro-differential equations.

In this paper, we consider the following general fourth-order FBVP:

$$v^{(4)}(x) + D^{\alpha} q(x) v(x) = g(x), \qquad x \in [a,b] \tag{1}$$

subject to the boundary conditions

$$v(a) = A_0, \quad v(b) = B_0, \quad v''(a) = A_1, \quad v''(b) = B_1$$

where A_0, A_1, B_0, B_1 are real constants and $q(x)$ is continuous function defined on $[a,b]$, and fractional integro-differential equation is given by

$$D^{-\alpha} v^{(4)}(x) + v(x) + \int_0^x v(t)dt = g(x), \qquad x \in [a,b] \tag{2}$$

subject to the boundary conditions $v(a) = A_0, \; v(b) = B_0, \; v''(a) = A_1, \; v''(b) = B_1$

where A_0, A_1, B_0, B_1 are real constants. Equation (2) is convert into the following FIBVPs:

$$D^{\alpha} v^{(4)}(x) + D^{\alpha} v(x) + D^{\alpha} \int_0^x v(t)dt = \tilde{g}(x), \qquad x \in [a,b] \tag{3}$$

where,

$$\tilde{g}(x) = D^{\alpha} g(x).$$

The main objective of this paper is to solve fractional order boundary value problems (BVP) using the Galerkin weighted residual method. We explain how to solve non-homogeneous fractional differential equations using this method. Two linear boundary value problems and two fourth-order integro-differential equations are tested by this technique. The suggested approach, which is more accurate and innovative than the prior method, produces an approximate answer. We are confident that the piecewise polynomial with weighted residual method will play a critical role in expanding the use of fractional differential equations.

The following is a representation of the paper: Section 2 defines some fundamental fractional calculus notions and notations. Section 3 is confined to discuss the matrix formulation to fourth order BVP of fractional order, followed by the numerical results and discussions. Finally, in Section 4, we develop the matrix formulation to integro-differential equations, of fractional order problems, for the numerical solutions.

2. Preliminaries

We consider some basic definitions and elementary facts of fractional calculus in this section. The common definitions of Gamma functions and some important lemma of Riemann-Liouville and Caputo's fractional derivatives are also discussed here [6].

Definition 1. (a) By D, we denote the operator that maps a differentiable function onto its derivative, i.e., $Df(x) := f'(x)$. (b) By J_a, we denote the operator that maps a function f, assumed to be Reimann integrable on the compact interval $[a,b]$, into its primitive centered at a, i.e., $J_a f(x) := \int f(t)dt$ for $a \le x \le b$. (c) For $n \in N$, we use the symbols D_n and J_a^n to denote the n-fold iterates of D and J_a, respectively, i.e., we set $D_1 := D$, $J_a^1 := J_a$, and $D_n := DD_{n-1}$ and $J_a^n := J_a J_{n-1}$ for $n \ge 2$.

Definition 2. The function $\Gamma(x) : (0,\infty) \to \Re$, defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \tag{4}$$

is called Euler's Gamma function or Euler's integral of the second kind [6].

$$J_a f(x) = \frac{1}{(n-1)!} \int (x-t)^{n-1} f(t) dt \tag{5}$$

Lemma If $f(x)$ is continuous and $\alpha, \beta > 0$, then the following relationships hold [1]

(i) $D^{-\alpha} D^{-\beta} f(x) = D^{-\beta} D^{-\alpha} f(x) = D^{-\alpha-\beta} f(x)$

(ii) $D^{-\alpha} x^m = \dfrac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha}$

Example $D^m f(x) = \dfrac{\Gamma(k+1)}{\Gamma(k+1-m)} (x-a)^{k-m}$ and $D_a^n f(x) = \dfrac{\Gamma(k+1)}{\Gamma(k+1-n)} (x-a)^{k-n}$

Comparing these two expression [6], we find in the limit $n \to m$:

$D_a^n f(x) \to D_m f(x)$ uniformly on $[a,b]$ if $m < k$.

For $m = k$ we obtain that $D_m f(x) = \Gamma(m+1)$, which is a non zero constant, whereas

$$D_a^n f(a) = \begin{cases} 0 \ if \quad n < m \\ \infty \ if \quad n > m \end{cases} \tag{6}$$

Modified Legendre Polynomials

The Rodrigues formula for the Legendre polynomials is

$$p_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - x)^n \tag{7}$$

To satisfy the condition $p_n(0) = p_n(1) = 0, \quad n \ge 1$, we modify the Legendre polynomials in equation (7) as [20]:

$$p_n(x) = [\frac{1}{n!} \frac{d^n}{dx^n} (x^2 - x)^n - (-1)^n](x-1) \tag{8}$$

Bernoulli Polynomials

The Bernoulli polynomials of degree *n* can be defined over the interval [0, 1] implicitly as [20]

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^{n-k} \tag{9}$$

where, b_k are Bernoulli numbers given by

$$b_0 = 1 \text{ and } b_k = -\int_0^1 B_k(x) dx \quad k \ge 1 \tag{10}$$

Also equations (10) can be written explicitly as $B_0(x) = 1$

$$B_m(x)=\sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^m-\sum_{k=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^m, \qquad m\geq 1 \tag{11}$$

3. Matrix Formulation of Fractional Fourth Order Differential Equations

We use Modified Legendre and Bernoulli polynomials as basis functions to obtain the approximate solutions to the BVPsin the Galerkin weighted residual (GWR) method. Let $v(x)$ be the exact solution while $\tilde{v}(x)$ be the approximate solutionto a BVP.

In this case, let us consider linear fourth order fractional order differential equation:

$$p_0(x)\frac{d^4v}{dx^4}+p_1(x)\frac{d^\alpha v}{dx^\alpha}+p_2(x)v(x)=g(x), \qquad 0<x<1 \tag{12}$$

subject to the boundary conditions

$$v(0)=A_0,\ v(1)=B_0,\ v''(0)=A_2,\ \ v''(1)=B_2$$

We assume an approximate solution in a form

$$\tilde{v}(x)=\phi_0(x)+\sum_{i=1}^{n}a_iN_i(x) \tag{13}$$

Choose $\phi_0(x)=0$ and $N_i(0)=N_i(1)=0$ for each $i=1,2,\ldots n$

Now the residual function is given by

$$\varepsilon(x)=p_0(x)\frac{d^4\tilde{v}}{dx^4}+p_1(x)\frac{d^\alpha\tilde{v}}{dx^\alpha}+p_2(x)\tilde{v}(x)-g(x) \tag{14}$$

The Galerkin weighted residual equations are

$$\int_0^1\varepsilon(x)N_j(x)dx=0$$

$$\int_0^1\left[p_0(x)\frac{d^4\tilde{v}}{dx^4}+p_1(x)\frac{d^\alpha\tilde{v}}{dx^\alpha}+p_2(x)\tilde{v}(x)-g(x)\right]N_j(x)dx=0$$

$$\int_0^1\left[p_0(x)\frac{d^4\tilde{v}}{dx^4}+p_1(x)\frac{d^\alpha\tilde{v}}{dx^\alpha}+p_2(x)\tilde{v}(x)\right]N_j(x)dx=\int_0^1 g(x)N_j(x)dx, \qquad j=1,2,.\ .n. \tag{15}$$

Integrating by parts on the left hand side of the equation (15), we obtain

$$\int_0^1 p_0\frac{d^4\tilde{v}}{dx^4}N_jdx=\left[p_0N_j\frac{d^3\tilde{v}}{dx^3}\right]_0^1-\int_0^1\frac{d}{dx}\left[p_0N_j\right]\frac{d^3\tilde{v}}{dx^3}dx \text{ [since } N_j(0)=N_j(1)=0\text{]}$$

$$=-\left[\frac{d}{dx}\left[p_0N_j\right]\frac{d^2\tilde{v}}{dx^2}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\left[p_0N_j\right]\frac{d^2\tilde{v}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\left[p_0N_j\right]\frac{d^2\tilde{v}}{dx^2}\right]_0^1+\left[\frac{d^2}{dx^2}\left[p_0N_j\right]\frac{d\tilde{v}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[p_0N_j\right]\frac{d\tilde{v}}{dx}dx$$

$$=-\left[\frac{d}{dx}\left[p_0N_j\right]\right]_{x=1}\times B_2+\left[\frac{d}{dx}\left[p_0N_j\right]\right]_{x=0}\times A_2+\left[\frac{d^2}{dx^2}\left[p_0N_j\right]\frac{d\tilde{v}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\left[p_0N_j\right]\frac{d\tilde{v}}{dx}dx \tag{16}$$

Again,

$$\int_0^1 p_1\frac{d^\alpha\tilde{v}}{dx^\alpha}N_jdx=\left[p_1N_j\int\frac{d^\alpha\tilde{v}}{dx^\alpha}\right]_0^1-\int_0^1\left\{\frac{d}{dx}\left[p_1N_j\right]\int\frac{d^\alpha\tilde{v}}{dx}dx\right\}dx=-\int_0^1\left\{\frac{d}{dx}\left[p_1N_j\right]\int\frac{d^\alpha\tilde{v}}{dx^\alpha}dx\right\}dx \tag{17}$$

[since $N_j(0) = N_j(1) = 0$]

Substituting equations (16) and (17) into equation (15) and using approximation for $\tilde{v}(x)$ given in equation (13), we get a system of equations in the matrix form as

$$\sum_{i=1}^{n} k_{i,j} a_i = G_j, \quad j = 1, 2, ..., n \tag{18}$$

where

$$k_{i,j} = \int_0^1 \left\{ \left[\frac{d^3}{dx^3}\left[p_0 N_j\right] \frac{dN_i}{dx} + \left[-\frac{d}{dx}\left[p_1 N_j\right] \int \frac{d^\alpha N_i}{dx^\alpha} dx \right] + N_i N_j \right] \right\} dx - \left[\frac{d^2}{dx^2}\left[p_0 N_j\right] \frac{dN_i}{dx} \right]_{x=0}^{x=1} \tag{19}$$

$$G_j = \int_0^1 g(x) N_j(x) dx + \left[\frac{d}{dx}\left[p_0 N_j\right] \right]_{x=1} \times B_2 - \left[\frac{d}{dx}\left[p_0 N_j\right] \right]_{x=0} \times A_2 \tag{20}$$

Solving the above system of equations specified by equation (18), we compute the values of the parameters, and then substituting into equation (13) to get the approximate solution of the desired FBVP (12).

Examples and Discussions

Here, we consider two numerical problems to verify the proposed method.

Example 1. Consider the following fourth order fractional BVP [22, 23]

$$v^{(4)}(x) + D^\alpha x v(x) = g(x), \quad \forall x \in [0,1] \tag{21}$$

with boundary conditions

$$v(0) = v(1) = 0, \quad v''(0) = 0, \quad v''(1) = 26(\alpha - 1)$$

The exact solution of this problem is, $v(x) = x^6(x^\alpha - x^{2-\alpha})$.

The obtained approximate solution is

$$\tilde{v}(x) = 0.00028x + 0.01695x^2 - 0.24147x^3 + 1.23534x^4 - 3.11381x^5 + 4.80121x^6 - 2.69851x^7.$$

We compute the absolute errors using first 6 polynomials of each type: Modified Legendre and Bernoulli in example 1. We compare the accuracy of the absolute errors versus the errors obtained by the Quintic spline collocation method [22], also by kernel Hilbert space
method [23], which are shown in Table 1. In this case, the present method is reliable with good accuracy.

Table 1. Absolute errors obtained by GWR method using Legendre polynomials for example 1, when $(\alpha = 0.1, 0.2, 0.3)$

	Absolute errors obtained by GWR method using 6 polynomials						
x	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		
	Legendre	Bernoulli	Legendre	Bernoulli	Legendre	Bernoulli	Ali &Esra [23]
0	0.	0.	0.	0.	0.	0.	1.54×10^{-11}
0.1	4.73×10^{-5}	4.71×10^{-5}	2.83×10^{-5}	2.88×10^{-5}	5.33×10^{-5}	5.27×10^{-5}	1.50×10^{-4}
0.2	3.08×10^{-5}	3.10×10^{-5}	4.48×10^{-5}	4.40×10^{-5}	2.14×10^{-5}	2.05×10^{-5}	2.92×10^{-4}
0.3	4.67×10^{-5}	4.69×10^{-5}	6.44×10^{-5}	6.35×10^{-5}	2.80×10^{-5}	2.72×10^{-5}	4.16×10^{-4}
0.4	5.89×10^{-5}	5.88×10^{-5}	1.74×10^{-5}	1.82×10^{-5}	1.08×10^{-4}	1.07×10^{-4}	5.10×10^{-4}
0.5	1.26×10^{-5}	1.26×10^{-5}	7.06×10^{-5}	7.12×10^{-5}	1.55×10^{-4}	1.54×10^{-4}	5.65×10^{-4}
0.6	5.64×10^{-5}	5.65×10^{-5}	1.49×10^{-5}	1.52×10^{-5}	1.03×10^{-4}	1.03×10^{-4}	5.71×10^{-4}
0.7	4.91×10^{-5}	4.89×10^{-5}	6.52×10^{-5}	6.52×10^{-5}	2.34×10^{-5}	2.38×10^{-5}	5.20×10^{-4}

0.8	3.25×10^{-5}	3.23×10^{-5}	4.37×10^{-5}	4.37×10^{-5}	1.78×10^{-5}	1.83×10^{-5}	4.05×10^{-4}
0.9	4.5×10^{-5}	4.51×10^{-5}	2.76×10^{-5}	2.76×10^{-5}	4.83×10^{-5}	4.87×10^{-5}	2.30×10^{-4}
1.	0.	2.188×10^{-15}	0.	1.936×10^{-15}	0.	1.687×10^{-15}	2.08×10^{-7}

Example 2. Consider the following fourth order FBVP [22, 23, 24]:

$$v^{(4)}(x)+0.05D^{\alpha}v(x)=g(x), \quad \forall x\in[0,1] \tag{22}$$

with boundary conditions

$$v(0)=v(1)=0,\ v''(0)=0,\ \ v''(1)=8$$

The exact solution of this problem is, $v(x)=x^4(x-1)$.

The required approximate solution is

$$\tilde{v}(x)=-0.000015x-0.0000069x^2+0.0000449x^3-1.000019x^4+1.0000084x^5-0.000024x^6$$
$$+0.000012x^7 .$$

In this example, we compute the absolute errors using first 6 polynomials of each type: Modified Legendre and Bernoulli. We compare the accuracy of the absolute errors versus the errors obtained by the Quintic spline collocation method [22] and kernel Hilbert space
method [23], respectively, are shown in Table 2. Our accuracy is reliable.

Table 2. Absolute errors obtained by GWR method using Legendre and Bernoulli polynomials for example 2, when $(\alpha=0.1,0.2,0.3,0.99)$

	Absolute errors obtained by GWR method using 6 polynomials								
x	$\alpha=0.1$		$\alpha=0.2$		$\alpha=0.3$		$\alpha=0.99$		
	Legendre	Bernoulli	Legendre	Bernoulli	Legendre	Bernoulli	Legendre	Bernoulli	Ali &Esra [23]
0	0	0	0	0	0	0	0	0	0.
0.1	1.60×10^{-6}	1.77×10^{-6}	3.95×10^{-6}	3.95×10^{-6}	6.69×10^{-6}	6.57×10^{-6}	4.43×10^{-5}	3.66×10^{-5}	1.17×10^{-4}
0.2	3.10×10^{-6}	3.35×10^{-6}	7.51×10^{-6}	7.48×10^{-6}	1.26×10^{-5}	1.24×10^{-5}	7.15×10^{-5}	6.99×10^{-5}	2.05×10^{-4}
0.3	4.29×10^{-6}	4.57×10^{-6}	1.02×10^{-5}	1.02×10^{-5}	1.71×10^{-5}	1.70×10^{-5}	8.48×10^{-5}	9.67×10^{-5}	2.50×10^{-4}
0.4	5.02×10^{-6}	5.30×10^{-6}	1.19×10^{-5}	1.18×10^{-5}	1.99×10^{-5}	1.98×10^{-5}	1.01×10^{-4}	1.14×10^{-4}	2.50×10^{-4}
0.5	5.21×10^{-6}	5.47×10^{-6}	1.23×10^{-5}	1.22×10^{-5}	2.06×10^{-5}	2.05×10^{-5}	1.19×10^{-4}	1.19×10^{-4}	2.13×10^{-4}
0.6	4.85×10^{-6}	5.08×10^{-6}	1.15×10^{-5}	1.14×10^{-5}	1.91×10^{-5}	1.90×10^{-5}	1.25×10^{-4}	1.12×10^{-4}	1.53×10^{-4}
0.7	4.01×10^{-6}	4.19×10^{-6}	9.53×10^{-6}	9.41×10^{-6}	1.58×10^{-5}	1.57×10^{-5}	1.05×10^{-4}	9.38×10^{-5}	8.94×10^{-5}
0.8	2.81×10^{-6}	2.94×10^{-6}	6.71×10^{-6}	6.60×10^{-6}	1.10×10^{-5}	1.10×10^{-5}	6.44×10^{-5}	6.60×10^{-5}	3.72×10^{-5}
0.9	1.43×10^{-6}	1.49×10^{-6}	3.42×10^{-6}	3.35×10^{-6}	5.61×10^{-6}	5.62×10^{-6}	2.56×10^{-5}	3.34×10^{-5}	7.85×10^{-6}
1.	0	2.22×10^{-16}	0	2.22×10^{-16}	0	2.22×10^{-16}	0	2.22×10^{-16}	1.59×10^{-8}

4. Formulation of Fractional Integro-differential Equation

We consider Modified Legendre and Bernoulli polynomials as basis functions to obtain the numerical solutions to the BVP exploiting the proposed method. Let $w(x)$ and $\tilde{w}(x)$ be the exact and approximate trial solution, respectively.

Fractional integro differential equation is given by

$$D^{-\alpha}w^{(4)}(x)+w(x)+\int_0^x w(t)dt=f(x), \qquad x\in[a,b] \tag{23}$$

Subject to the boundary conditions $w(a)=A_0$, $w(b)=B_0$, $w''(a)=A_1$, $w''(b)=B_1$

Where A_0, A_1, B_0, B_1 are real constants. Equation (23) is convert into the following FIBVPs:

$$w^{(4)}(x)+D^{\alpha}w(x)+D^{\alpha}\int_0^x w(t)d\ \ =t\tilde{f}(x), \qquad 0<x<1 \tag{24}$$

where

$$\tilde{f}(x)=D^{\alpha}f(x).$$

We assume that an approximate solution of the form:

$$\tilde{w}(x)=\phi_0(x)+\sum_{i=1}^{n}a_iL_i(x) \tag{25}$$

Choose $\phi_0(x)=0$ and $L_i(0)=L_i(1)=0$ for each $i=1,2,...n$

Now the residual function is given by

$$\varepsilon(x)=w^{(4)}(x)+D^{\alpha}w(x)+D^{\alpha}\int_0^x w(t)d\ \ -t\tilde{f}(x) \tag{26}$$

The Galerkin weighted residual equations are

$$\int_0^1 \varepsilon(x)L_j(x)dx=0$$

$$\int_0^1\Big[p_0(x)\frac{d^4\tilde{w}}{dx^4}+p_1(x)\frac{d^{\alpha}\tilde{w}}{dx^{\alpha}}+p_2(x)\int_0^x\ \tilde{w}(t)dt-\tilde{f}(x)\Big]L_j(x)dx=0$$

$$\int_0^1\Big[p_0(x)\frac{d^4\tilde{w}}{dx^4}+p_1(x)\frac{d^{\alpha}\tilde{w}}{dx^{\alpha}}+p_2(x)\int_0^x\ \tilde{w}(t)dt\Big]L_j(x)dx=\int_0^1\tilde{f}(x)L_j(x)dx, \qquad j=1,2,.\ .n. \tag{27}$$

Integrating by parts on the left hand side of the equation (27), we obtain

$$\int_0^1 p_0\frac{d^4\tilde{w}}{dx^4}L_jdx=\left[p_0L_j\frac{d^3\tilde{w}}{dx^3}\right]_0^1-\int_0^1\frac{d}{dx}\big[p_0L_j\big]\frac{d^3\tilde{w}}{dx^3}dx \quad [\text{since } \ L_j(0)=L_j(1)=0\,]$$

$$=-\left[\frac{d}{dx}\big[p_0L_j\big]\frac{d^2\tilde{w}}{dx^2}\right]_0^1+\int_0^1\frac{d^2}{dx^2}\big[p_0L_j\big]\frac{d^2\tilde{w}}{dx^2}dx$$

$$=-\left[\frac{d}{dx}\big[p_0L_j\big]\frac{d^2\tilde{w}}{dx^2}\right]_0^1+\left[\frac{d^2}{dx^2}\big[p_0L_j\big]\frac{d\tilde{w}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\big[p_0L_j\big]\frac{d\tilde{w}}{dx}dx$$

$$=-\left[\frac{d}{dx}\big[p_0L_j\big]\right]_{x=1}\times B_2+\left[\frac{d}{dx}\big[p_0L_j\big]\right]_{x=0}\times A_2+\left[\frac{d^2}{dx^2}\big[p_0L_j\big]\frac{d\tilde{w}}{dx}\right]_0^1-\int_0^1\frac{d^3}{dx^3}\big[p_0L_j\big]\frac{d\tilde{w}}{dx}dx \tag{28}$$

Again

$$\int_0^1 p_1\frac{d^{\alpha}\tilde{w}}{dx^{\alpha}}L_jdx=\left[p_1L_j\int\frac{d^{\alpha}\tilde{w}}{dx^{\alpha}}\right]_0^1-\int_0^1\left\{\frac{d}{dx}\big[p_1L_j\big]\int\frac{d^{\alpha}\tilde{w}}{dx}dx\right\}dx=-\int_0^1\left\{\frac{d}{dx}\big[p_1L_j\big]\int\frac{d^{\alpha}\tilde{w}}{dx^{\alpha}}dx\right\}dx \tag{29}$$

[since $L_j(0)=L_j(1)=0$]

$$p_2\int_0^x[\tilde{w}(t)dt]L_j=\int_0^1[\int_0^x p_2L_j(t)dt]dx \tag{30}$$

Substituting equation (28), (29) and (30) into equation (27) and using approximation for $\tilde{w}(x)$ given in equation (26), we get a system of equations in the matrix form as

$$\sum_{i=1}^{n}k_{i,j}a_i=F_j, \qquad j=1,2,...,n \tag{31}$$

where

$$k_{i,j} = \int_0^1 \left\{ \left[\frac{d^3}{dx^3}\left[p_0 L_j\right] \frac{dL_i}{dx} + \left[-\frac{d}{dx}\left[p_1 L_j\right] \right] \int \frac{d^\alpha L_i}{dx^\alpha} dx \right] + L_i L_j + [\int_0^x p_2 L_j(t)dt] L_i \right\} dx - \left[\frac{d^2}{dx^2}\left[p_0 L_j\right] \frac{dL_i}{dx} \right]_{x=0}^{x=1} \quad (32)$$

$$F_j = \int_0^1 \tilde{f}(x) L_j(x) dx + \left[\frac{d}{dx}\left[p_0 L_j\right] \right]_{x=1} \times B_2 - \left[\frac{d}{dx}\left[p_0 L_j\right] \right]_{x=0} \times A_2 \quad (33)$$

Solving the system of equations specified by equation (31), we find the values of parameters and then substituting into equation (25), we get the approximate solution of the desired FBVP (24).

5. Results and Discussion

In this case, we consider two problems with non-homogeneous boundary conditions to verify the proposed method. The approximate results using the present method are computed for the problems of equation (34).

Example 3. Consider the fractional integro-differential boundary value problem [21]

$$D^{-\alpha} w^{(4)}(x) + w(x) + \int_0^x w(t)\,dt = f(x), \quad \forall x \in [0\ 1] \quad (34)$$

with boundary conditions

$$w(0) = 1, \quad w''(0) = 2, \quad w(1) = 1 + e, \quad w''(1) = 3e$$

where

$$f(x) = x(1 + e^x) + 3e^x$$

The exact solution of this problem is $w(x) = 1 + xe^x$.

The approximate solution of the given problem is

$$\tilde{w}(x) = e^x - 0.13994x + 0.50016x^2 + 0.58308x^3 + 0.02334x^4 + 0.03261x^5 + 0.00204x^6 - 0.00130x^7 .$$

In this problem, we represent the absolute error using first 6 polynomials of each type: Modified Legendre and Bernoulli polynomials. We compare the accuracy of the obtained results with the spline solutions [21] which are shown in Table 3. In this case, the exactness is found with good acceptance compare to the solutions of the existing method and exact solutions.

Table 3. Absolute errors obtained by GWR method using Legendre and Bernoulli polynomials for example 3, when $(\alpha = 0, 0.25, 0.5, 0.75)$

	Absolute errors obtained by GWR method using 6 polynomials											
x	$\alpha = 0$			$\alpha = 0.25$			$\alpha = 0.5$			$\alpha = 0.75$		
	Legendre	Bernoulli	W. K. ZAHRA[21]	Legendre	Bernoulli	Zahra [21]	Legendre	Bernoulli	Zahra [21]	Legendre	Bernoulli	Zahra [21]
0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	1.42×10^{-4}	1.33×10^{-4}	5.17×10^{-4}	1.43×10^{-3}	1.44×10^{-3}	4.04×10^{-3}	3.62×10^{-3}	3.61×10^{-3}	1.12×10^{-2}	5.88×10^{-3}	5.86×10^{-3}	1.82×10^{-2}
0.2	2.38×10^{-4}	2.30×10^{-4}	6.03×10^{-3}	2.78×10^{-3}	2.80×10^{-3}	6.90×10^{-4}	6.88×10^{-3}	6.88×10^{-3}	1.24×10^{-2}	1.11×10^{-2}	1.10×10^{-2}	2.53×10^{-2}
0.3	2.78×10^{-4}	2.73×10^{-4}	8.46×10^{-3}	3.94×10^{-3}	3.96×10^{-3}	8.23×10^{-3}	9.49×10^{-3}	9.49×10^{-3}	1.33×10^{-2}	1.51×10^{-2}	1.51×10^{-2}	3.07×10^{-2}
0.4	2.63×10^{-4}	2.60×10^{-4}	1.01×10^{-2}	4.78×10^{-3}	4.82×10^{-3}	1.29×10^{-2}	1.12×10^{-2}	1.11×10^{-2}	1.39×10^{-2}	1.76×10^{-2}	1.76×10^{-2}	3.40×10^{-2}
0.5	2.07×10^{-4}	2.04×10^{-4}	1.08×10^{-2}	5.22×10^{-3}	5.26×10^{-3}	1.45×10^{-2}	1.18×10^{-2}	1.18×10^{-2}	1.41×10^{-2}	1.83×10^{-2}	1.83×10^{-2}	3.50×10^{-2}
0.6	1.31×10^{-4}	1.28×10^{-4}	1.05×10^{-2}	5.16×10^{-3}	5.21×10^{-3}	1.27×10^{-2}	1.13×10^{-2}	1.13×10^{-2}	1.40×10^{-2}	1.73×10^{-2}	1.73×10^{-2}	3.36×10^{-2}
0.7	5.84×10^{-5}	5.63×10^{-5}	8.70×10^{-3}	4.56×10^{-3}	4.62×10^{-3}	7.86×10^{-3}	9.75×10^{-3}	9.74×10^{-3}	1.35×10^{-2}	1.47×10^{-2}	1.47×10^{-2}	3.01×10^{-2}

0.8	8.71×10^{-6}	8.10×10^{-6}	6.80×10^{-3}	3.44×10^{-3}	3.49×10^{-3}	3.02×10^{-3}	7.16×10^{-3}	7.15×10^{-3}	1.26×10^{-2}	1.06×10^{-2}	1.06×10^{-2}	2.46×10^{-2}
0.9	8.10×10^{-6}	8.29×10^{-6}	3.61×10^{-3}	1.86×10^{-3}	1.89×10^{-3}	9.28×10^{-3}	3.80×10^{-3}	3.79×10^{-3}	1.14×10^{-2}	5.62×10^{-3}	5.62×10^{-3}	1.77×10^{-2}
1.	0	0	1.82×10^{-6}	0	0	1.82×10^{-6}	0	0	1.71×10^{-7}	0	0	1.82×10^{-6}

Example 4. Consider the fractional integro-differential boundary value problem [30]

$$D^{-\alpha} w^{(4)}(x) - \int_0^x e^{-t} w^2(t)\, dt = 1, \quad \forall x \in [0.1] \tag{35}$$

with boundary conditions

$$w(0) = 1, \; w''(0) = 1, \; w(1) = e, \; w''(1) = e$$

The exact solution of this problem is $w(x) = e^x$.

The approximate solution of the given problem is

$$\tilde{w}(x) = e^x - 1.77 \times 10^{-6} x + 0.0483x^2 - 0.4353x^3 + 1.4030x^4 - 2.0999x^5$$
$$+ 1.4903x^6 - 0.4064x^7.$$

In this problem, we represented absolute error using first 6 polynomials of each type: Modified Legendre and Bernoulli polynomials. We compare the accuracy of the obtained result with the spline solutions [30] which are shown in Table 4. In this case, a good agreement is found to the solutions of the existing method and exact solutions.

Table 4. Absolute errors obtained by GWR method using Legendre and Bernoulli polynomials for example 4, when $(\alpha = 0, 0.25, 0.5, 0.75)$

Absolute errors obtained by GWR method using 6 polynomials												
x	$\alpha = 0$			$\alpha = 0.25$			$\alpha = 0.5$			$\alpha = 0.75$		
	Legendre	Bernoulli	Momani&Noor [30]	Legendre	Bernoulli	Momani&Noor [30]	Legendre	Bernoulli	Momani&Noor [30]	Legendre	Bernoulli	Momani&Noor [30]
0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	5.93×10^{-9}	8.14×10^{-8}	1.09×10^{-5}	9.54×10^{-7}	8.21×10^{-8}	9.79×10^{-4}	1.68×10^{-4}	1.95×10^{-6}	1.57×10^{-3}	1.54×10^{-4}	1.75×10^{-6}	1.37×10^{-3}
0.2	7.61×10^{-8}	1.48×10^{-7}	2.27×10^{-5}	6.47×10^{-8}	1.50×10^{-7}	1.81×10^{-3}	1.14×10^{-4}	4.57×10^{-7}	2.91×10^{-3}	1.08×10^{-4}	4.47×10^{-7}	2.52×10^{-3}
0.3	8.18×10^{-8}	2.01×10^{-7}	3.88×10^{-5}	1.79×10^{-6}	2.03×10^{-7}	2.44×10^{-3}	1.43×10^{-4}	2.89×10^{-6}	3.89×10^{-3}	1.20×10^{-3}	2.39×10^{-6}	3.34×10^{-3}
0.4	2.47×10^{-8}	2.37×10^{-7}	4.46×10^{-5}	1.99×10^{-6}	2.39×10^{-6}	2.80×10^{-3}	2.72×10^{-4}	3.22×10^{-6}	4.44×10^{-3}	2.52×10^{-3}	3.12×10^{-6}	3.77×10^{-3}
0.5	2.17×10^{-7}	2.55×10^{-7}	4.12×10^{-5}	2.54×10^{-7}	2.58×10^{-6}	2.88×10^{-3}	1.51×10^{-4}	2.21×10^{-8}	4.54×10^{-3}	1.31×10^{-3}	2.11×10^{-8}	3.82×10^{-3}
0.6	4.17×10^{-7}	2.54×10^{-7}	4.88×10^{-5}	1.50×10^{-6}	2.57×10^{-6}	2.70×10^{-3}	7.12×10^{-5}	3.28×10^{-6}	4.23×10^{-3}	6.52×10^{-3}	3.02×10^{-6}	3.53×10^{-3}
0.7	5.23×10^{-7}	2.31×10^{-7}	4.27×10^{-5}	1.36×10^{-6}	2.33×10^{-6}	2.26×10^{-3}	1.68×10^{-4}	2.97×10^{-6}	3.53×10^{-3}	1.48×10^{-3}	2.17×10^{-6}	2.92×10^{-3}
0.8	4.59×10^{-7}	1.83×10^{-7}	3.09×10^{-5}	3.95×10^{-7}	1.85×10^{-6}	1.62×10^{-3}	7.32×10^{-5}	3.66×10^{-7}	2.53×10^{-3}	7.01×10^{-3}	3.46×10^{-7}	2.08×10^{-3}
0.9	2.31×10^{-7}	1.07×10^{-7}	2.31×10^{-5}	1.14×10^{-6}	1.08×10^{-6}	3.46×10^{-4}	3.28×10^{-5}	1.90×10^{-6}	1.32×10^{-3}	3.28×10^{-5}	1.30×10^{-6}	1.08×10^{-3}
1.	0	0	1.82×10^{-6}	0	0	1.82×10^{-6}	0	0	1.71×10^{-7}	0	0	1.82×10^{-6}

6. Conclusions

The goal of this approach is to solve fractional fourth-order fractional boundary value problems in a simple and consistent manner. The Galerkin approach mentioned in this study is also implemented to fourth-order integro-differential equations. The precise and approximate solutions for fourth-order fractional integro-differential equations are compared to linear BVPs. This procedure is more exact and provides more accurate findings. This approach can be extended for fractional partial differential equations in the Caputo sense.

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