Journal of Applied Mathematics and Computation, 2022, 6(3), 332-342
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

# Riemann-Stieltjes Operators Between Zygmund-Type Spaces

**Zhitao Guo**

School of Science, Henan Institute of Technology, Xinxiang, Henan, China.

**How to cite this paper:** Zhitao Guo. (2022) Riemann-Stieltjes Operators Between Zygmund-Type Spaces. *Journal of Applied Mathematics and Computation*, 6(3), 332-342.
DOI: 10.26855/jamc.2022.09.005

**Received:** July 1, 2022
**Accepted:** July 28, 2022
**Published:** September 8, 2022

***Corresponding author:** Zhitao Guo, School of Science, Henan Institute of Technology, Xinxiang, Henan, China.
**Email:** guotao60698@163.com

## Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The integral operators $J_g$ and $I_g$, called the Riemann-Stieltjes operators or the Volterra type operators are defined by

$$J_g f(z) = \int_0^z f(\xi) g'(\xi) \mathrm{d}\xi \quad \text{and} \quad I_g f(z) = \int_0^z f'(\xi) g(\xi) \mathrm{d}\xi,$$

where $g$ and $f$ are analytic functions on $\mathbb{D}$ and $z \in \mathbb{D}$. Suppose $X$ and $Y$ are Banach spaces and $T: X \to Y$ is a linear operator. If there exists a positive constant $C$ such that $\|TX\| \le C\|X\|$ for every $x \in X$, then we say that $T$ is a bounded linear operator from $X$ into $Y$. If $T$ maps every bounded set of $X$ to a relatively compact set of $Y$, then $T$ is called a compact operator. The boundedness and compactness of a linear operator acting between the spaces of analytic functions are basic questions in operator theory, which has been extensively studied by many researchers. In this paper, we mainly investigate the boundedness and compactness of the above two Riemann-Stieltjes operators between Zygmund-type space.

## Keywords

Riemann-Stieltjes operator, Zygmund-type space, boundedness, compactness

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Denote by $\mathbb{N}$ the set of positive integers. For $0 < \alpha < \infty$, the Bloch-type space $B^\alpha$, which also can be called the $\alpha$-Bloch space, consists of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

$B^\alpha$ becomes a Banach space normed by $\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)|$. It is well-known that for $0 < \alpha < 1$, $B^\alpha$ is a subspace of $H^\infty$, the space of bounded analytic functions on $\mathbb{D}$. Moreover, $B^\alpha$ equals to the analytic Lipschitz space of order $1-\alpha$ whenever $0 < \alpha < 1$. When $\alpha = 1$, we get the classical Bloch space $B$.

We denote by $Z^\alpha$ the Zygmund-type space (or the $\alpha$-Zygmund space), which consists of all functions $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f''(z)| < \infty.$$

Under the norm $\|f\|_{Z^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)|$, $Z^\alpha$ is a Banach space. When $\alpha = 1$, $Z^\alpha$ reduce to the classical Zygmund space. For some results on $B^\alpha$, $Z^\alpha$ and a variety of operators on them, see, for example, [1-15].

Suppose that $g \in H(\mathbb{D})$. The integral operator $J_g$, called the Riemann-Stieltjes operator (see [16]) or the Volterra type operator (see [17]) is defined by

$$J_g f(z) = \int_0^z f\mathrm{d}g = \int_0^1 f(tz)zg'(tz)\mathrm{d}t = \int_0^z f(\xi)g'(\xi)\mathrm{d}\xi,$$

where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. Note that $J_g$ can be viewed as a generalization of the well-known Cesàro operator, and their study was initiated by Alemann and Siskakis [18], where they showed that $J_g$ is bounded (compact) on Hardy space $H^p$, $1 \le p < \infty$, if and only if $g \in$ BMOA ($g \in$ VMOA).

Another natural Riemann-Stieltjes operator $I_g$ is defined as follows

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)\mathrm{d}\xi.$$

Recently, many researchers considered the Riemann-Stieltjes operators and characterized their boundedness and compactness between various spaces of analytic functions, as well as their $n$-dimensional extensions. For instance, Li and Stević in [3] studied the boundedness and compactness of $J_g$ and $I_g$ acting on Zygmund space, and the conditions for the boundedness and compactness of the two operators from $H^\infty$ into the $\alpha$-Bloch space and the little $\alpha$-Bloch space were given in [7]. Stević in [12] investigated the boundedness of the integral operator of the form

$$T_g f(z) = \int_0^1 f(tz)(\mathcal{R}g)(tz)\frac{\mathrm{d}t}{t},$$

where $g$ is analytic on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and $\mathcal{R}g$ is the standard radial derivative, on $\alpha$-Bloch spaces, and the results also yield estimates on the operator norm of $T_g$, in terms of the growth of the radial derivative $\mathcal{R}g$. In [10], Liu and Yu characterized the boundedness (resp. compactness) of the Riemann-Stieltjes operator

$$L_g f(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\frac{\mathrm{d}t}{t}$$

with sup (resp. lim) conditions on the symbol $g$ from mixed norm spaces to Zygmund-type spaces on the unit ball. Some more related results can be found (see, e.g., [4, 6, 9, 11, 19-24] and the references therein).

In this paper, we are devoted to investigating the boundedness and compactness of $J_g$ and $I_g$ between different Zygmund-type spaces.

Throughout this paper, we use the letter $C$ to denote a positive constant whose value may change its value at each occurrence. For two nonnegative quantities $X$ and $Y$, the abbreviation $X \prec Y$ or $Y \succ X$ means that there is a positive constant $C$ such that $X \le CY$.

## 2. Preliminaries

In this section, we formulate some auxiliary results which will be used in the proof of the main results. First, we show that every $f \in Z^\alpha$ must belong to $B^\alpha$.

**Lemma 2.1** For any $0 < \alpha < \infty$, we have $Z^\alpha \subseteq B^\alpha$.

*Proof*. For every $f \in Z^\alpha$, we have $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)| < \infty$, where $0 < \alpha < \infty$. It is sufficient to show that $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty$. Let $t \in [0,1]$, by a usual calculation, we get

$$\int_0^1 (1-|z|^2)^\alpha f''(tz)\mathrm{d}t = \frac{(1-|z|^2)^\alpha}{z}(f'(z) - f'(0)).$$

Note that $(1-|z|^2)^\alpha \le (1-|tz|^2)^\alpha$, then

$$(1-|z|^2)^\alpha |f'(z)| \le (1-|z|^2)^\alpha |f'(0)| + \int_0^1 (1-|tz|^2)^\alpha |f''(tz)|\mathrm{d}t.$$

Taking the supremum in above inequality over all $z \in \mathbb{D}$, the conclusion follows.

The following lemma can be found in [1, Lemma 1.1].

**Lemma 2.2** For every $\alpha > 0$ and $f \in Z^\alpha$ we have:

(i) for $0 < \alpha < 1$, $|f'(z)| \le \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha}$ and $|f(z)| \le \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha}$,

(ii) for $\alpha = 1$, $|f'(z)| \le \|f\|_{\mathcal{Z}} \log\frac{e}{1-|z|^2}$ and $|f(z)| \le \|f\|_{\mathcal{Z}}$,

(iii) for $\alpha > 1$, $|f'(z)| \le \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$,

(iv) for $1 < \alpha < 2$, $|f(z)| \le \frac{2}{(\alpha-1)(2-\alpha)}\|f\|_{\mathcal{Z}^\alpha}$,

(v) for $\alpha = 2$, $|f(z)| \le 2\|f\|_{\mathcal{Z}^2} \log\frac{e}{1-|z|^2}$,

(vi) for $\alpha > 2$, $|f(z)| \le \frac{2}{(\alpha-1)(\alpha-2)}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-2}}$.

In order to investigate the compactness of $J_g$, $I_g : Z^\alpha \to Z^\beta$, where $\alpha, \beta > 0$, we need the following two lemmas. The first one characterizes the compactness in terms of sequential convergence, whose proof is similar to that of [25, Proposition 3.11] and we omit the details.

**Lemma 2.3** Let $\alpha$, $\beta > 0$, $T = J_g$ or $I_g$. Then $T : Z^\alpha \to Z^\beta$ is compact if and only if $T : Z^\alpha \to Z^\beta$ is bounded and for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $Z^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\|T f_n\|_{Z^\beta} \to 0$ as $n \to \infty$.

The following lemma was essentially proved in [13, Lemma 2.5].

**Lemma 2.4** Fix $0 < \alpha < 2$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $Z^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ .Then $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n(z)| = 0$. Moreover, for $0 < \alpha < 1$, if $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $Z^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, then $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n'(z)| = 0$.

## 3. Main Results

In this section we characterize the boundedness and compactness of the Riemann-Stieltjes operators $J_g$, $I_g : Z^\alpha \to Z^\beta$. For the boundedness of $J_g : Z^\alpha \to Z^\beta$, we need to break the problem into five different cases: $0 < \alpha < 1$, $\alpha = 1$, $1 < \alpha < 2$, $\alpha = 2$ and $\alpha > 2$.

**Theorem 3.1** Let $g \in H(\mathbb{D})$ and $\alpha$, $\beta > 0$. Then $J_g : Z^\alpha \to Z^\beta$ is bounded if and only if

(a) $g \in Z^\beta$, for $0 < \alpha < 1$;

(b) $g \in Z^\beta$ and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g'(z)| \log\frac{e}{1-|z|^2} < \infty, \tag{1}$$

for $\alpha = 1$;

(c) $g \in Z^\beta$ and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|<\infty, \tag{2}$$

for $1<\alpha<2$;

(d)
$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|g''(z)|\log\frac{e}{1-|z|^2}<\infty, \tag{3}$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-1}|g'(z)|<\infty, \tag{4}$$

for $\alpha=2$;

(e)
$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-\alpha+2}|g''(z)|<\infty, \tag{5}$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|<\infty, \tag{6}$$

for $\alpha>2$.

***Proof.*** Suppose that $J_g : Z^\alpha \to Z^\beta$ is bounded. Take $f_1(z)=1\in Z^\alpha$, then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|g''(z)|=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|(J_g f_1)''(z)|\le\left\|J_g f_1\right\|_{\mathcal{Z}^\beta}<\infty.$$

Thus $g\in Z^\beta$ is necessary for all cases, and we also have that $g\in B^\beta$ by Lemma 2.1.

For a fixed $w\in\mathbb{D}$ such that $\frac{1}{2}<|w|<1$, consider the function

$$f_{2,w}(z)=\frac{h(\bar{w}z)}{\bar{w}}\left(\log\frac{e}{1-|w|^2}\right)^{-1}-\frac{h(|w|^2)}{\bar{w}}\left(\log\frac{e}{1-|w|^2}\right)^{-1},$$

where $h(z)=(z-1)\left[\left(1+\log\frac{e}{1-z}\right)^2+1\right]$ and $z\in\mathbb{D}$. By a direct calculation, we get

$$\left|f''_{2,w}(z)\right|\le\frac{2}{1-|z|}\left(\log\frac{e}{1-|w|}\right)\left(\log\frac{e}{1-|w|^2}\right)^{-1}\prec\frac{1}{1-|z|},$$

for $\frac{1}{2}<|w|<1$ and $\sup_{\frac{1}{2}<|w|<1}\left\|f_{2,w}\right\|_Z<\infty$. Moreover, we have

$$f_{2,w}(w)=0,\quad f'_{2,w}(w)=\log\frac{e}{1-|w|^2}.$$

By the boundedness of $J_g : Z\to Z^\beta$, we have

$$\begin{aligned}\infty&>\left\|J_g f_{2,w}\right\|_{\mathcal{Z}^\beta}\\&\ge\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|f_{2,w}(z)g''(z)+f'_{2,w}(z)g'(z)\right|\\&\ge\sup_{\frac{1}{2}<|w|<1}(1-|w|^2)^\beta|g'(w)|\log\frac{e}{1-|w|^2}.\end{aligned} \tag{7}$$

On the other hand, by using $g\in B^\beta$

$$\sup_{|w|\le\frac{1}{2}}(1-|w|^2)^\beta|g'(w)|\log\frac{e}{1-|w|^2}\le\log\frac{4e}{3}\sup_{|w|\le\frac{1}{2}}(1-|w|^2)^\beta|g'(w)|$$

$$\prec\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta|g'(w)|<\infty. \tag{8}$$

From (7) and (8) it follows that (1) holds.

Now let $\alpha>1$. For a fixed $w\in\mathbb{D}$ such that $\frac{1}{2}<|w|<1$, set

$$f_{3,w}(z)=\frac{1}{\overline{w}^2}\left[\frac{(1-|w|^2)^2}{(1-\overline{w}z)^\alpha}-\frac{1-|w|^2}{(1-\overline{w}z)^{\alpha-1}}\right],\ z\in\mathbb{D}.$$

Then we have $\sup_{\frac{1}{2}<|w|<1}\|f_{3,w}\|_{Z^\alpha}<\infty$ (see [1]), and

$$f_{3,w}(w)=0,\ \ f'_{3,w}(w)=\frac{1}{\overline{w}(1-|w|^2)^{\alpha-1}},$$

which along with the boundedness of $J_g:Z^\alpha\to Z^\beta$ implies that

$$\infty\ge\|J_g f_3,w\|_{\mathcal{Z}^\beta}\ge\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|f_{3,w}(z)g''(z)+f'_{3,w}(z)g'(z)\right|$$

$$\ge\sup_{\frac{1}{2}<|w|<1}(1-|w|^2)^{\beta-\alpha+1}|g'(w)|\frac{1}{|w|}.$$

Thus,

$$\sup_{\frac{1}{2}<|w|<1}(1-|w|^2)^{\beta-\alpha+1}|g'(w)|<\infty. \tag{9}$$

On the other hand, since $g\in B^\beta$,

$$\sup_{|w|\le\frac{1}{2}}(1-|w|^2)^{\beta-\alpha+1}|g'(w)|\le\left(\frac{4}{3}\right)^{\alpha-1}\sup_{|w|\le\frac{1}{2}}(1-|w|^2)^\beta|g'(w)|<\infty. \tag{10}$$

By (9) and (10) we can see that (2), (4) and (6) hold.

For $z,w\in\mathbb{D}$, let

$$f_{4,w}(z)=\log\frac{e}{1-\overline{w}z}.$$

It is clear that $f_{4,w}(z)\in Z^2$ and

$$f_{4,w}(w)=\log\frac{e}{1-|w|^2},\quad f'_{4,w}(w)=\frac{\overline{w}}{1-|w|^2}.$$

By the boundedness of $J_g:Z^2\to Z^\beta$, we have

$$\begin{aligned}\infty &\geq \left\|J_g f_{4,w}\right\|_{\mathcal{Z}^\beta} \\ &\geq \sup_{w\in\mathbb{D}}(1-|w|^2)^\beta \left|f_{4,w}(z)g''(z)+f'_{4,w}(z)g'(z)\right| \\ &\geq \sup_{w\in\mathbb{D}}(1-|w|^2)^\beta |g''(w)|\log\frac{e}{1-|w|^2}-\sup_{w\in\mathbb{D}}(1-|w|^2)^{\beta-1}|g'(w)||w|.\end{aligned}$$

which along with (4) and the fact that $|w|<1$ we get that (3) holds.

For $\alpha>2$, take

$$f_{5,w}(z)=\frac{(1-|w|^2)^2}{(1-\overline{w}z)^\alpha},$$

where $z,w\in\mathbb{D}$. Then $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f_5''(z)|\leq 4\alpha\cdot 2\alpha\cdot(\alpha+1)$, which says that $f_{5,w}(z)\in Z^\alpha$. Moreover, we have

$$f_{5,w}(w)=\frac{1}{(1-|w|^2)^{\alpha-2}},\quad f_{5,w}{}'(w)=\frac{\alpha\overline{w}}{(1-|w|^2)^{\alpha-1}}.$$

By the boundedness of $J_g:Z^\alpha\to Z^\beta$, we have

$$\begin{aligned}\infty &\geq \left\|J_g f_{5,w}\right\|_{\mathcal{Z}^\beta} \\ &\geq \sup_{w\in\mathbb{D}}(1-|w|^2)^\beta \left|f_{5,w}(z)g''(z)+f'_{5,w}(z)g'(z)\right| \\ &\geq \sup_{w\in\mathbb{D}}(1-|w|^2)^{\beta-\alpha+2}|g''(w)|-\sup_{w\in\mathbb{D}}(1-|w|^2)^{\beta-\alpha+1}|g'(w)||\alpha w|.\end{aligned}$$

which along with (6) and the fact that $|w|<1$ we get that (5) holds.

Conversely, for the case $0<\alpha<1$, suppose that $g\in Z^\beta$, which yields $g\in B^\beta$ by Lemma 2.1. For any $f\in\mathcal{Z}^\alpha$, by Lemma 2.2 (i), we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|(J_g f)''(z)\right|\leq\frac{2}{1-\alpha}\|f\|_{Z^\alpha}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|g''(z)\right|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|g'(z)\right|\right)\prec\|f\|_{Z^\alpha}.$$

Furthermore, it is easy to see that $\left|(J_g f)(0)\right|=0$ and

$$\left|(J_g f)'(0)\right|=\left|f(0)g'(0)\right|\leq\frac{2|g'(0)|}{1-\alpha}\|f\|_{Z^\alpha}.$$

Thus we get $\left\|J_g f\right\|_{Z^\beta}\prec\|f\|_{Z^\alpha}$, which implies that $J_g:Z^\alpha\to Z^\beta$ is bounded. For other cases, we can use Lemma 2.2 to get the conclusion similarly.

For the boundedness of $I_g:Z^\alpha\to Z^\beta$, we need to break the problem into three different cases: $0<\alpha<1$, $\alpha=1$ and $\alpha>1$, whose proof is similar to that of Theorem 3.1 and the details are omitted.

**Theorem 3.2** Let $g\in H(\mathbb{D})$ and $\alpha,\ \beta>0$. Then $I_g:Z^\alpha\to Z^\beta$ is bounded if and only if

(a)
$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(z)|<\infty,$$
$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-\alpha}|g(z)|<\infty,$$

for $0<\alpha<1$;

(b) $$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|g'(z)|\log\frac{e}{1-|z|^2}<\infty,$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-1}|g(z)|<\infty,$$

for $\alpha=1$;

(c) $$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|<\infty,$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-1}|g(z)|<\infty,$$

for $\alpha>1$.

Now we characterize the compactness of the Riemann-Stieltjes operators $J_g, I_g: Z^{\alpha}\to Z^{\beta}$.

**Theorem 3.3** Let $g\in H(\mathbb{D})$ and $\alpha,\ \beta>0$. Then $J_g: Z^{\alpha}\to Z^{\beta}$ is compact if and only if

(a) $g\in Z^{\beta}$, for $0<\alpha<1$;

(b) $g\in Z^{\beta}$ and

$$\lim_{|z|\to1}(1-|z|^2)^{\beta}|g'(z)|\log\frac{e}{1-|z|^2}=0, \tag{11}$$

for $\alpha=1$;

(c) $g\in Z^{\beta}$ and

$$\lim_{|z|\to1}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|=0, \tag{12}$$

for $1<\alpha<2$;

(d) $$\lim_{|z|\to1}(1-|z|^2)^{\beta}|g''(z)|\log\frac{e}{1-|z|^2}=0, \tag{13}$$

$$\lim_{|z|\to1}(1-|z|^2)^{\beta-1}|g'(z)|=0, \tag{14}$$

for $\alpha=2$;

(e) $$\lim_{|z|\to1}(1-|z|^2)^{\beta-\alpha+2}|g''(z)|=0, \tag{15}$$

$$\lim_{|z|\to1}(1-|z|^2)^{\beta-\alpha+1}|g'(z)|=0, \tag{16}$$

for $\alpha>2$.

***Proof.*** Suppose that $J_g: Z^{\alpha}\to Z^{\beta}$ is compact, then it is immediate that $J_g: Z^{\alpha}\to Z^{\beta}$ is bounded. By Theorem3.1, we get that $g\in Z^{\beta}$, which is necessary for all cases. Moreover, $g\in B^{\beta}$ by using Lemma 2.1. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|z_n|\to1$ as $n\to\infty$. Without loss of generality, we may assume that $|z_n|>\frac{1}{2}$.

For the case $\alpha=1$, set

$$p_n(z)=\frac{h(\overline{z}_n z)}{\overline{z}_n}\left(\log\frac{e}{1-|z_n|^2}\right)^{-1}-\frac{h(|z_n|^2)}{\overline{z}_n}\left(\log\frac{e}{1-|z_n|^2}\right)^{-1},$$

where $h(z)=(z-1)\left[\left(1+\log\frac{e}{1-z}\right)^2+1\right]$ and $z\in\mathbb{D}$. Then $\sup_{n\in\mathbb{N}}\|p_n\|_Z<\infty$ by the proof of Theorem 3.1, and $p_n\to 0$ uniformly on compact subsets of $\mathbb{D}$ by a direct calculation. Applying Lemma 2.3 we conclude that $\lim_{n\to\infty}\|J_g p_n\|_{Z^\beta}=0$. Moreover, we have

$$p_n(z_n)=0,\ \ p_n'(z_n)=\log\frac{e}{1-|z_n|^2}.$$

Then

$$\|J_g p_n\|_{Z^\beta}\ge\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|(J_g p_n)''(z)\right|\ge(1-|z_n|^2)^\beta\left|g'(z_n)\right|\log\frac{e}{1-|z_n|^2}.$$

It follows that

$$\lim_{n\to\infty}(1-|z_n|^2)^\beta\,|g'(z_n)|\log\frac{e}{1-|z_n|^2}=\lim_{|z|\to 1}(1-|z|^2)^\beta\,|g'(z)|\log\frac{e}{1-|z|^2}=0,$$

that is, (11) holds.

Let $\alpha>1$ and consider the function

$$q_n(z)=\frac{1}{\overline{z}_n^{\,2}}\left[\frac{(1-|\overline{z}_n|^2)^2}{(1-\overline{z}_n z)^\alpha}-\frac{1-|z_n|^2}{(1-\overline{z}_n z)^{\alpha-1}}\right].$$

Then $\sup_{n\in\mathbb{N}}\|q_n\|_{Z^\alpha}<\infty$, and $q_n\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Lemma 2.3 shows that $\lim_{n\to\infty}\|J_g q_n\|_{Z^\beta}=0$. Moreover, we have

$$q_n(z_n)=0,\ q_n'(z_n)=\frac{1}{\overline{z}_n(1-|z_n|^2)^{\alpha-1}}.$$

Hence

$$\|J_g q_n\|_{Z^\beta}\ge\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|(J_g q_n)''(z)\right|\ge(1-|z_n|^2)^\beta\left|g'(z_n)\right|\frac{1}{|z_n|}. \tag{17}$$

Letting $n\to\infty$ in (17), we can get (12), (14) and (16).

For the case $\alpha=2$, let

$$s_n(z)=\left(1+\left(\log\frac{e}{1-\overline{z}_n z}\right)^2\right)\left(\log\frac{e}{1-|z_n|^2}\right)^{-1},$$

It is easy to show that $\{s_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $Z^2$, and $s_n\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. We now apply Lemma 2.3 again to obtain $\lim_{n\to\infty}\|J_g s_n\|_{Z^\beta}=0$. Moreover,

$$s_n(z_n)=\log\frac{e}{1-|z_n|^2}+\left(\log\frac{e}{1-|z_n|^2}\right)^{-1},\ \ s_n'(z_n)=\frac{2\overline{z}_n}{1-\overline{z}_n z}.$$

Therefore,

$$\begin{aligned}\|J_g s_n\|_{Z^\beta} &\ge \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(J_g s_n)''(z)| \\ &\ge (1-|z_n|^2)^{\beta-\alpha+1}|g''(z_n)|\left|\log\frac{e}{1-|z_n|^2}+\left(\log\frac{e}{1-|z_n|^2}\right)^{-1}\right| \\ &\quad -2|z_n|(1-|z_n|^2)^{\beta-1}|g'(z_n)|.\end{aligned} \tag{18}$$

Since $\lim_{|z_n|\to 1}\left(\log\frac{e}{1-|z_n|^2}\right)^{-1}=0$, which along with (14), letting $n\to\infty$ in (18) yields (13).

For $\alpha>2$, take

$$t_n(z)=\frac{(1-|z_n|^2)^2}{(1-\overline{z}_n z)^\alpha},$$

which is bounded in $\mathcal{Z}^\alpha$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Since $J_g: Z^\alpha\to Z^\beta$ is compact, we have $\lim_{n\to\infty}\|J_g t_n\|_{\mathcal{Z}^\beta}=0$. Moreover,

$$t_n(z_n)=\frac{1}{(1-|z_n|^2)^{\alpha-2}},\quad t_n'(z_n)=\frac{\alpha\overline{z}_n}{(1-|z_n|^2)^{\alpha-1}}.$$

Thus,

$$\begin{aligned}\|J_g t_n\|_{Z^\beta} &\ge \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(J_g t_n)''(z)| \\ &\ge (1-|z_n|^2)^{\beta-\alpha+2}|g''(z_n)|-\alpha|z_n|(1-|z_n|^2)^{\beta-\alpha+1}|g'(z_n)|.\end{aligned} \tag{19}$$

Letting $n\to\infty$ in (19) and using (16), we can get (15).

Conversely, for the case $0<\alpha<1$. Assume that $g\in Z^\beta$, which implies $g\in B^\beta$ and $J_g: Z^\alpha\to Z^\beta$ is bounded. Let $\{h_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $Z^\alpha$, and $h_n\to\infty$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Then

$$\begin{aligned}\|J_g h_n\|_{Z^\beta} &= |h_n(0)g'(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(J_g h_n)''(z)| \\ &\le |h_n(0)g'(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g''(z)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(z)||h_n'(z)| \\ &\le \|g\|_{\mathcal{Z}^\beta}|h_n(0)|+\|g\|_{\mathcal{Z}^\beta}\sup_{z\in\mathbb{D}}|h_n(z)|+\|g\|_{\mathcal{B}^\beta}\sup_{z\in\mathbb{D}}|h_n'(z)|.\end{aligned}$$

We see that $\lim_{n\to\infty}\|J_g h_n\|_{\mathcal{Z}^\beta}=0$ by Lemma 2.4. Employing Lemma 2.3 we have $J_g: Z^\alpha\to Z^\beta$ is compact.

If $\alpha=1$, suppose that $g\in Z^\beta$ and (11) holds. Then for every $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$(1-|z|^2)^\beta|g'(z)|\log\frac{e}{1-|z|^2}<\varepsilon. \tag{20}$$

Assume that $\{h_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $Z$ such that $\sup_{n\in\mathbb{N}}\|h_n\|_{\mathcal{Z}}\le L$, and $h_n\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Let $K=\{z\in\mathbb{D}:|z|\le\delta\}$, from Lemma 2(ii) and (20) it follows that

$$
\begin{aligned}
\|J_g h_n\|_{Z^\beta} &= |h_n(0)g'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(J_g h_n)''(z)| \\
&\le |h_n(0)g'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |g''(z)||h_n(z)| \\
&\quad + \sup_{z\in K}(1-|z|^2)^\beta |g'(z)||h_n'(z)| + \sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta |g'(z)||h_n(z)|_{\mathcal{Z}} \log\frac{e}{1-|z|^2} \\
&\le \|g\|_{\mathcal{Z}^\beta}|h_n(0)| + \|g\|_{\mathcal{Z}^\beta}\sup_{z\in\mathbb{D}}|h_n(z)| + \|g\|_{\mathcal{B}^\beta}\sup_{z\in K}|h_n'(z)| + L\varepsilon.
\end{aligned}
$$

Applying Lemma 2.4 we have $\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|h_n(z)|=0$, and Cauchy's estimation gives that $h_n'\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$, which along with the arbitrariness of $\varepsilon$ it follows that $\lim_{n\to\infty}\|J_g h_n\|_{\mathcal{Z}^\beta}=0$. Employing Lemma 2.3 the implication follows. For other cases, we can use Lemma 2.2 and Lemma 2.3 to show that $J_g: Z^\alpha \to Z^\beta$ is compact similarly.

**Remark 3.4** From Theorem 3.1 and Theorem 3.3, we see that the boundedness and compactness of $J_g: Z^\alpha\to Z^\beta$ are equivalent when $0<\alpha<1$.

For $I_g: Z^\alpha\to Z^\beta$, we need to break the problem into three different cases: $0<\alpha<1$, $\alpha=1$ and $\alpha>1$, whose proof is similar to that of Theorem 3.3 and we omit the details.

**Theorem 3.5** Let $g\in H(\mathbb{D})$ and $\alpha,\ \beta>0$. Then $I_g: Z^\alpha\to Z^\beta$ is compact if and only if

(a)
$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(z)|=0,$$
$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\alpha} |g(z)|=0,$$

for $0<\alpha<1$;

(b)
$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(z)|\log\frac{e}{1-|z|^2}=0,$$
$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-1} |g(z)|=0,$$

for $\alpha=1$;

(c)
$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\alpha+1} |g'(z)|=0,$$
$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-1} |g(z)|=0,$$

for $\alpha>1$.

## Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 12101188) and Doctoral Fund of Henan Institute of Technology (No. KQ2003).

## References

[1] Esmaeili, K. and Lindström, M. (2013). Weighted composition operators between Zygmund type spaces and their essential norms. Integral Equations and Operator Theory, 75, 473-490.

[2] Li, H. and Guo, Z. (2015). On a product-type operator from Zygmund-type spaces to Bloch-Orlicz spaces. Journal of Inequalities and Applications, 132, 18.

[3] Li, S. and Stević, S. (2007). Volterra-type operators on Zygmund spaces. Journal of Inequalities and Applications, Art. ID 32124,

10.

[4] Li, S. and Stević, S. (2008). Compactness of Riemann-Stieltjes operators between $F(p,q,s)$ spaces and $\alpha$-Bloch spaces. Publicationes Mathematicae Debrecen, 72, 111-128.

[5] Li, S. and Stević, S. (2008). Generalized composition operators on Zygmund spaces and Bloch type spaces. Journal of Mathematical Analysis and Applications, 338, 1282-1295.

[6] Li, S. and Stević, S. (2008). Products of Volterra type operator and composition operator from $H^{\infty}$ and Bloch spaces to Zygmund spaces. Journal of Mathematical Analysis and Applications, 345, 40-52.

[7] Li, S. and Stević, S. (2008). Riemann-Stieltjes operators from $H^{\infty}$ space to $\alpha$-Bloch spaces. Integral Transforms and Special Functions, 19, 767-776.

[8] Li, S. and Stević, S. (2008). Weighted composition operators from Zygmund spaces into Bloch spaces. Applied Mathematics and Computation, 206, 825-831.

[9] Li, S. and Stević, S. (2009). Riemann-Stieltjes operators between $\alpha$-Bloch spaces and Besov spaces. Mathematische Nachrichten, 282, 899-911.

[10] Liu, Y. and Yu, Y. (2013). Riemann-Stieltjes operator from mixed norm spaces to Zygmund-type spaces on the unit ball. Taiwanese Journal of Mathematics, 17, 1751-1764.

[11] Liu, Y., Yu, Y. and Liu, X. (2015). Riemann-Stieltjes operator from the general space to Zygmund-type spaces on the unit ball. Complex Analysis and Operator Theory, 9, 985-997.

[12] Stević, S. (2005). On an integral operator on the unit ball in $\mathbb{C}^n$. Journal of Inequalities and Applications, 1, 81-88.

[13] Stević, S. (2010). On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball. Abstract and Applied Analysis, Art. ID 198608, 7.

[14] Lin, Q. (2019). Volterra type operators between Bloch type spaces and weighted Banach spaces. Integral Equations and Operator Theory, 91, 20 pp.

[15] Ye, S. and Lin, C. (2017). Essential norms of Volterra type operators between Zygmund type spaces. Journal of Function Spaces, Art. ID 1409642, 14 pp.

[16] Xiao J. (2004). Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball. Journal of the London Mathematical Society, 70, 199-214.

[17] Siskakis, A. G. and Zhao, R. (1999). A Volterra type operator on spaces of analytic functions. Contemporary Mathematics, 232, 299-311.

[18] Aleman, A. and Siskakis, A. G. (1995). An integral operator on $H^p$. Complex Variables Theory and Application, 28, 149-158.

[19] Ueki, S. (2022). Volterra type integral operator acting between Fock spaces. Mathematical Inequalities & Applications, 25, 307-318.

[20] Yang, L. and Qian, R. (2021). Volterra integral operator and essential norm on Dirichlet type spaces. AIMS Mathematics, 6, 10092-10104.

[21] Liang, Y., Zeng, H., and Zhou, Z. H. (2020) Volterra-type operators from $F(p,q,s)$ space to Bloch-Orlicz and Zygmund-Orlicz spaces. Filomat, 34, 1359-1381.

[22] Lin, Q., Liu, J., and Wu, Y. (2018). Volterra type operators on $S^p(\mathbb{D})$ spaces. Journal of Mathematical Analysis and Applications, 461, 1100-1114.

[23] Liang, Y. X. (2017). Volterra-type operators from weighted Bergman-Orlicz space to β-Zygmund-Orlicz and γ-Bloch-Orlicz spaces. Monatsheftefür Mathematik, 182, 877-897.

[24] He, Z. H., Cao, G. F. and He, L. (2016). Volterra composition operators from Bloch-type spaces to $F(p,q,s)$ spaces on the unit ball. Advances in Mathematics (China), 45, 263-270.

[25] Cowen, C. C. and MacCluer, B. D. (1995). Composition operators on spaces of analytic functions. Studies in Advanced Mathematics, CRC Press, Boca Raton.