Journal of Applied Mathematics and Computation, 2022, 6(3), 290-309
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

On Triple Laplace-Aboodh-Sumudu Transform and Its Properties with Applications

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How to cite this paper: Ali Al-Aati, Mona Hunaiber, Yasmin Ouideen. (2022) On Triple Laplace-Aboodh-Sumudu Transform and Its Properties with Applications. *Journal of Applied Mathematics and Computation*, 6(3), 290-309.
DOI: 10.26855/jamc.2022.09.003

Received: May 22, 2022
Accepted: June 20, 2022
Published: July 28, 2022

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Abstract

In the literature, there are many different types of integral transforms such as Laplace transform, Aboodh transform, Sumudu transform, Ezaki transform, Shehu transform, and so on. These kinds of integral transforms have many applications in various fields of mathematical sciences and engineering such as physics, mechanics, etc. We combine the Laplace transform, Aboodh transform and Sumudu transform to give another transform which is called the triple Laplace-Aboodh-Sumudu transform. This interesting transform reduces a linear partial differential equation with unknown function of three variables to an algebraic equation, which can then be solved by applying the inverse triple Laplace-Aboodh-Sumudu transform. In this paper, we introduce double Laplace-Aboodh transform (DLAT), double Aboodh-Sumudu transform (DAST) and triple Laplace-Aboodh-Sumudu transform (TLAST). We presented some fundamental properties and theorems about them. Moreover, we use the triple Laplace-Aboodh-Sumudu transform to solve some kinds of partial differential equations.

Keywords

Double Laplace-Aboodh transform, Double Aboodh-Sumudu transform, Triple Laplace-Aboodh-Sumudu transform, Laplace transform, partial differential equations

1. Introduction

The topic of partial differential equations is one of the most important subjects in mathematics and other sciences. Therefore, it is very important to know methods to solve such partial differential equations. One of most popular and rather method for solving partial differential equations is the integral transform method. In the literature, several different transforms are introduced and applied to find the solution of partial differential equations such as Laplace transform [1, 2], Sumudu transform [3], Aboodh transform [4, 5], and so on. Aboodh transform has deeper connection with Laplace and Sumudu transforms [6], and the Sumudu transform is a simple variant of the Laplace transform.

In recent years, great attention has been given to deal with double and triple integral transforms, see for example [7-10]. Aboodh [4] in 2013 introduced a new integral transform called Aboodh transform, which is derived from the Fourier integral and similar to Laplace transform, and applied it to solve ordinary differential equations, after that he introduced the double Aboodh transform and used it to solve integral differential equation and partial differential equation [11]. Recently, in 2020, the authors in [8] introduced a new double integral transform called Laplace-Sumudu transform and applied it to solve partial differential equations. In [7], the concept of triple Laplace transform was used to solve third order partial differential equations and the properties have been determined and studied also. For further detail and theories about triple integrals transform and their characteristics, see [2, 10, 12, 13].

The aim of this work is to study a new operator integral transform called triple Laplace-Aboodh-Sumudu transform with its main properties, studied double Laplace-Aboodh transform and double Aboodh-Sumudu transform and their properties.

In order to illustrate the applicability and efficiency of the triple Laplace-Aboodh-Sumudu transform, we apply this interesting transform to solve some kinds of partial differential equations.

1.1 Definition

The Laplace transform [14] of the continuous function $f(x)$ is defined by

$$L[f(x)] = F(p) = \int_0^{\infty} e^{-px} f(x)dx. \tag{1.1}$$

The inverse Laplace transform is defined by

$$L^{-1}[F(p)] = f(x) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{px} F(p)dp, \tag{1.2}$$

where κ is a real constant.

1.2 Definition

The Aboodh transform [4] of the real function $f(y)$ of exponential order is defined over the set of functions,

$$\mathcal{M} = \left\{ f(y) : \exists K, \tau_1, \tau_2 > 0, |f(y)| < Ke^{|y|\tau_i},\ y \in (-1)^i \times [0,\infty),\ i = 1,2 \right\},$$

by the following integral

$$A[f(y)] = F(q) = \frac{1}{q}\int_0^{\infty} e^{-qy} f(y)dy,\ \tau_1 \le q \le \tau_2. \tag{1.3}$$

The inverse Aboodh transform is

$$A^{-1}[F(q)] = f(y) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} qe^{qx} F(q)dq,\ \omega \ge 0. \tag{1.4}$$

1.3 Definition

[3] The Sumudu transform of the function $f(t)$ is defined over the set of functions,

$$\mathcal{N} = \left\{ f(t) : \exists M, \rho_1, \rho_2 > 0, |f(t)| < Me^{\frac{|t|}{\rho_j}},\ t \in (-1)^j \times [0,\infty),\ j = 1,2 \right\},$$

by

$$S[f(t)] = F(r) = \frac{1}{r}\int_0^{\infty} e^{-\frac{t}{r}} f(t)dt. \tag{1.5}$$

And the inverse Sumudu transform is

$$S^{-1}[F(r)] = f(t) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{1}{r}e^{\frac{t}{r}} F(r)dr,\ \omega \ge 0. \tag{1.6}$$

1.4 Definition

The double Laplace-Sumudu transform [8] of the continuous function $f(x,t)$ and $x, t > 0$ is defined by

$$\begin{aligned} L_xS_t[f(x,t)] = F(p,r) &= \frac{1}{r}\int_0^{\infty}\int_0^{\infty} e^{-(px+\frac{t}{r})} f(x,t)dxdt \\ &= \frac{1}{r}\lim_{\alpha\to\infty,\beta\to\infty}\int_0^{\alpha}\int_0^{\beta} e^{-(px+\frac{t}{r})} f(x,t)dxdt. \end{aligned} \tag{1.7}$$

It converges if the limit of the integral exists, and diverges if not. The inverse of double Laplace-Sumudu transform is given by

$$f(x,t) = L_x^{-1}S_t^{-1}[F(p,r)] = \frac{1}{(2\pi i)^2}\int_{\omega_1-i\infty}^{\omega_1+i\infty} e^{px}\left\{\int_{\omega_2-i\infty}^{\omega_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}F(p,r)dr\right\}dp, \tag{1.8}$$

where ω_1 and ω_2 are real constants.

For further detail about double Laplace-Sumudu transform and its characteristics see [8, 15].

2. Double Laplace-Aboodh Transform (DLAT)

In this section, the definition of the double Laplace-Aboodh transform and its fundamental properties of some basic functions are presented, and we prove the existence and uniqueness of the double Laplace-Aboodh transform.

2.1 Definition

The double Laplace-Aboodh transform of the continuous function $f(x, y)$ and $x, y > 0$ is denoted by the operator $L_xA_y[f(x, y)] = F(p, q)$ and defined by

$$L_xA_y[f(x,y)] = F(p,q) = \frac{1}{q}\int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)dxdy, \tag{2.1}$$

provided the integral exists.

The inverse double Laplace-Aboodh transform is defined by

$$f(x,y) = L_x^{-1}A_y^{-1}[F(p,q)] = \frac{1}{(2\pi i)^2}\int_{\bar{\omega}_1 - i\infty}^{\bar{\omega}_1 + i\infty} e^{px}\Big\{\int_{\bar{\omega}_2 - i\infty}^{\bar{\omega}_2 + i\infty} q e^{qy} F(p,q)dq\Big\}dp, \tag{2.2}$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are real constants.

2.2 Definition

[16] A function $f(x, y)$ is said to be of exponential order $\alpha > 0, \beta > 0$, on $0 \le x < \infty$, $0 \le y < \infty$, if there are positive constants K, X and Y such that

$$|f(x,y)| \le Ke^{(\alpha x + \beta y)}, \ \ for \ all \ x > X, \ y > Y,$$

and we write

$$f(x,y) = o(e^{(\alpha x + \beta y)}), \ \ as \ x, y \to \infty.$$

Or, equivalently,

$$\lim_{x\to\infty, y\to\infty} e^{-(px+qy)}|f(x,y)| = K \lim_{x\to\infty, y\to\infty} e^{-(p-\alpha)x}e^{-(q-\beta)y} = 0, \ \ p > \alpha, \ \ q > \beta.$$

2.3 Theorem

[12] Let $f(x, y)$ be a continuous function in every finite intervals $(0, X)$ and $(0, Y)$ and of exponential order $e^{(\alpha x+\beta y)}$, then the double Laplace-Aboodh transform of $f(x, y)$ exists for all $p > \alpha$ and $q > \beta$.

Proof. Let $f(x, y)$ be of exponential order $e^{(\alpha x+\beta y)}$ such that

$$|f(x,y)| \le Ke^{(\alpha x + \beta y)}, \ \forall \ x > X, \ y > Y.$$

Using definition of double Laplace-Aboodh transform, we have

$$\begin{aligned}
|F(p,q)| &= \left|\frac{1}{q}\int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)dxdy\right| \\
&\le \frac{1}{q}\int_0^\infty \int_0^\infty e^{-(px+qy)} |f(x,y)|dxdy \\
&\le \frac{K}{q}\int_0^\infty \int_0^\infty e^{-(px+qy)} e^{(\alpha x + \beta y)}dxdy \\
&= \frac{K}{q}\int_0^\infty e^{-(p-\alpha)x}dx \int_0^\infty e^{-(q-\beta)y}dy \\
&= \frac{K}{q(p-\alpha)(q-\beta)}.
\end{aligned}$$

Thus, the proof is complete.

2.4 Theorem

Let $F_1(p, q)$ and $F_2(p, q)$ be the double Laplace-Aboodh transform of the continuous functions $f_1(x, y)$ and $f_2(x, y)$ defined for $x, y \ge 0$ respectively. If $F_1(p, q) = F_2(p, q)$, then $f_1(x, y) = f_2(x, y)$.

Proof. Assume that $\bar{\kappa}$ and $\bar{\omega}$ are sufficiently large, since

$$f(x,y)=L_x^{-1}A_y^{-1}[F(p,q)]=\frac{1}{(2\pi i)^2}\int_{\bar{\kappa}-i\infty}^{\bar{\kappa}+i\infty}e^{px}\Big\{\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty}qe^{qy}F(p,q)dq\Big\}dp,$$

we deduce that

$$\begin{aligned} f_1(x,y) &= \frac{1}{(2\pi i)^2}\int_{\bar{\kappa}-i\infty}^{\bar{\kappa}+i\infty}e^{px}\Big\{\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty}qe^{qy}F_1(p,q)dq\Big\}dp \\ &= \frac{1}{(2\pi i)^2}\int_{\bar{\kappa}-i\infty}^{\bar{\kappa}+i\infty}e^{px}\Big\{\int_{\bar{\omega}-i\infty}^{\bar{\omega}+i\infty}qe^{qy}F_2(p,q)dq\Big\}dp \\ &= f_2(x,y), \end{aligned}$$

and this proves the uniqueness of the double Laplace-Aboodh transform.

2.5 Some Properties of The Double Laplace-Aboodh Transform.

2.5.1 Linearity property

If $f(x,y)$ and $g(x,y)$ be two functions such that

$$L_xA_y[f(x,y)]=F(p,q),$$
$$L_xA_y[g(x,y)]=G(p,q).$$

Then for any constants α and β, we have

$$L_xA_y[\alpha f(x,y)+\beta g(x,y)]=\alpha L_xA_y[f(x,y)]+\beta L_xA_y[g(x,y)]. \tag{2.3}$$

Proof. Using the definition of double Laplace-Aboodh transform, we deduce

$$\begin{aligned} L_xA_y[\alpha f(x,y)+\beta g(x,y)] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}\Big(\alpha f(x,y)+\beta g(x,y)\Big)dxdy \\ &= \frac{\alpha}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}f(x,y)dxdy \\ &+ \frac{\beta}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}g(x,y)dxdy \\ &= \alpha L_xA_y[f(x,y)]+\beta L_xA_y[g(x,y)]. \end{aligned}$$

2.5.2 Shifting property

If the double Laplace-Aboodh transform of $f(x,y)$ is $F(p,q)$, then for real constants a and b, we have

$$L_xA_y[e^{(ax+by)}f(x,y)]=\frac{q-b}{q}F(p-a,q-b). \tag{2.4}$$

Proof. Using the definition of double Laplace-Aboodh transform, we get

$$\begin{aligned} L_xA_y\big[e^{(ax+by)}f(x,y)\big] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}e^{(ax+by)}f(x,y)dxdy \\ &= \frac{q-b}{q(q-b)}\int_0^\infty\int_0^\infty e^{-\big((p-a)x+(q-b)y\big)}f(x,y)dxdy \\ &= \frac{q-b}{q}F(p-a,q-b). \end{aligned} \tag{2.5}$$

2.5.3 Changing of scale property

Let $f(x,y)$ be a function such that

$$L_xA_y[f(x,y)]=F(p,q).$$

Then for $a,b>0$, we have

$$L_xA_y\big[f(ax,by)\big]=\frac{1}{ab^2}F(\frac{p}{a},\frac{q}{b}). \tag{2.6}$$

Proof. Using the definition of double Laplace-Aboodh transform, we deduce

$$L_xA_y\big[f(ax,by)\big]=\frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}f(ax,by)dxdy.$$

Let $\tau = ax$, $\upsilon = by$, then

$$\begin{aligned} L_xA_y\big[f(\tau,\upsilon)\big] &= \frac{1}{abq}\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\tau+\frac{q}{b}\upsilon)}f(\tau,\upsilon)d\tau d\upsilon \\ &= \frac{1}{ab^2\frac{q}{b}}\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\tau+\frac{q}{b}\upsilon)}f(\tau,\upsilon)d\tau d\upsilon \\ &= \frac{1}{ab^2}F(\frac{p}{a},\frac{q}{b}). \end{aligned}$$

2.5.4 Derivatives properties

If $L_xA_y[f(x,y)] = F(p,q)$, then

(1)
$$L_xA_y\Big[\frac{\partial f(x,y)}{\partial x}\Big]=pF(p,q)-A[f(0,y)]. \tag{2.7}$$

Proof.

$$\begin{aligned} L_xA_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}\frac{\partial f(x,y)}{\partial x}dxdy \\ &= \frac{1}{q}\int_0^\infty e^{-qy}dy\Big\{\int_0^\infty e^{-px}f_x(x,y)dx\Big\}. \end{aligned}$$

Using integration by parts, let $u = e^{-px}$, $dv = f_x(x,y)d_x = \frac{\partial f(x,y)}{\partial x}dx$, then we obtain

$$\begin{aligned} L_xA_y\Big[\frac{\partial f(x,y)}{\partial x}\Big] &= \frac{1}{q}\int_0^\infty e^{-qy}dy\Big\{-f(0,y)+p\int_0^\infty e^{-px}f(x,y)dx\Big\} \\ &= pF(p,q)-A[f(0,y)] \end{aligned}$$

(2)
$$L_xA_y\Big[\frac{\partial f(x,y)}{\partial y}\Big]=qF(p,q)-\frac{1}{q}L[f(x,0)]. \tag{2.8}$$

Proof.

$$\begin{aligned} L_xA_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}\frac{\partial f(x,y)}{\partial y}dxdy \\ &= \frac{1}{q}\int_0^\infty e^{-px}dx\Big\{\int_0^\infty e^{-qy}f_y(x,y)dy\Big\} \end{aligned}$$

Using integration by parts, let $u = e^{-qy}$, $dv = f_y(x,y)d_y = \frac{\partial f(x,y)}{\partial y}dy$, then we obtain

$$\begin{aligned} L_xA_y\Big[\frac{\partial f(x,y)}{\partial y}\Big] &= \frac{1}{q}\int_0^\infty e^{-px}dx\Big\{-f(x,0)+q\int_0^\infty e^{-qy}f(x,y)dy\Big\} \\ &= qF(p,q)-\frac{1}{q}L[f(x,0)]. \end{aligned}$$

Similarly, we can prove that:

$$L_xA_y\Big[\frac{\partial^2 f(x,y)}{\partial x^2}\Big]=p^2F(p,q)-pA[f(0,y)]-A[f_x(0,y)],$$
$$L_xA_y\Big[\frac{\partial^2 f(x,y)}{\partial y^2}\Big]=q^2F(p,q)-L[f(x,0)]-\frac{1}{q}L[f_y(x,0)],$$
$$L_xA_y\Big[\frac{\partial^2 f(x,y)}{\partial x\partial y}\Big]=pqF(p,q)-\frac{p}{q}L[f(x,0)]-A[f_y(0,y)].$$

2.6 The Double Laplace-Aboodh Transform of some Functions

(1) If $f(x, y) = 1$, then

$$L_xA_y[f(x,y)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}dxdy = \frac{1}{pq^2}. \tag{2.9}$$

(2) If $f(x, y) = xy$, then

$$L_xA_y[f(x,y)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}xydxdy = \frac{1}{p^2q^3}. \tag{2.10}$$

(3) If $f(x, y) = x^n y^m$, $n, m = 0, 1, 2, \ldots$, then

$$L_xA_y[f(x,y)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}x^n y^m dxdy = \frac{n!}{p^{n+1}}\frac{m!}{q^{m+2}}. \tag{2.11}$$

(4) If $f(x, y) = x^\sigma y^\nu$, $\sigma \geq -1$, $\nu \geq -1$, then

$$L_xA_y[f(x,y)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}x^\sigma y^\nu dxdy \;=\; \int_0^\infty e^{-px}x^\sigma dx\int_0^\infty \frac{1}{q}e^{-qy}y^\nu dy,$$

let $\theta = px$ and $\varphi = qy$

$$\begin{aligned} L_xA_y[f(x,y)] &= \frac{1}{p^{\sigma+1}}\int_0^\infty e^{-\theta}\theta^\sigma d\theta\Big\{\frac{1}{q^{\nu+2}}\int_0^\infty e^{-\varphi}\varphi^\nu d\varphi\Big\} \\ &= \frac{1}{p^{\sigma+1}}\Gamma(\sigma+1)\frac{1}{q^{\nu+2}}\Gamma(\nu+1), \end{aligned} \tag{2.12}$$

where, $\Gamma(.)$ is the Euler gamma function.

(5) If $f(x, y) = e^{(nx+my)}$, $n, m = 0, 1, 2, \ldots$, then

$$L_xA_y[f(x,y)] = \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}e^{(nx+my)}dxdy = \frac{1}{(p-n)}\frac{1}{q(q-m)}. \tag{2.13}$$

Similarly,

$$\begin{aligned} L_xA_y[e^{i(nx+my)}] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}e^{i(nx+my)}dxdy = \frac{1}{(p-in)}\frac{1}{q(q-im)} \\ &= \frac{(pq-mn)+i(mp+nq)}{q(p^2+n^2)(q^2+m^2)}. \end{aligned} \tag{2.14}$$

Consequently,

$$L_xA_y[\cos(nx+my)] = \frac{pq-mn}{q(p^2+n^2)(q^2+m^2)},$$

$$L_xA_y[\sin(nx+my)] = \frac{mp+nq}{q(p^2+n^2)(q^2+m^2)}.$$

(6) If $f(x, y) = \sinh(nx + my)$ or $\cosh(nx + my)$, $n, m = 0, 1, 2, \ldots$.

Recall that

$$\sinh(nx+my) = \frac{e^{(nx+my)}-e^{-(nx+my)}}{2},\quad \cosh(nx+my) = \frac{e^{(nx+my)}+e^{-(nx+my)}}{2}.$$

Therefore,

$$L_xA_y[\sinh(nx+my)] = \frac{pq+mn}{q(p^2-n^2)(q^2-m^2)},$$

$$L_xA_y[\cosh(nx+my)] = \frac{mp+nq}{q(p^2-n^2)(q^2-m^2)}.$$

(7) If $f(x, y) = f_1(x)f_2(y)$, then

$$\begin{aligned} L_xA_y[f(x,y)] &= \frac{1}{q}\int_0^\infty\int_0^\infty e^{-(px+qy)}\Big\{f_1(x)f_2(y)\Big\}dxdy \\ &= \int_0^\infty e^{-px}f_1(x)dx\left\{\frac{1}{q}\int_0^\infty e^{-qy}f_2(y)dy\right\} \\ &= L_x[f_1(x)]\ A_y[f_2(y)]. \end{aligned} \tag{2.15}$$

Therefore,

$$L_xA_y[\sin(ax)\sin(by)] = \frac{a}{(p^2+a^2)}\frac{b}{q(q^2+b^2)},$$

$$L_xA_y[\cos(ax)\cos(by)] = \frac{p}{(p^2+a^2)}\frac{1}{(q^2+b^2)}.$$

3. Double Aboodh-Sumudu Transform (DAST)

In this section, we introduce the definition, some properties of the double Aboodh-Sumudu transform. Moreover, we prove the existence and uniqueness of the double Aboodh-Sumudu transform.

3.1 Definition

The double Aboodh-Sumudu transform of the continuous function $h(y, t)$, $y, t > 0$ is denoted by the operator $A_yS_t[h(y, t)] = H(q, r)$ and defined by

$$A_yS_t[h(y,t)] = H(q,r) = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}h(y,t)dydt. \tag{3.1}$$

And the inverse double Aboodh-Sumudu transform is defined by

$$h(y,t) = A_y^{-1}S_t^{-1}[H(q,r)] = \frac{1}{(2\pi i)^2}\int_{\bar{\gamma}_1-i\infty}^{\bar{\gamma}_1+i\infty} qe^{qy}\Big\{\int_{\bar{\gamma}_2-i\infty}^{\bar{\gamma}_2+i\infty}\frac{1}{r}e^{\frac{t}{r}}H(q,r)dr\Big\}dq, \tag{3.2}$$

where $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are real constants.

3.2 Theorem

[12] Let $h(y, t)$ be a continuous function in every finite intervals $(0, Y)$ and $(0, T)$, and of exponential order $e^{(\alpha y+\beta t)}$, then the double Aboodh- Sumudu transform of $h(y, t)$ exists for all $q > \alpha$ and $\frac{1}{r} > \beta$.

Proof. Let $h(y, t)$ be of exponential order $e^{(\alpha y+\beta t)}$ such that

$$\left|h(y,t)\right| \le Ke^{(\alpha y+\beta t)},\ \forall\ y > Y,\ t > T.$$

Then, from the definition of double Aboodh-Sumudu transform, we have

$$\begin{aligned} |H(q,r)| &= \left|\frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}h(y,t)dydt\right| \\ &\le \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}|h(y,t)|dydt \\ &\le \frac{K}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}e^{(\alpha y+\beta t)}dydt \\ &= \frac{K}{qr}\int_0^\infty e^{-(q-\alpha)y}dy\int_0^\infty e^{-(\frac{1}{r}-\beta)t}dt \\ &= \frac{K}{q(q-\alpha)(1-\beta r)} \end{aligned}$$

Thus, the proof is complete.

3.3 Theorem

Let $h_1(y, t)$ and $h_2(y, t)$ be continuous functions defined for $y, t \geq 0$ and having the double Aboodh-Sumudu transform

$H_1(q, r)$ and $H_2(q, r)$ respectively. If $H_1(q, r) = H_2(q, r)$, then $h_1(y, t) = h_2(y, t)$.

Proof. Assume that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are sufficiently large, since

$$h(y,t) = A_y^{-1} S_t^{-1}[H(q,r)] = \frac{1}{(2\pi i)^2} \int_{\bar{\gamma}_1 - i\infty}^{\bar{\gamma}_1 + i\infty} q e^{qy} \Big\{ \int_{\bar{\gamma}_2 - i\infty}^{\bar{\gamma}_2 + i\infty} \frac{1}{r} e^{\frac{t}{r}} H(q,r) dr \Big\} dq,$$

we deduce that

$$\begin{aligned} h_1(y,t) &= \frac{1}{(2\pi i)^2} \int_{\bar{\gamma}_1 - i\infty}^{\bar{\gamma}_1 + i\infty} q e^{qy} \Big\{ \int_{\bar{\gamma}_2 - i\infty}^{\bar{\gamma}_2 + i\infty} \frac{1}{r} e^{\frac{t}{r}} H_1(q,r) dr \Big\} dq \\ &= \frac{1}{(2\pi i)^2} \int_{\bar{\gamma}_1 - i\infty}^{\bar{\gamma}_1 + i\infty} q e^{qy} \Big\{ \int_{\bar{\gamma}_2 - i\infty}^{\bar{\gamma}_2 + i\infty} \frac{1}{r} e^{\frac{t}{r}} H_2(q,r) dr \Big\} dq \\ &= h_2(y,t). \end{aligned}$$

This ends the proof of the theorem.

3.4 Some Properties of The Double Aboodh-Sumudu Transform

3.4.1 Linearity property

If $h(y, t)$ and $g(y, t)$ be two functions such that

$$A_y S_t[h(y,t)] = H(q,r),$$
$$A_y S_t[g(y,t)] = G(q,r).$$

Then for any constants α and β, we have

$$A_y S_t[\alpha h(y,t) + \beta g(y,t)] = \alpha A_y S_t[h(y,t)] + \beta A_y S_t[g(y,t)]. \tag{3.3}$$

Proof. Using the definition of double Aboodh-Sumudu transform, we obtain

$$\begin{aligned} A_y S_t[\alpha h(y,t) + \beta g(y,t)] &= \frac{1}{qr} \int_0^\infty \int_0^\infty e^{-(qy + \frac{t}{r})} \Big(\alpha h(y,t) + \beta g(y,t) \Big) dy dt \\ &= \frac{\alpha}{qr} \int_0^\infty \int_0^\infty e^{-(qy + \frac{t}{r})} h(y,t) dy dt \\ &+ \frac{\beta}{qr} \int_0^\infty \int_0^\infty e^{-(qy + \frac{t}{r})} g(y,t) dy dt \\ &= \alpha A_y S_t[h(y,t)] + \beta A_y S_t[g(y,t)]. \end{aligned}$$

3.4.2 Shifting property

If $A_y S_t[h(y, t)] = H(q, r)$, then for any pair of real constants b, $c > 0$

$$A_y S_t[e^{(by+ct)} h(y,t)] = \frac{q-b}{q(1-cr)} H\big(q-b, \frac{r}{1-cr}\big). \tag{3.4}$$

Proof. Using the definition of double Aboodh-Sumudu transform, we get

$$\begin{aligned} A_y S_t\big[e^{(by+ct)} h(y,t)\big] &= \frac{1}{qr} \int_0^\infty \int_0^\infty e^{-(qy + \frac{t}{r})} e^{(by+ct)} h(y,t) dy dt \\ &= \frac{1}{qr} \int_0^\infty \int_0^\infty e^{-\big((q-b)y + (\frac{1}{r} - b)t\big)} h(y,t) dy dt. \end{aligned}$$

Put $z = \frac{r}{1-cr}$, then

$$\begin{aligned} A_y S_t\big[e^{(by+ct)} h(y,t)\big] &= \frac{1}{qz(1-br)} \int_0^\infty \int_0^\infty e^{-\big((q-b)y + \frac{t}{z}\big)} h(y,t) dy dt \\ &= \frac{q-b}{q(1-cr)} \frac{1}{(q-b)z} \int_0^\infty \int_0^\infty e^{-\big((q-b)y + \frac{t}{z}\big)} h(y,t) dy dt \\ &= \frac{q-b}{q(1-cr)} H\big(q-b, z\big) = \frac{q-b}{q(1-cr)} H\big(q-b, \frac{r}{1-cr}\big). \end{aligned}$$

3.4.3 Changing of scale property

Let $h(y, t)$ be a function such that

$$A_yS_t[h(y,t)] = H(q,r).$$

Then for b and c are positive constants, we have

$$A_yS_t[h(by,ct)] = \frac{1}{b^2}H\Big(\frac{q}{b}, cr\Big). \tag{3.5}$$

Proof. Using the definition of double Aboodh-Sumudu transform, we deduce

$$A_yS_t[h(by,ct)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}h(by,ct)dydt,$$

Let $\tau = by$, $\upsilon = ct$, then

$$\begin{aligned} A_yS_t[h(\tau,\upsilon)] &= \frac{1}{bcqr}\int_0^\infty\int_0^\infty e^{-(\frac{q}{b}\tau+\frac{1}{cr}\upsilon)}h(\tau,\upsilon)d\tau d\upsilon \\ &= \frac{1}{b^2\frac{q}{b}cr}\int_0^\infty\int_0^\infty e^{-(\frac{q}{b}\tau+\frac{1}{cr}\upsilon)}h(\tau,\upsilon)d\tau d\upsilon \\ &= \frac{1}{b^2}H\Big(\frac{q}{b}, cr\Big). \end{aligned}$$

3.4.4 Derivatives properties

If $A_yS_t[h(y, t)] = H(q, r)$, then:

$$(1)\quad A_yS_t\Big[\frac{\partial h(y,t)}{\partial y}\Big] = qH(q,r) - \frac{1}{q}S[h(0,t)]. \tag{3.6}$$

Proof.

$$\begin{aligned} A_yS_t\Big[\frac{\partial h(y,t)}{\partial y}\Big] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}\frac{\partial h(y,t)}{\partial y}dydt \\ &= \frac{1}{qr}\int_0^\infty e^{-\frac{t}{r}}dt\Big\{\int_0^\infty e^{-qy}\frac{\partial h(y,t)}{\partial y}dy\Big\}. \end{aligned}$$

Using integration by parts, let $u = e^{-qy}$, $dv = \frac{\partial h(y,t)}{\partial y}dy$, then we obtain

$$\begin{aligned} A_yS_t\Big[\frac{\partial h(y,t)}{\partial y}\Big] &= \frac{1}{qr}\int_0^\infty e^{-\frac{t}{r}}dt\Big\{-h(0,t) + q\int_0^\infty e^{-qy}h(y,t)dy\Big\} \\ &= qH(q,r) - \frac{1}{q}S[h(0,t)]. \end{aligned}$$

$$(2)\quad A_yS_t\Big[\frac{\partial h(y,t)}{\partial t}\Big] = \frac{1}{r}H(q,r) - \frac{1}{r}A[h(y,0)]. \tag{3.7}$$

Proof.

$$\begin{aligned} A_yS_t\Big[\frac{\partial h(y,t)}{\partial t}\Big] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}\frac{\partial h(y,t)}{\partial t}dydt \\ &= \frac{1}{qr}\int_0^\infty e^{-qy}dy\Big\{\int_0^\infty e^{-\frac{t}{r}}\frac{\partial h(y,t)}{\partial t}dt\Big\}. \end{aligned}$$

Using integration by parts, let $u = e^{-\frac{t}{r}}$, $dv = \frac{\partial h(y,t)}{\partial t}dt$, then we obtain

$$\begin{aligned} A_yS_t\Big[\frac{\partial h(y,t)}{\partial t}\Big] &= \frac{1}{qr}\int_0^\infty e^{-qy}dy\Big\{-h(y,0) + \frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}h(y,t)dt\Big\} \\ &= \frac{1}{r}H(q,r) - \frac{1}{r}A[h(y,0)]. \end{aligned}$$

Similarly, we can prove

$$A_yS_t\Big[\frac{\partial^2 h(y,t)}{\partial y^2}\Big]= q^2H(q,r) - S[h(0,t)] - \frac{1}{q}S[h_y(0,t)],$$
$$A_yS_t\Big[\frac{\partial^2 h(y,t)}{\partial t^2}\Big]= \frac{1}{r^2}H(q,r) - \frac{1}{r^2}A[h(y,0)] - \frac{1}{r}A[h_t(y,0)],$$
$$A_yS_t\Big[\frac{\partial^2 h(y,t)}{\partial y\partial t}\Big]= \frac{q}{r}H(q,r) - \frac{q}{r}A[h(y,0)] - \frac{1}{q}S[h_t(0,t)].$$

3.5 The Double Aboodh-Sumudu Transform of some Functions

(1) Let $h(y, t) = 1$, then

$$A_yS_t[h(y,t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}dydt = \frac{1}{q^2}. \tag{3.8}$$

(2). Let $h(y, t) = yt$, then

$$A_yS_t[h(y,t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}ytdydt= \frac{r}{q^3} \tag{3.9}$$

(3) Let $h(y, t) = y^m t^k$, $m, k = 0,1,2,...$, then

$$A_yS_t[h(y,t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}y^m t^k dydt= \frac{m!\ k!\ r^k}{q^{m+2}}. \tag{3.10}$$

(4) Let $h(y, t) = y^\nu t^\rho$, $\nu \geq -1, \rho \geq -1$, then

$$A_yS_t[h(y,t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}y^\nu t^\rho dydt \quad = \quad \int_0^\infty \frac{1}{q}e^{-qy}y^\nu dy\int_0^\infty \frac{1}{r}e^{-\frac{t}{r}}t^\rho dt,$$

let $\theta = qy$ and $\varphi = \frac{t}{r}$

$$\begin{aligned} A_yS_t[h(y,t)] &= \frac{1}{q^{\nu+2}}\int_0^\infty e^{-\theta}\theta^\nu d\theta\ \Big\{r^\rho\int_0^\infty e^{-\varphi}\varphi^\rho d\varphi\Big\} \\ &= \frac{\Gamma(\nu+1)}{q^{\nu+2}}\ \Gamma(\rho+1)r^\rho, \end{aligned} \tag{3.11}$$

where, $\Gamma(.)$ is the Euler gamma function.

(5) Let $h(y, t) = e^{(my+kt)}$, $m, k = 0, 1, 2,...$, then

$$A_yS_t[h(y,t)] = \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}e^{(my+kt)}dydt = \frac{1}{q(q-m)(1-kr)}. \tag{3.12}$$

Similarly,

$$\begin{aligned} A_yS_t[e^{i(my+kt)}] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}e^{i(my+kt)}dydt = \frac{1}{q(q-im)}\frac{1}{(1-ikr)} \\ &= \frac{(q-mkr)+i(m+kqr)}{q(q^2+m^2)(1+k^2r^2)}. \end{aligned} \tag{3.13}$$

Consequently,

$$A_yS_t[\cos(my+kt)] = \frac{q-mkr}{q(q^2+m^2)(1+k^2r^2)},$$
$$A_yS_t[\sin(my+kt)] = \frac{m+kqr}{q(q^2+m^2)(1+k^2r^2)}.$$

As in the previous section, it is easy to prove

$$A_yS_t[\cosh(my+kt)] = \frac{q+mnr}{q(q^2-n^2)(1-m^2r^2)},$$
$$A_yS_t[\sinh(my+kt)] = \frac{kqr+m}{q(q^2-m^2)(1-k^2r^2)}.$$

(7) If $h(y,t) = h_1(y)h_2(t)$, then

$$\begin{aligned} A_yS_t[h(y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty e^{-(qy+\frac{t}{r})}\Big\{h_1(y)h_2(t)\Big\}dydt \\ &= \frac{1}{q}\int_0^\infty e^{-qy}f_1(y)dy\left\{\frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}f_2(t)dt\right\} \\ &= A_y[h_1(y)]\ S_t[h_2(t)]. \end{aligned} \tag{3.14}$$

Therefore,

$$A_yS_t[\sin(by)\sin(ct)] = \frac{b}{q(q^2+b^2)}\frac{cr}{(1+c^2r^2)},$$
$$A_yS_t[\cos(by)\cos(ct)] = \frac{1}{(q^2+b^2)}\frac{1}{(1+c^2r^2)}.$$

4. Triple Laplace-Aboodh-Sumudu Transform (TLAST)

First, we introduce the definition of triple Laplace-Aboodh-Sumudu transform.

4.1 Definition

Let f be a continuous function of three variables say $x, y, t > 0$; then, the triple Laplace-Aboodh-Sumudu transform of $f\,(x,y,t)$ is denoted by the operator $L_xA_yS_t[f(x,y,t)] = F(p,q,r)$ and defined by

$$L_xA_yS_t[f(x,y,t)] = F(p,q,r) = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}f(x,y,t)dxdydt \tag{4.1}$$

Provided the integral exists.

The inverse triple Laplace-Aboodh-Sumudu transform is defined by

$$\begin{aligned} f(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}[F(p,q,r)] \\ &= \frac{1}{(2\pi i)^3}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty} e^{px}\left\{\int_{\kappa_2-i\infty}^{\kappa_2+i\infty} qe^{qy}\left\{\int_{\kappa_3-i\infty}^{\kappa_3+i\infty}\frac{1}{r}e^{\frac{t}{r}}F(p,q,r)dr\right\}dq\right\}dp. \end{aligned}$$

where κ_1, κ_2 and κ_3 are real constants.

4.2 Existence and uniqueness of triple Laplace-Aboodh-Sumudu transform

4.2.1 Definition. A function $f(x,y,t)$ is said to be of exponential order $e^{(ax+by+ct)}$, $a,b,c > 0$ on $[0,\infty)$, if there are positive constants K, X, Y and T such that

$$|f(x,y,t)| \le Ke^{(ax+by+ct)}, \qquad for\ all\ x > X,\ y > Y,\ t > T,$$

and we write

$$f(x,y,t) = o(e^{(ax+by+ct)}) \qquad as\ (x,y,t\to\infty).$$

Or, equivalently,

$$\sup_{x,y,t>0}\left(\frac{|f(x,y,t)|}{e^{(ax+by+ct)}}\right) < \infty.$$

4.3 Theorem

Let $f(x,y,t)$ be a continuous function on the interval $[0,\infty)$ and of exponential order $e^{(ax+by+ct)}$. Then the triple Laplace-Aboodh-Sumudu transform of $f(x,y,t)$ exists for all $p > a, q > b, r < \frac{1}{c}$.

Proof. Let $f(x,y,t)$ be of exponential order $e^{(ax+by+ct)}$ such that

$$|f(x,y,t)| \le Ke^{(ax+by+ct)},\ \ for\ all\ x > X,\ y > Y,\ t > T.$$

Then, we have

$$\begin{aligned}
\left|L_xA_yS_t[f(x,y,t)]\right| &= \left|\frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}f(x,y,t)dxdydt\right| \\
&\le \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}|f(x,y,t)|dxdydt \\
&\le K\frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}e^{(ax+by+ct)}dxdydt \\
&= \frac{K}{qr}\int_0^\infty e^{-(\frac{1-cr}{r})t}\left\{\frac{1}{q}\int_0^\infty e^{-(q-b)y}\left\{\int_0^\infty e^{-(p-a)x}dx\right\}dy\right\}dt \\
&= \frac{K}{q(p-a)(q-b)(1-cr)}.
\end{aligned}$$

4.4 Theorem

Let $F_1(p,q,r)$ and $F_2(p,q,r)$ be the triple Laplace-Aboodh-Sumudu transforms of the continuous functions $f_1(x,y,t)$ and $f_2(x,y,t)$ defined for $x,y,t \ge 0$ respectively. If $F_1(p,q,r) = F_2(p,q,r)$, then $f_1(x,y,t) = f_2(x,y,t)$.

Proof. Assume that κ_1, κ_2 and κ_3 are sufficiently large, since

$$f(x,y,t) = \frac{1}{(2\pi i)^3}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty} e^{px}\left\{\int_{\kappa_2-i\infty}^{\kappa_2+i\infty} qe^{qy}\left\{\int_{\kappa_3-i\infty}^{\kappa_3+i\infty}\frac{1}{r}e^{\frac{t}{r}}F(p,q,r)dr\right\}dq\right\}dp,$$

we deduce that

$$\begin{aligned}
f_1(x,y,t) &= \frac{1}{(2\pi i)^3}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty} e^{px}\left\{\int_{\kappa_2-i\infty}^{\kappa_2+i\infty} qe^{qy}\left\{\int_{\kappa_3-i\infty}^{\kappa_3+i\infty}\frac{1}{r}e^{\frac{t}{r}}F_1(p,q,r)dr\right\}dq\right\}dp \\
&= \frac{1}{(2\pi i)^3}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty} e^{px}\left\{\int_{\kappa_2-i\infty}^{\kappa_2+i\infty} qe^{qy}\left\{\int_{\kappa_3-i\infty}^{\kappa_3+i\infty}\frac{1}{r}e^{\frac{t}{r}}F_2(p,q,r)dr\right\}dq\right\}dp \\
&= f_2(x,y,t).
\end{aligned}$$

This proves the uniqueness of the triple Laplace-Aboodh-Sumudu Transform.

4.5 Some Properties of Triple Laplace-Aboodh-Sumudu Transform

4.5.1 Linearity property

If $f(x,y,t)$ and $g(x,y,t)$ be two functions such that

$$L_xA_yS_t[f(x,y,t)] = F(p,q,r),$$
$$L_xA_yS_t[g(x,y,t)] = G(p,q,r).$$

Then for any constants α and β, we have

$$L_xA_yS_t[\alpha f(x,y,t) + \beta g(x,y,t)] = \alpha L_xA_yS_t[f(x,y,t)] + \beta L_xA_yS_t[g(x,y,t)]. \tag{4.2}$$

Proof. Using definition of triple Laplace-Aboodh-Sumudu transform, we obtain

$$\begin{aligned}
L_xA_yS_t[\alpha f(x,y,t) + \beta g(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}\left\{\alpha f(x,y,t) + \beta g(x,y,t)\right\}dxdydt \\
&= \frac{\alpha}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}f(x,y,t)dxdydt \\
&+ \frac{\beta}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}g(x,y,t)dxdydt \\
&= \alpha L_xA_yS_t[f(x,y,t)] + \beta L_xA_yS_t[g(x,y,t)].
\end{aligned}$$

4.5.2 Shifting property

If the triple Laplace-Aboodh-Sumudu transform of $f(x,y,t)$ is $F(p,q,r)$, then for real constants a, b and c, we have

$$L_xA_yS_t[e^{(ax+by+ct)}f(x,y,t)] = \frac{q-b}{q(1-cr)}F(p-a,q-b,\frac{r}{1-cr}). \tag{4.3}$$

Proof. Using the definition of triple Laplace-Aboodh-Sumudu transform, we get

$$\begin{aligned} L_xA_yS_t[e^{(ax+by+ct)}f(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}e^{(ax+by+ct)}f(x,y,t)dxdydt \\ &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left((p-a)x+(q-b)y+(\frac{1}{r}-c)t\right)}f(x,y,t)dxdydt \\ &= \frac{q-b}{q(q-b)\frac{(1-cr)r}{(1-cr)}}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left((p-a)x+(q-b)y+(\frac{1}{r}-c)t\right)}f(x,y,t)dxdydt \\ &= \frac{q-b}{q(1-cr)}F(p-a,q-b,\frac{r}{1-cr}). \end{aligned}$$

4.5.3 Changing of scale property

Let $f(x,y,t)$ be a function such that

$$L_xA_yS_t[f(x,y,t)] = F(p,q,r).$$

Then for a, b and c are positive constants, we have

$$L_xA_yS_t[f(ax,by,ct)] = \frac{1}{ab^2}F(\frac{p}{a},\frac{q}{b},cr). \tag{4.4}$$

Proof. Using the definition of triple Laplace-Aboodh-Sumudu transform, we deduce

$$L_xA_yS_t[f(ax,by,ct)] = \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}f(ax,by,ct)dxdydt,$$

Let $\tau = ax$; $\upsilon = by$; $\varphi = ct$, then

$$\begin{aligned} L_xA_yS_t[f(\tau,\upsilon,\varphi)] &= \frac{1}{abcqr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\tau+\frac{q}{b}\upsilon+\frac{\varphi}{cr})}f(\tau,\upsilon,\varphi)d\tau d\upsilon d\varphi \\ &= \frac{1}{ab^2\frac{q}{b}cr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{p}{a}\tau+\frac{q}{b}\upsilon+\frac{\varphi}{cr})}f(\tau,\upsilon,\varphi)d\tau d\upsilon d\varphi \\ &= \frac{1}{ab^2}F(\frac{p}{a},\frac{q}{b},cr). \end{aligned}$$

4.5.4 Derivatives properties.

If $L_xA_yS_t[f(x,y,t)] = F(p,q,r)$, then:

$$(1)\quad L_xA_yS_t\left[\frac{\partial f(x,y,t)}{\partial x}\right] = pF(p,q,r) - A_yS_t[f(0,y,t)] \tag{4.5}$$

Proof.

$$\begin{aligned} L_xA_yS_t\left[\frac{\partial f(x,y,t)}{\partial x}\right] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}\frac{\partial f(x,y,t)}{\partial x}dxdydt \\ &= \frac{1}{qr}\int_0^\infty e^{-qy}dy\int_0^\infty e^{-\frac{t}{r}}dt\left\{\int_0^\infty e^{-px}f_x(x,y)dx\right\}. \end{aligned}$$

Using integration by parts, let $u = e^{-px}$, $dv = \frac{\partial f(x,y,t)}{\partial x}dx$, then we obtain

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial x}\Big] &= \frac{1}{qr}\int_0^\infty e^{-qy}dy\int_0^\infty e^{-\frac{t}{r}}dt\Big\{-f(0,y,t)+p\int_0^\infty e^{-px}f(x,y,t)dx\Big\} \\ &= pF(p,q,r)-A_yS_t[f(0,y,t)] \end{aligned}$$

(2) $$L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial y}\Big]=qF(p,q,r)-\frac{1}{q}L_xS_t[f(x,0,t)]. \tag{4.6}$$

Proof.

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial y}\Big] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}\frac{\partial f(x,y,t)}{\partial y}dxdydt \\ &= \frac{1}{qr}\int_0^\infty e^{-px}dx\int_0^\infty e^{-\frac{t}{r}}dt\Big\{\int_0^\infty e^{-qy}f_y(x,y,t)dy\Big\}. \end{aligned}$$

Using integration by parts, let $u=e^{-qy}$, $dv=\frac{\partial f(x,y,t)}{\partial y}dy$, then we obtain

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial y}\Big] &= \frac{1}{qr}\int_0^\infty e^{-px}dx\int_0^\infty e^{-\frac{t}{r}}dt\Big\{-f(x,0,t)+q\int_0^\infty e^{-qy}f(x,y,t)dy\Big\} \\ &= qF(p,q,r)-\frac{1}{q}L_xS_t[f(x,0,t)]. \end{aligned}$$

Similarly, we can prove that:

$$\begin{aligned} L_xA_yS_t\Big[\frac{\partial f(x,y,t)}{\partial t}\Big] &= \frac{1}{r}F(p,q,r)-\frac{1}{r}L_xA_y[f(x,y,0)], \\ L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial x^2}\Big] &= p^2F(p,q,r)-pA_yS_t[f(0,y,t)]-A_yS_t[f_x(0,y,t)], \\ L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial y^2}\Big] &= q^2F(p,q,r)-L_xS_t[f(x,0,t)]-\frac{1}{q}L_xS_t[f_y(x,0,t)], \\ L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial t^2}\Big] &= \frac{1}{r^2}F(p,q,r)-\frac{1}{r^2}L_xA_y[f(x,y,0)]-\frac{1}{r}L_xA_y[f_t(x,y,0)], \\ L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial x\partial y}\Big] &= pqF(p,q,r)-\frac{p}{q}L_xS_t[f(x,0,t)]-A_yS_t[f_y(0,y,t)], \\ L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial x\partial t}\Big] &= \frac{p}{r}F(p,q,r)-\frac{p}{r}L_xA_y[f(x,y,0)]-A_yS_t[f_t(0,y,t)], \\ L_xA_yS_t\Big[\frac{\partial^2 f(x,y,t)}{\partial y\partial t}\Big] &= \frac{q}{r}F(p,q,r)-\frac{q}{r}L_xA_y[f(x,y,0)]-\frac{1}{q}L_xS_t[f_t(x,0,y)]. \end{aligned}$$

4.6 Triple Laplace-Aboodh-Sumudu Transform of Some Elementary Functions

(1) If $f(x,y,t)=1$, then

$$L_xA_yS_t[f(x,y,t)]=\frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}dxdydt=\frac{1}{pq^2}.$$

(2) If $f(x,y,t)=xyt$, then

$$L_xA_yS_t[f(x,y,t)]=\frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}xytdxdydt=\frac{r}{p^2q^3}.$$

(3) If $f(x,y,t)=x^ny^mt^k$, $n,m,k=0,1,2,\ldots$, then

$$\begin{aligned} L_xA_yS_t[f(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}x^ny^mt^kdxdydt \\ &= \frac{n!}{p^{n+1}}\cdot\frac{m!}{q^{m+2}}\cdot k!r^k. \end{aligned}$$

(4) If $f(x,y,t)=x^{\sigma}y^{\nu}t^{\rho}$, $\sigma\geq -1$, $\nu\geq -1$ $\rho\geq -1$, then

$$\begin{aligned} L_xA_yS_t[f(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}x^{\sigma}y^{\nu}t^{\rho}dxdydt \\ &= \int_0^\infty e^{-px}x^{\sigma}dx\ \frac{1}{q}\int_0^\infty e^{-qy}y^{\nu}dy\ \frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}t^{\rho}dt, \end{aligned}$$

let $\xi=px$, $\varsigma=qy$ and $\eta=\frac{t}{r}$

$$\begin{aligned} L_xA_yS_t[f(x,y,t)] &= \frac{1}{p^{\sigma+1}}\int_0^\infty e^{-\xi}\xi^{\sigma}d\xi\ \frac{1}{q^{\nu+2}}\int_0^\infty e^{-\zeta}\zeta^{\nu}d\zeta\ r^{\rho}\int_0^\infty e^{-\eta}\eta^{\rho}d\eta \\ &= \frac{\Gamma(\sigma+1)}{p^{\sigma+1}}\ \frac{\Gamma(\nu+1)}{q^{\nu+2}}\ \Gamma(\rho+1)r^{\rho}, \end{aligned}$$

where $\Gamma(.)$ is the Euler gamma function.

(5) If $f(x,y,t)=e^{(ax+by+ct)}$, then

$$\begin{aligned} L_xA_yS_t[f(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}e^{(ax+by+ct)}dxdydt \\ &= \frac{1}{q(p-a)(q-b)(1-cr)}. \end{aligned}$$

(6) If $f(x,y,t)=\sinh(ax+by+ct)$ or $\cosh(ax+by+ct)$.

Recall that

$$\sinh(ax+by+ct)=\frac{e^{(ax+by+ct)}-e^{-(ax+by+ct)}}{2},\quad \cosh(ax+by+ct)=\frac{e^{(ax+by+ct)}+e^{-(ax+by+ct)}}{2}.$$

Therefore,

$$L_xA_yS_t[\cosh(ax+by+ct)]=\frac{pq+ab+bcpr+acqr}{q(p^2-a^2)(q^2-b^2)(1-c^2r^2)},$$

$$L_xA_yS_t[\sinh(ax+by+ct)]=\frac{bp+aq+cpqr+abcr}{q(p^2-a^2)(q^2-b^2)(1-c^2r^2)}.$$

(7) If $f(x,y,t)=e^{i(ax+by+ct)}$, then

$$\begin{aligned} L_xA_yS_t[f(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}e^{i(ax+by+ct)}dxdydt \\ &= \frac{1}{q(p-ia)(q-ib)(1-icr)} \\ &= \frac{(pq-bcpr-acqr-ab)+i(cpqr+bp+aq-abcr)}{q(p^2+a^2)(q^2+b^2)(1+c^2r^2)}. \end{aligned}$$

Consequently,

$$L_xA_yS_t[\cos(ax+by+ct)]=\frac{pq-bcpr-acqr-ab}{q(p^2+a^2)(q^2+b^2)(1+c^2r^2)},$$

$$L_xA_yS_t[\sin(ax+by+ct)]=\frac{cpqr+bp+aq-abcr}{q(p^2+a^2)(q^2+b^2)(1+c^2r^2)}.$$

(8) If $f(x,y,t)=f_1(x)f_2(y)f_3(t)$, then

$$\begin{aligned} L_xA_yS_t[f(x,y,t)] &= \frac{1}{qr}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+\frac{t}{r})}\Big(f_1(x)f_2(y)f_3(t)\Big)dxdydt \\ &= \int_0^\infty e^{-px}f_1(x)\ \left\{\frac{1}{q}\int_0^\infty e^{-qy}f_2(y)\ \left\{\frac{1}{r}\int_0^\infty e^{-\frac{t}{r}}f_3(t)dt\right\}dy\right\}dx \\ &= L_x[f_1(x)]\ A_y[f_2(y)]\ S_t[f_3(t)]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
L_xA_yS_t[\cos(ax)\cos(by)\cos(ct)] &= \frac{p}{(p^2+a^2)}\frac{1}{(q^2+b^2)}\frac{1}{(1+c^2r^2)},\\
L_xA_yS_t[\sin(ax)\sin(by)\sin(ct)] &= \frac{a}{(p^2+a^2)}\frac{b}{q(q^2+b^2)}\frac{cr}{(1+c^2r^2)}.
\end{aligned}
$$

5. Applications

In this section, we apply the triple Laplace-Aboodh-Sumudu transform (TLAST) operator for solving some kinds of linear partial differential equations.

Example 5.1. Consider the following nonhomogeneous Heat equation

$$U_t(x,y,t) = U_{xx}(x,y,t) + U_{yy}(x,y,t) + 2\cos(x+y),\ \ (x,y)\in\mathbb{R}^2_+,\ \ t>0, \tag{5.1}$$

subject to the boundary and initial conditions

$$U(0,y,t) = e^{-2t}\sin y + \cos y,\ \ U_x(0,y,t) = e^{-2t}\cos y - \sin y, \tag{5.2}$$

$$U(x,0,t) = e^{-2t}\sin x + \cos x,\ \ U_y(x,0,t) = e^{-2t}\cos x - \sin x, \tag{5.3}$$

$$U(x,y,0) = \sin(x+y) + \cos(x+y). \tag{5.4}$$

Solution. Applying TLAST on both sides of Eq. (5.1), we have

$$L_xA_yS_t[U_t(x,y,t)] = L_xA_yS_t[U_{xx}(x,y,t) + U_{yy}(x,y,t) + 2\cos(x+y)]. \tag{5.5}$$

By linearity property and partial derivative properties of TLAST, we get

$$
\begin{aligned}
\frac{1}{r}F(p,q,r) - \frac{1}{r}L_xA_y[U(x,y,0)] &= p^2F(p,q,r) - pA_yS_t[U(0,y,t)] - A_yS_t[U_x(0,y,t)] + q^2F(p,q,r)\\
&- L_xS_t[U(x,0,t)] - \frac{1}{q}L_xS_t[U_y(x,0,t)] + \frac{2(pq-1)}{q(p^2+1)(q^2+1)},
\end{aligned} \tag{5.6}
$$

where

$$L_xA_yS_t[2\cos(x+y)] = \frac{2(pq-1)}{q(p^2+1)(q^2+1)}.$$

Rearranging the terms, we have

$$
\begin{aligned}
F(p,q,r) &= \frac{r}{(p^2r+q^2r-1)}\Big\{pA_yS_t[U(0,y,t)] + A_yS_t[U_x(0,y,t)] + L_xS_t[U(x,0,t)]\\
&+ \frac{1}{q}L_xS_t[U_y(x,0,t)] - \frac{1}{r}L_xA_y[U(x,y,0)] - \frac{2(pq-1)}{q(p^2+1)(q^2+1)}\Big\}
\end{aligned} \tag{5.7}
$$

Using DAST for equations (5.2), DLST for equations (5.3) and DLAT for equation (5.4), we obtain

$$A_yS_t[U(0,y,t)] = \frac{1}{q(q^2+1)(1+2r)} + \frac{1}{(q^2+1)}, \tag{5.8}$$

$$A_yS_t[U_x(0,y,t)] = \frac{1}{(q^2+1)(1+2r)} - \frac{1}{q(q^2+1)}, \tag{5.9}$$

$$L_xS_t[U(x,0,t)] = \frac{1}{(p^2+1)(1+2r)} + \frac{p}{(p^2+1)}, \tag{5.10}$$

$$L_xS_t[U_y(x,0,t)] = \frac{p}{(p^2+1)(1+2r)} - \frac{1}{(p^2+1)}, \tag{5.11}$$

$$L_xA_y[U(x,y,0)] = \frac{p+q}{q(p^2+1)(q^2+1)} + \frac{pq-1}{q(p^2+1)(q^2+1)}. \tag{5.12}$$

Substitute equations (5.8)-(5.12) into equation (5.7) and simplify to obtain

$$\begin{aligned} F(p,q,r) &= \frac{r}{(p^2r+q^2r-1)}\left\{\frac{(p+q)(p^2r+q^2r-1)}{qr(p^2+1)(q^2+1)(1+2r)}+\frac{(pq-1)(p^2r+q^2r-1)}{qr(p^2+1)(q^2+1)}\right\} \\ &= \frac{p+q}{q(p^2+1)(q^2+1)(1+2r)}+\frac{pq-1}{q(p^2+1)(q^2+1)}. \end{aligned} \tag{5.13}$$

Taking the inverse TLAST of equation (5.13), we get

$$\begin{aligned} U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\Big[\frac{p+q}{q(p^2+1)(q^2+1)(1+2r)}+\frac{pq-1}{q(p^2+1)(q^2+1)}\Big] \\ &= e^{-2t}\sin(x+y)+\cos(x+y). \end{aligned} \tag{5.14}$$

Which is the desired solution of (5.1)

Example 5.2. Consider the following nonhomogeneous Wave equation

$$2U_{tt}(x,y,t)=U_{xx}(x,y,t)+U_{yy}(x,y,t)+24t^2+4y,\ (x,y)\in\mathbb{R}_+^2,\ t>0, \tag{5.15}$$

subject to the boundary and initial conditions

$$\begin{cases} U(0,y,t)=t^4+yt^2, & U_x(0,y,t)=\sin y\sin t, \\ U(x,0,t)=t^4, & U_y(x,0,t)=t^2+\sin x\sin t, \\ U(x,y,0)=0, & U_t(x,y,0)=\sin x\sin y. \end{cases} \tag{5.16}$$

Solution. Taking the TLAST to both sides of equation (5.15) and rearranging the terms, we get

$$\begin{aligned} F(p,q,r) &= \frac{r^2}{(p^2r^2+q^2r^2-2)}\Big\{pA_yS_t[U(0,y,t)]+A_yS_t[U_x(0,y,t)] \\ &+ L_xS_t[U(x,0,t)]+\frac{1}{q}L_xS_t[U_y(x,0,t)]-\frac{2}{r^2}L_xA_y[U(x,y,0)] \\ &- \frac{2}{r}L_xA_y[U_t(x,y,0)]-\frac{48r^2}{pq^2}-\frac{4}{pq^3}\Big\}, \end{aligned} \tag{5.17}$$

where

$$L_xA_yS_t[24t^2+4y]=\frac{48r^2}{pq^2}+\frac{4}{pq^3}.$$

Applying DLAT, DLST and DAST to the given conditions, we obtain

$$\begin{aligned} F(p,q,r) &= \frac{r^2}{(p^2r^2+q^2r^2-2)}\Big\{\frac{24pr^4}{q^2}+\frac{2pr^2}{q^3} \\ &+ \frac{r}{q(q^2+1)(1+r^2)}+\frac{24r^4}{p}+\frac{2r^2}{pq}+\frac{r}{q(p^2+1)(1+r^2)} \\ &- \frac{2}{qr(p^2+1)(q^2+1)}-\frac{48r^2}{pq^2}-\frac{4}{pq^3}\Big\}, \end{aligned} \tag{5.18}$$

by simple computation, we get

$$F(p,q,r)=\frac{24r^4}{pq^2}+\frac{2r^2}{pq^3}+\frac{r}{q(p^2+1)(q^2+1)(1+r^2)}. \tag{5.19}$$

Applying the inverse TLAST on equation (5.19), we get

$$\begin{aligned} U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\Big[\frac{24r^4}{pq^2}+\frac{2r^2}{pq^3}+\frac{r}{q(p^2+1)(q^2+1)(1+r^2)}\Big] \\ &= t^4+yt^2+\sin x\sin y\sin t. \end{aligned} \tag{5.20}$$

Which is the required solution of (5.15).

Example 5.3. Consider the following boundary Laplace equation

$$U_{xx}(x,y,t)+U_{yy}(x,y,t)+U_{tt}(x,y,t)=0,\ \ (x,y,t)\in\mathbb{R}^3_+, \tag{5.21}$$

subjected to the conditions

$$\begin{cases} U(0,y,t)=0, & U_x(0,y,t)=\sin y\ \ \sinh\sqrt{2}t,\\ U(x,0,t)=0, & U_y(x,0,t)=\sin x\ \ \sinh\sqrt{2}t,\\ U(x,y,0)=0, & U_t(x,y,0)=\sqrt{2}\sin x\ \ \sin y. \end{cases} \tag{5.22}$$

Solution. Applying TLAST to the equation gives and by linearity property and partial Derivative properties of TLAST, we get

$$\begin{aligned} 0 &= p^2F(p,q,r)-pA_yS_t[U(0,y,t)]-A_yS_t[U_x(0,y,t)]\\ &+ q^2F(p,q,r)-L_xS_t[U(x,0,t)]-\frac{1}{q}L_xS_t[U_y(x,0,t)]\\ &+ \frac{1}{r^2}F(p,q,r)-\frac{1}{r^2}L_xA_y[U(x,y,0)]-\frac{1}{r}L_xA_y[U_t(x,y,0)]. \end{aligned} \tag{5.23}$$

Substituting

$$\begin{aligned} A_yS_t[U_x(0,y,t)] &= \frac{\sqrt{2}r}{q(q^2+1)(1-2r^2)}, L_xS_t[U_y(x,0,t)]=\frac{\sqrt{2}r}{(p^2+1)(1-2r^2)},\\ L_xA_y[U_t(x,y,0)] &= \frac{\sqrt{2}}{q(p^2+1)(q^2+1)}, \end{aligned}$$

in equation (5.23) and simplifying, we get

$$\begin{aligned} F(p,q,r) &= \frac{r^2}{(p^2r^2+q^2r^2+1)}\left\{\frac{\sqrt{2}(p^2r^2+q^2r^2+1)}{qr(p^2+1)(q^2+1)(1-2r^2)}\right\}\\ &= \frac{\sqrt{2}r}{q(p^2+1)(q^2+1)(1-2r^2)}. \end{aligned} \tag{5.24}$$

Taking the inverse of TLAST, we get

$$\begin{aligned} U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\left[\frac{\sqrt{2}r}{q(p^2+1)(q^2+1)(1-2r^2)}\right]\\ &= \sin x\sin y\sinh\sqrt{2}t. \end{aligned} \tag{5.25}$$

Which is the required solution of the considered Laplace equation.

Example 5.4. Consider the following Poisson partial differential equation

$$U_{xx}(x,y,t)+U_{yy}(x,y,t)+U_{tt}(x,y,t)=2\sin x\cos y\sinh 2t,\ \ (x,y,t)\in\mathbb{R}^3_+, \tag{5.26}$$

subjected to the conditions

$$\begin{cases} U(0,y,t)=0, & U_x(0,y,t)=\cos y\sinh 2t,\\ U(x,0,t)=\sin x\sinh 2t, & U_y(x,0,t)=0,\\ U(x,y,0)=0, & U_t(x,y,0)=2\sin x\cos y. \end{cases} \tag{5.27}$$

Solution. Applying TLAST on both sides of equation (5.26) and by using properties of TLAST, then we have

$$\begin{aligned} (p^2+q^2+\frac{1}{r^2})F(p,q,r) &= pA_yS_t[U(0,y,t)]+A_yS_t[U_x(0,y,t)]+L_xS_t[U(x,0,t)]\\ &+ \frac{1}{q}L_xS_t[U_y(x,0,t)]+\frac{1}{r^2}L_xA_y[U(x,y,0)]+\frac{1}{r}L_xA_y[U_t(x,y,0)]\\ &+ \frac{4r}{(p^2+1)(q^2+1)(1-4r^2)}, \end{aligned} \tag{5.28}$$

where

$$L_x A_y S_t[2\sin x\cos y\sinh 2t] = \frac{4r}{(p^2+1)(q^2+1)(1-4r^2)}.$$

Substituting

$$\begin{aligned} A_y S_t[U_x(0,y,t)] &= \frac{2r}{(q^2+1)(1-4r^2)}, \quad L_x S_t[U(x,0,t)] = \frac{2r}{(p^2+1)(1-4r^2)}, \\ L_x A_y[U_t(x,y,0)] &= \frac{2}{(p^2+1)(q^2+1)}, \end{aligned}$$

in equation (5.28), we obtain

$$\begin{aligned} (p^2+q^2+\frac{1}{r^2})F(p,q,r) &= \left\{\frac{4r}{(p^2+1)(q^2+1)(1-4r^2)} + \frac{2r}{(q^2+1)(1-4r^2)}\right. \\ &+ \left.\frac{2r}{(p^2+1)(1-4r^2)} + \frac{2}{r(p^2+1)(q^2+1)}\right\}. \end{aligned} \tag{5.29}$$

After some simple algebraic operations, we get

$$\begin{aligned} F(p,q,r) &= \frac{r^2}{(p^2r^2+q^2r^2+1)}\left\{\frac{2(p^2r^2+q^2r^2+1)}{r(p^2+1)(q^2+1)(1-4r^2)}\right\} \\ &= \frac{2r}{(p^2+1)(q^2+1)(1-4r^2)}. \end{aligned} \tag{5.30}$$

Taking $L_x^{-1}A_y^{-1}S_t^{-1}$ for equation (5.30), we get

$$\begin{aligned} U(x,y,t) &= L_x^{-1}A_y^{-1}S_t^{-1}\left[\frac{2r}{(p^2+1)(q^2+1)(1-4r^2)}\right] \\ &= \sin x\cos y\sinh 2t. \end{aligned} \tag{5.31}$$

Which is the required solution of Poisson equation (5.26).

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