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Estimate of an Error for a Finite Difference a Phase Field Model for the Euler-Crank-Nicolson Methods

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Abstract

In order to understand phase transition processes like solidification, phase field models are frequently used. The energy (heat) equation for temperature is coupled with another nonlinear parabolic p.d.e. that includes a second unknown, the phase, which takes characteristic values, such as zero in the solid phase and one in the liquid phase. We consider the parabolic system of p.d.e' s

$$q(\phi)\phi_t = \nabla \cdot (A(\phi)\nabla\phi) + f(\phi, u),$$
$$u_t = \Delta u + [p(\phi)]_t,$$

which may be considered as a simplified phase field model. Here, $\phi = \phi(x, y, t)$ is the phase indicator function, $u = u(x, y, t)$ is the temperature, q, p, and f are given scalar functions, and A is a 2×2 diagonal matrix of given functions of ϕ. This system is posed for $t \geq 0$ on a rectangle in the x, y plane with appropriate boundary and initial conditions. We solve the system using a finite difference method that uses for first equation recursive Euler and Crank-Nicolson method the other equation. We prove a convergence result for the method and show results of numerical experiments verifying its order of accuracy.

Keywords

Finite Difference methods, simplified Phase Field Models, Parabolic System, Euler Crank-Nicolson Method, Error Estimates

1. Introduction

Phase field models, [1], are commonly employed to understand phase transition events like solidification. The energy (heat) equation for temperature is coupled with another nonlinear parabolic p.d.e. involving an additional unknown, the phase, which takes characteristic values, such as zero in the solid phase and one in the liquid phase, and exhibits a large gradient in a small neighborhood of the interface between the two phases. The solution to this system, which evolves from proper beginning circumstances, should show a sharp moving front representing the interface and, where applicable, depict the evolution of complicated geometric patterns (dendrites) that occur in practical solidification issues.The accurate numerical solution of this coupled system of p.d.e' s requires fine spatial discretizations around the interface and small time steps and is quite time consuming even in two space dimensions.

In this note we consider a specific system in two space dimensions, simplifying a model due to McFadden, Wheeler, Sekerka, Wang et al., [2]−[5]. This "semi-anisotropic" model consists of a system of two p.d.e's of the form

$$\begin{aligned} &\phi_t = \nabla\cdot(A(\phi)\nabla\phi) + f(\phi,u), \\ &u_t = \Delta u + [p(\phi)]_t, \end{aligned} \tag{1}$$

where $\phi = \phi(x,y,t)$ is the phase indicator function and $u = u(x,y,t)$ is the temperature, both defined on a rectangle Ω in the x,y plane for $t \geq 0$. The functions f and p are given smooth scalar functions of their arguments and A is a 2×2 matrix given by

$$A(\phi) = \begin{pmatrix} a(\phi) & -g(\phi) \\ g(\phi) & b(\phi) \end{pmatrix},$$

where $a,\ b,\ and\ g$ are smooth function of ϕ. The system (1) is supplemented by given initial conditions $\phi(x,y,0) = \phi_0(x,y)$, $u(x,y,0) = u_0(x,y)$, $(x,y)\in\Omega$, and boundary conditions of Neumann or Dirichlet type for ϕ and u on the boundary $\partial\Omega$ of Ω for $t \geq 0$.

In its fully nonlinear "anisotropic" version (where A is a function of $\nabla\phi$) the system (1) has been solved numerically by Wang, [4], and Wang and Sekerka, [5], by an 'explicit-implicit' finite difference scheme that uses the explicit Euler method for advancing the phase field over a temporal step by the first p.d.e. of (1), and then uses an ADI (Alternating Direction Implicit) scheme in the second p.d.e to solve for the temperature field. These two references contain many interesting numerical computations and measurements of the efficiency of the underlying numerical technique. In [6]−[8] Rappaz and his collaborators considered similar systems to (1), for which they proved existence and uniqueness of weak solutions. They also constructed and implemented fully discrete, adaptive finite element methods and used them to simulate dendritic growth in the anisotropic case. In [9]−[12] we have constructed and implemented Euler-Crank Nicolson finite difference scheme for equations in the general, anisotropic case and showed how they may be used in a parallel algorithm to solve the dendrite generation problem fast and efficiently.

In this paper, we solve the semi-anisotropic system (1) numerically using a finite difference discretization of the recursive Euler first equation and Crank-Nicolson method the other equation, for parabolic problem, whose parameters depend on the derivative of the solution. We consider the initial-boundary-value problem for (1) with homogeneous Dirichlet boundary condition for ϕ and u in the boundary of Ω. (The Neumann problem may be similarly analyzed.) In Section 2 we state the p.d.e problem and the assumptions on its coefficients. In subsection 2.1 we introduce the basic symbolism, formulate the shape of finite differences and prove some basic Lemmata. In subsection 2.2 we will prove that the finite difference method, for this non-linear system, existence a solution. In subsection 2.5 we prove some preliminary results, stability, on various finite difference approximations and in subsection 2.6 we state and prove the main result of the paper establishing that the finite difference approximations to u and ϕ converge, in the discrete L_2 norms respectively, with bounds of order $\Delta t + h$, where Δt is the time step and h the (uniform) spatial discretization meshlength under a stability condition of the form $\frac{\Delta t}{h} \leq \sigma$. We close,In Section 3, by presenting some simple numerical experiments verifying orders of convergence.

2. The numerical Method

Let $\Omega = (\alpha,\beta)\times(\alpha,\beta)$. For $(x,y,t)\in\ \overline{\Omega}\times[0,T]$, where $T>0$, we consider the initial-boundary-value problem

$$\begin{cases} \frac{\partial}{\partial t}\phi - div(A(\nabla\phi)\nabla\phi) = f(\phi,u), & in\ \overline{\Omega}\times[0,T] \\ \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial y^2}u = q(\phi), & in\ \overline{\Omega}\times[0,T] \\ \phi(x,y,t) = 0 & (x,y,t)\in\partial\Omega\times[0,T] \\ \phi(x,y,0) = \phi_0(x,y) & (x,y)\in\overline{\Omega} \\ u(x,y,t) = 0 & (x,y,t)\in\partial\Omega\times[0,T] \\ u(x,y,0) = u_0(x,y) & (x,y)\in\overline{\Omega} \end{cases} \tag{2}$$

with $A(\theta) = \begin{pmatrix} a & -g \\ g & b \end{pmatrix}$, where $a = a(\phi,\phi_x,\phi_y)$, $b = b(\phi,\phi_x,\phi_y)$, $g = g(\phi,\phi_x,\phi_y)$, are smooth functions of ϕ, such that $0 < c_a^0 \leq a(\xi) \leq c_a^1, 0 < c_b^0 \leq b(\xi) \leq c_b^1$, for $\xi\in$ IR, and f and p are smooth functions defined on IR^2 and IR respectively, with $f(0,0) = 0$ and $p(0) = 0$. We suppose that $\phi_0 = u_0 = 0$ on $\partial\Omega$. In addition we assume (to simplify the proofs of the error estimates) that the functions f,a,b,g and p are globally Lipschitz functions of their arguments, i.e. that there exists a constant C such that:

$$|f(z_1,z_2) - f(Z_1,Z_2)| \leq L_{f,1}|z_1 - Z_1| + L_{f,2}|z_2 - Z_2| \tag{3}$$

$$|a(z_1,z_2,z_3)-a(Z_1,Z_2,Z_3)|\le L_{a,1}|z_1-Z_1|+L_{a,2}|z_2-Z_2|+L_{a,3}|z_3-Z_3| \quad (4)$$
$$|b(z_1,z_2,z_3)-b(Z_1,Z_2,Z_3)|\le L_{b,1}|z_1-Z_1|+L_{b,2}|z_2-Z_2|+L_{b,3}|z_3-Z_3| \quad (5)$$
$$|g(z_1,z_2,z_3)-g(Z_1,Z_2,Z_3)|\le L_{g,1}|z_1-Z_1|+L_{g,2}|z_2-Z_2|+L_{g,3}|z_3-Z_3| \quad (6)$$
$$|q(z_1)-q(Z_1)|\le L_q|z_1-Z_1| \quad (7)$$
$$\forall\ z_1,z_2,z_3,Z_1,Z_2,Z_3\in \mathrm{IR}.$$

We will assume that the initial-boundary-value problem ((2)) has a unique solution (u,ϕ), smooth enough for the purpose of the numerical approximations.

2.1 Introduction Symbolism and Basic Lemmas

We discretize the initial-boundary-value problem ((2)) as follows. Let $h:=\frac{\beta-\alpha}{J+1}$ $with\ J\in$ IN, and $x_i:=\alpha+ih\ with\ i=0,\dots,J+1, y_j:=\alpha+jh\ with\ j=0,\dots,J+1$. We define $\Omega_h:=\{(x_i,y_j)\ i,j=1,\dots,J\}$ and $\partial\Omega_h:=\{(x_i,y_j)\ i=0\ \ or\ \ i=J+1,\ or\ \ j=0\ \ or\ \ j=J+1\}$. Still $t^{n+1}=t^n+\Delta t,\ n=0,\dots,N-1$ for $\Delta t:=\frac{T}{N}\ with\ N\in$ IN, and $t^0=0$. We also define $t^{n+\frac{1}{2}}=t^n+\frac{\Delta t}{2}, x_{i\pm\frac{1}{2}}=x_i\pm\frac{h}{2}$, and $X_h^0:=\{U=(U_{0,0},\dots,U_{J+1,J+1})^T\in$ IR2(J+2):Uij=0, i=0 or i=J+1 or j=0 or j=J+1.

We approximate the solution u,ϕ of ((2)) by mesh functions $U^n,\Phi^n\in X_h^0$ as follows: For $n=0,1,2,\dots,N$, we approximate the vectors $(u(x_0,y_0,t^n),\dots,u(x_{J+1},y_{J+1},t^n))^T,(\phi(x_0,y_0,t^n),\dots,\phi(x_{J+1},y_{J+1},t^n))^T$ by $U^n,\Phi^n\in X_h^0$ satisfying the following finite difference scheme.

2.2 Finite Difference Scheme

The finite difference scheme, which is a recursive Euler for firt Eq. and a Crank-Nicolson method for second Eq, for the nonlinear problem defined by Eq. ((2)), whose parameters depend on the derivative of the solution, is defined as follows:

$$\left.\begin{array}{ll}
(a)\Phi_{ij}^0=\phi_0(x_i,y_j),\Phi^0\in X_h^0 & (x_i,y_j)\in\Omega_h\cup\partial\Omega_h\\
(b)U_{ij}^0=u_0(x_i,y_j),U^0\in X_h^0 & (x_i,y_j)\in\Omega_h\cup\partial\Omega_h\\
For n=0,1,\dots,N-1: & \\
(c)\frac{\Phi_{ij}^{n+1}-\Phi_{ij}^n}{\Delta t}-(L_h^{n+1}\Phi^{n+1})_{ij}=f_{ij}^n, & (x_i,y_j)\in\Omega_h\\
(d)\Phi_{ij}^{n+1}=0 & (x_i,y_j)\in\partial\Omega_h\\
(e)\frac{U_{ij}^{n+1}-U_{ij}^n}{\Delta t}-\Delta_h\left(\frac{U_{ij}^{n+1}+U_{ij}^n}{2}\right)=\frac{1}{2}(q_{ij}^{n+1}+q_{ij}^n), & (x_i,y_j)\in\Omega_h\\
(f)U_{ij}^{n+1}=0 & (x_i,y_j)\in\partial\Omega_h
\end{array}\right\} \quad (8)$$

where $\delta_x v_{ij}=\frac{v_{i+1,j}-v_{i,j}}{h},\delta_{\bar{x}}v_{ij}=\frac{v_{i,j}-v_{i-1,j}}{h},\delta_y v_{ij}=\frac{v_{i,j+1}-v_{i,j}}{h},\delta_{\bar{y}}v_{ij}=\frac{v_{i,j}-v_{i,j-1}}{h}$,

$$\delta_{x\bar{x}}v_{ij}=\delta_x\delta_{\bar{x}}v_{ij}=\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{h^2},\delta_{y\bar{y}}v_{ij}=\delta_y\delta_{\bar{y}}v_{ij}=\frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{h^2},$$
$$\Delta_h v_{ij}:=\delta_{x\bar{x}}v_{ij}+\delta_{y\bar{y}}v_{ij},$$

We difine $L_h^n:X_h^0\to X_h^0$ for $0\le n\le N$ and $v\in X_h^0$

$$(L_h^n v)_{i,j}:=\begin{cases}\delta_x(a_{i-\frac{1}{2},j}^n\delta_{\bar{x}}v_{ij})+\delta_y(b_{i,j-\frac{1}{2}}^n\delta_{\bar{y}}v_{ij})-\delta_x(g_{i-\frac{1}{2},j-\frac{1}{2}}\delta_{\bar{y}}v_{ij})+\delta_y(g_{i-\frac{1}{2},j-\frac{1}{2}}\delta_{\bar{x}}v_{ij}) & (x_i,y_j)\in\Omega_h\\ 0 & (x_i,y_j)\in\partial\Omega_h\end{cases}$$

$$a_{i-\frac{1}{2},j}^n:=a(\frac{\Phi_{i,j}^n+\Phi_{i-1,j}^n}{2},\frac{\Phi_{i,j}^n-\Phi_{i-1,j}^n}{h},\frac{(\Phi_{i,j+1}^n+\Phi_{i-1,j+1}^n)-(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{4h}),$$
$$b_{i,j-\frac{1}{2}}^n:=b(\frac{\Phi_{i,j}^n+\Phi_{i,j-1}^n}{2},\frac{(\Phi_{i+1,j}^n+\Phi_{i+1,j-1}^n)-(\Phi_{i-1,j}^n+\Phi_{i-1,j-1}^n)}{4h},\frac{\Phi_{i,j}^n-\Phi_{i,j-1}^n}{h}),$$
$$g_{i-\frac{1}{2},j-\frac{1}{2}}^n:=g(\frac{(\Phi_{i,j}^n+\Phi_{i-1,j}^n)+(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{4},\frac{(\Phi_{i,j}^n+\Phi_{i,j-1}^n)-(\Phi_{i-1,j}^n+\Phi_{i-1,j-1}^n)}{2h},\frac{(\Phi_{i,j}^n+\Phi_{i-1,j}^n)-(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{2h}),$$

Also, $f_{ij}^n := f(\Phi_{ij}^n, U_{ij}^n), q_{ij}^n := q(\Phi_{ij}^n), (x_i, y_j) \in \Omega_h \cup \partial\Omega_h$.

2.3 Basic Lemmas

Lemma 2.1

a) Let $\phi \in \mathrm{IR}^{2(J+2)}, \psi \in \mathrm{IR}^{2(J+2)}$ $with$ $\psi_{ij} = 0$ for $i = 0$ or $i = J+1$.
Then $\sum_{i=1}^{J}(\phi_{i+1,j} - \phi_{i,j})\psi_{ij} = -\sum_{i=1}^{J+1}\phi_{ij}(\psi_{i,j} - \psi_{i-1,j})$

b) Let $\phi \in \mathrm{IR}^{2(J+2)}$, $\psi \in \mathrm{IR}^{2(J+2)} with \psi_{ij} = 0$ $forj = 0$ $orj = J+1$.
Then $\sum_{j=1}^{J}(\phi_{i,j+1} - \phi_{i,j})\psi_{ij} = -\sum_{j=1}^{J+1}\phi_{ij}(\psi_{i,j} - \psi_{i,j-1})$

c) If $\phi, v \in \mathrm{IR}^{2(J+2)}$ then for $(x_i, y_j) \in \Omega_h$:

i. $\delta_{x\bar{x}}v_{ij} := \delta_x\delta_{\bar{x}}v_{ij} = \delta_{\bar{x}}\delta_x v_{ij}$, $\delta_{y\bar{y}}v_{ij} := \delta_y\delta_{\bar{y}}v_{ij} = \delta_{\bar{y}}\delta_y v_{ij}$,

ii. $\delta_x\delta_y v_{ij} = \delta_y\delta_x v_{ij}, \delta_{\bar{x}}\delta_y v_{ij} = \delta_y\delta_{\bar{x}}v_{ij}, \delta_{\bar{y}}\delta_x v_{ij} = \delta_x\delta_{\bar{y}}v_{ij}$,

iii. $\delta_{\bar{x}\bar{y}}v_{ij} := \delta_{\bar{x}}\delta_{\bar{y}}v_{ij} = \delta_{\bar{y}}\delta_{\bar{x}}v_{ij}$,

iv. $\delta_{\bar{x}}\left(\phi_{ij}\left(\delta_{\bar{y}}v_{ij}\right)\right) = \left[\delta_{\bar{x}}\phi_{ij}\,\delta_{\bar{y}}v_{ij} + \phi_{ij}\,\delta_{\bar{x}\bar{y}}v_{ij}\right] - h\delta_{\bar{x}}\phi_{i,j}\,\delta_{\bar{x}\bar{y}}v_{i,j}$,

$$\delta_{\bar{y}}\left(\phi_{ij}\left(\delta_{\bar{x}}v_{ij}\right)\right) = \left[\delta_{\bar{y}}\phi_{ij}\,\delta_{\bar{x}}v_{ij} + \phi_{ij}\,\delta_{\bar{x}\bar{y}}v_{ij}\right] - h\delta_{\bar{y}}\phi_{i,j}\,\delta_{\bar{x}\bar{y}}v_{i,j}.$$

v. $\delta_x(\phi_{ij}v_{ij}) = v_{ij}\,\delta_x\phi_{ij} + \phi_{ij}\,\delta_x v_{ij} + h(\delta_x\phi_{i,j})(\delta_x v_{i,j})$.

We define the inner product $(v, w)_h := h^2\sum_{i=1}^{J}\sum_{j=1}^{J}v_{ij}w_{ij}$, on X_h^0, with corresponding norm

$$\| v \|_h := h\left(\sum_{i=1}^{J}\sum_{j=1}^{J}v_{ij}^2\right)^{\frac{1}{2}}.$$

Lemma 2.2 *Define* $\Delta_h : X_h^0 \to X_h^0$

$$(\Delta_h)_{i,j} := \begin{cases} \dfrac{v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{ij}}{h^2}, & (x_i, y_j) \in \Omega_h \\ 0 & (x_i, y_j) \in \partial\Omega_h \end{cases}$$

Then,

$$(\Delta_h v, w)_h = (v, \Delta_h w)_h \;\; \forall \;\; v, w \in X_h^0 \tag{9}$$

and

$$(\Delta_h v, v)_h = -|v|_{1,h}^2 \;\; \forall \;\; v \in X_h^0 \tag{10}$$

where $|v|_{1,h}^2 := h^2\sum_{i=1}^{J+1}\sum_{j=1}^{J+1}[(\delta_{\bar{x}}v_{ij})^2 + (\delta_{\bar{y}}v_{ij})^2]$.

Proof.

For proof, see the Entry below Lemma 2.3 for $a \equiv 1, b \equiv 1, and\ g \equiv 0$.

Lemma 2.3 *We define* $L_h^n : X_h^0 \to X_h^0$

$$(L_h^n v)_{i,j} := \begin{cases} \delta_x(a_{i-\frac{1}{2},j}^n\,\delta_{\bar{x}}v_{ij}) + \delta_y(b_{i,j-\frac{1}{2}}^n\delta_{\bar{y}}v_{ij}) - \delta_x(g_{i-\frac{1}{2},j-\frac{1}{2}}^n\delta_{\bar{y}}v_{ij}) + \delta_y(g_{i-\frac{1}{2},j-\frac{1}{2}}^n\delta_{\bar{x}}v_{ij}) & (x_i, y_j) \in \Omega_h \\ 0 & (x_i, y_j) \in \partial\Omega_h \end{cases}$$

Then,

$$(L_h^n v, w)_h = (v, L_h^n w)_h + 2h^2\sum_{i=1}^{J+1}\sum_{j=1}^{J+1}g_{i-\frac{1}{2},j-\frac{1}{2}}^n\left[(\delta_{\bar{y}}v)_{ij}(\delta_{\bar{x}}w)_{ij} - (\delta_{\bar{x}}v)_{ij}(\delta_{\bar{y}}w)_{ij}\right] \;\forall\; v, w \in X_h^0 \tag{11}$$

and

$$c_0|v|_{1,h}^2 \le -(L_h^n v, v)_h = h^2\sum_{j=1}^{J+1}\sum_{i=1}^{J+1}\{\tilde{a}_{ij}^n(\delta_{\bar{x}}v_{ij})^2 + \tilde{b}_{ij}^n(\delta_{\bar{y}}v_{ij})^2\} \le C_1|v|_{1,h}^2 \tag{12}$$

where $C_1 = \max(c_a^1, c_b^1) > 0, c_0 = \min(c_a^0, c_b^0) > 0$, and $\tilde{a}_{ij} := a_{i-\frac{1}{2},j}^n, \tilde{b}_{ij} := b_{i,j-\frac{1}{2}}^n$.

Proof.

Skipping the time indicator n, we have for $v, w \in X_h^0$, with $\tilde{g}_{ij} := g_{i-\frac{1}{2},j-\frac{1}{2}}^n$.

$$(L_h v, w)_h = (\delta_x(\tilde{a}\delta_{\bar{x}}v), w)_h + (\delta_y(\tilde{b}\delta_{\bar{y}}v), w)_h - (\delta_x(\tilde{g}\delta_{\bar{y}}v), w)_h + (\delta_y(\tilde{g}\delta_{\bar{x}}v), w)_h.$$

Now,

$$
\begin{aligned}
(\delta_x(\tilde{a}\delta_{\bar{x}}v), w)_h &= h^2 \sum_{i=1}^{J} \sum_{j=1}^{J} \frac{1}{h}\left[(\tilde{a}\delta_{\bar{x}}v)_{i+1,j} - (\tilde{a}\delta_{\bar{x}}v)_{i,j}\right] w_{ij} \\
&= h \sum_{j=1}^{J} \left\{ \sum_{i=1}^{J} \left[(\tilde{a}\delta_{\bar{x}}v)_{i+1,j} - (\tilde{a}\delta_{\bar{x}}v)_{i,j}\right] w_{ij} \right\} \overset{Lemma\ 2.1.a}{=} \\
&- h \sum_{j=1}^{J} \left\{ \sum_{i=1}^{J+1} (\tilde{a}\delta_{\bar{x}}v)_{i,j} \left(w_{ij} - w_{i-1,j}\right) \right\} \\
&= -h^2 \sum_{j=1}^{J} \sum_{i=1}^{J+1} \tilde{a}_{ij} (\delta_{\bar{x}}v_{ij})(\delta_{\bar{x}}w_{ij}) \overset{(\delta_{\bar{x}}w)_{i,J+1}=0}{=} - h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{a}_{ij} (\delta_{\bar{x}}v_{ij})(\delta_{\bar{x}}w_{ij}).
\end{aligned}
$$

Hence

$$(\delta_x(\tilde{a}\delta_{\bar{x}}v), w)_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{a}_{ij} \left(\delta_{\bar{x}}v_{ij}\right)\left(\delta_{\bar{x}}w_{ij}\right) (+1)$$

Similarly

$$(\delta_y(\tilde{b}\delta_{\bar{y}}v), w)_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{b}_{ij} \left(\delta_{\bar{y}}v_{ij}\right)\left(\delta_{\bar{y}}w_{ij}\right) (+2)$$

Let

$$
\begin{aligned}
(\delta_x(\tilde{g}\delta_{\bar{y}}v), w)_h &= h^2 \sum_{i=1}^{J} \sum_{j=1}^{J} \frac{1}{h}\left[(\tilde{g}\delta_{\bar{y}}v)_{i+1,j} - (\tilde{g}\delta_{\bar{y}}v)_{i,j}\right] w_{ij} \\
&= h \sum_{j=1}^{J} \left\{ \sum_{i=1}^{J} \left[(\tilde{g}\delta_{\bar{y}}v)_{i+1,j} - (\tilde{g}\delta_{\bar{y}}v)_{i,j}\right] w_{ij} \right\} \overset{Lemma\ 2.1.a}{=} \\
&- h \sum_{j=1}^{J} \left\{ \sum_{i=1}^{J+1} (\tilde{g}\delta_{\bar{y}}v)_{i,j} \left(w_{ij} - w_{i-1,j}\right) \right\} \\
&= -h^2 \sum_{j=1}^{J} \sum_{i=1}^{J+1} \tilde{g}_{ij} (\delta_{\bar{y}}v_{ij})(\delta_{\bar{x}}w_{ij}) \overset{(\delta_{\bar{x}}w)_{i,J+1}=0}{=} - h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{g}_{ij} (\delta_{\bar{y}}v_{ij})(\delta_{\bar{x}}w_{ij}).
\end{aligned}
$$

Hence

$$(\delta_x(\tilde{g}\delta_{\bar{y}}v), w)_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{g}_{ij} (\delta_{\bar{y}}v_{ij})(\delta_{\bar{x}}w_{ij}). (+3)$$

Similar

$$(\delta_y(\tilde{g}\delta_{\bar{x}}v), w)_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{g}_{ij} (\delta_{\bar{x}}v_{ij})(\delta_{\bar{y}}w_{ij}). (+4)$$

Therefore

$$(L_h v, w)_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \left\{ \tilde{a}_{ij} (\delta_{\bar{x}}v)_{ij} (\delta_{\bar{x}}w)_{ij} + \tilde{b}_{ij} (\delta_{\bar{y}}v)_{ij} (\delta_{\bar{y}}w)_{ij} - \tilde{g}_{ij} (\delta_{\bar{y}}v)_{ij} (\delta_{\bar{x}}w)_{ij} + \tilde{g}_{ij} (\delta_{\bar{x}}v)_{ij} (\delta_{\bar{y}}w)_{ij} \right\}. (+5)$$

Now,

$$(v, L_h w)_h = (v, \delta_x(\tilde{a}\delta_{\bar{x}}w))_h + (v, \delta_y(\tilde{b}\delta_{\bar{y}}w))_h - (v, \delta_x(\tilde{g}\delta_{\bar{y}}w))_h + (v, \delta_y(\tilde{g}\delta_{\bar{x}}w))_h.$$

As above, we have

$$(v, \delta_x(\tilde{a}\delta_{\bar{x}}w))_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{a}_{ij} (\delta_{\bar{x}}v_{ij})(\delta_{\bar{x}}w_{ij}) (+6)$$

and

$$(v, \delta_y(\tilde{b}\delta_{\bar{y}}w))_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{b}_{ij} (\delta_{\bar{y}}v_{ij})(\delta_{\bar{y}}w_{ij}). (+7)$$

Also,

$$(v, \delta_x(\tilde{g}\delta_{\bar{y}}w))_h = -h^2 \sum_{j=1}^{J+1} \sum_{i=1}^{J+1} \tilde{g}_{ij} (\delta_{\bar{x}}v_{ij})(\delta_{\bar{y}}w_{ij}) (+8)$$

and

$$(v,\delta_y(\tilde{g}\delta_{\bar{x}}w))_h = -h^2 \sum_{j=1}^{J+1}{}^{\dagger}\sum_{i=1}^{J+1}{}^{\dagger}\tilde{g}_{ij}(\delta_{\bar{y}}v_{ij})(\delta_{\bar{x}}w_{ij}).\ (+9)$$

Therefore

$$(v,L_h w)_h = -h^2 \sum_{j=1}^{J+1}{}^{\dagger}\sum_{i=1}^{J+1}{}^{\dagger}\{\tilde{a}_{ij}(\delta_{\bar{x}}v_{ij})(\delta_{\bar{x}}w_{ij}) + \tilde{b}_{ij}(\delta_{\bar{y}}v_{ij})(\delta_{\bar{y}}w_{ij}) - \tilde{g}_{ij}(\delta_{\bar{x}}v_{ij})(\delta_{\bar{y}}w_{ij}) + \tilde{g}_{ij}(\delta_{\bar{y}}v_{ij})(\delta_{\bar{x}}w_{ij})\}(+10)$$

Finally, we have the Eq. ((11)) is obtained from Eqs. (+5) and (+10).
When $v = w$, the (+5) gives

$$-(L_h v, v)_h = h^2 \sum_{j=1}^{J+1}{}^{\dagger}\sum_{i=1}^{J+1}{}^{\dagger}\{\tilde{a}_{ij}(\delta_{\bar{x}}v)_{ij}^2 + \tilde{b}_{ij}(\delta_{\bar{y}}v)_{ij}^2\}.$$

Therefore, from inequalities $0 < c_a^0 \le a(\xi_1,\xi_2,\xi_3) \le c_a^1, 0 < c_b^0 \le b(\xi_1,\xi_2,\xi_3) \le c_b^1$ resulting the ((12))

2.4 Existence of solutions of the finite difference method

Solving the difference equations that define quantities $\Phi_{ij}^{n+1}, (x_i,y_j) \in \Omega_h$ on ((8).(c)), knowing the Φ_{ij}^n and U_{ij}^n on $(x_i,y_j) \in \Omega_h$ requires solving a nonlinear system. In this paragraph we will prove that this non-linear system has a solution.

We symbolize with $\Phi^* = \Phi^{n+1} \in X_h^0$, we write the ((8).(c)), in the form

$$\Gamma(\Phi^*) = 0 \tag{13}$$

where the illustration $\Gamma: X_h^0 \to X_h^0$ given by the formula

$$\Gamma(\Phi) := \Phi - \Delta t L_h^{n+1}\Phi - X^n, \tag{14}$$

where

$$X^n := \Phi^n - f(\Phi^n, U^n). \tag{15}$$

The proof of the existence of a solution Φ^* the systems ((12)) is based on the entry which is a form of the know this fixed point theorem Browder [13].

Lemma 2.4 *Let H real vector space, finite dimension, with interior product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Let $\gamma: H \to H$ continuous display as it exists $\alpha > 0$ such that for each $x \in H$ with $\| x \| = \alpha$, to apply that* $(\gamma(x), x) \ge 0$. *Then, exist $x^* \in H$ such that $\gamma(x^*) = 0$ and $\| x^* \| \le \alpha$.*

In our case, we consider, the vector space X_h^0 with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Because of our assumptions, knowing $\Phi_{ij}^n, U_{ij}^n \in X_h^0$, or $\Gamma: X_h^0 \to X_h^0$ where define the ((14)) is continuous function. In addition we have for each $\Phi \in X_h^0$

$$(\Gamma(\Phi),\Phi)_h = \| \Phi \|_h^2 - \Delta t (L_h^{n+1}\Phi, \Phi)_h - (X^n, \Phi)_h$$

$$\overset{(12)}{\ge} \| \Phi \|_h^2 + \Delta t |\Phi|_{1,h}^2 - \| X^n \|_h \| \Phi \|_h$$

$$\ge \| \Phi \|_h^2 - \| X^n \|_h \| \Phi \|_h \ge (\| \Phi \|_h - \| X^n \|_h) \| \Phi \|_h$$

Therefore, for each $\Phi \in X_h^0$ with $\| \Phi \|_h = \| X^n \|_h + \varepsilon$, and $\varepsilon > 0$, it is true that

$$(\Gamma(\Phi),\Phi)_h > 0.$$

So, from Lemma 2.4 it follows that it exists $\Phi^* \in X_h^0$ such as $\Gamma(\Phi^*) = 0$ $(\| \Phi^* \|_h \le \| X^n \|_h + \varepsilon)$. We conclude that the system $\Gamma(\Phi) = 0$ has a solution $\Phi^* = \Phi^{n+1} \in X_h^0$.

The existence-uniqueness of the linear system solution ((8).(e)) the method Crank-Nicolson for the U^{n+1}, knowing that $\Phi_{ij}^n, \Phi_{ij}^{n+1}$ and U_{ij}^n on $(x_i,y_j) \in \Omega_h$ is classic.

2.5 Stability

Lemma 2.5 *(Stability) Let Φ_{ij}^n, U_{ij}^n the solutions of the distinct shape ((8)). Then, if Δt small enough, we have*

$$\max_{0 \le n \le N} \left(\| \Phi_{ij}^n \|_h + \| U_{ij}^n \|_h \right) \le C\left(\| \Phi_{ij}^0 \|_h + \| U_{ij}^0 \|_h \right) \tag{16}$$

Proof.

We assume that $(i,j) \in \Omega_h$ and $n \le M \le N$.

From ((8).c) we have

$$\Phi_{ij}^{n+1} - \Phi_{ij}^n - \Delta t (L_h^{n+1}\Phi^{n+1})_{ij} = \Delta t f(\Phi_{ij}^n, U_{ij}^n)$$

$$(\Phi^{n+1}-\Phi^n,\Phi^{n+1})_h-\Delta t(L_h^{n+1}\Phi^{n+1},\Phi^{n+1})_h=\Delta t(f(\Phi^n,U^n),\Phi^{n+1})_h$$
$$\|\Phi^{n+1}\|_h^2-(\Phi^n,\Phi^{n+1})_h-\Delta t(L_h^{n+1}\Phi^{n+1},\Phi^{n+1})_h=\Delta t(f(\Phi^n,U^n),\Phi^{n+1})_h$$

However $\|\Phi^{n+1}-\Phi^n\|_h^2=\|\Phi^{n+1}\|_h^2+\|\Phi^n\|_h^2-2(\Phi^{n+1},\Phi^n)_h$. Therefore

$$\frac{1}{2}\|\Phi^{n+1}-\Phi^n\|_h^2+\frac{1}{2}(\|\Phi^{n+1}\|_h^2-\|\Phi^n\|_h^2)-\Delta t(L_h^{n+1}\Phi^{n+1},\Phi^{n+1})_h=\Delta t(f(\Phi^n,U^n),\Phi^{n+1})_h$$

From the relationship ((12)), we have

$$\frac{1}{2}\|\Phi^{n+1}-\Phi^n\|_h^2+\frac{1}{2}(\|\Phi^{n+1}\|_h^2-\|\Phi^n\|_h^2)+c_0\Delta t|\Phi^{n+1}|_{1,h}^2\le\Delta t(f(\Phi^n,U^n),\Phi^{n+1})_h$$

From the treaty Lipschitz ((3)) for f we have

$$\frac{1}{2}\|\Phi^{n+1}-\Phi^n\|_h^2+\frac{1}{2}(\|\Phi^{n+1}\|_h^2-\|\Phi^n\|_h^2)+c_0\Delta t|\Phi^{n+1}|_{1,h}^2\le\Delta tC(\|\Phi^{n+1}\|_h^2+\|\Phi^n\|_h^2+\|U^n\|_h^2)$$

Therefore

$$\frac{1}{2}(\|\Phi^{n+1}\|_h^2-\|\Phi^n\|_h^2)\le\Delta tC(\|\Phi^{n+1}\|_h^2+\|\Phi^n\|_h^2+\|U^n\|_h^2)$$

Adding by $n=0$ to $M-1$, we have

$$\sum_{n=0}^{M-1}(\|\Phi^{n+1}\|_h^2-\|\Phi^n\|_h^2)\le\Delta tC\sum_{n=0}^{M-1}(\|\Phi^{n+1}\|_h^2+\|\Phi^n\|_h^2+\|U^n\|_h^2)$$

Therefore

$$\|\Phi^M\|_h^2\le\Delta tC\sum_{n=0}^{M}\|\Phi^n\|_h^2+\Delta tC\sum_{n=0}^{M-1}\|U^n\|_h^2+\|\Phi^0\|_h^2.(*1)$$

Similar to above ((8).e) we have

$$\frac{U_{ij}^{n+1}-U_{ij}^n}{\Delta t}-\Delta_h\frac{U_{ij}^{n+1}+U_{ij}^n}{2}=\frac{q_{ij}^{n+1}+q_{ij}^n}{2}$$
$$U_{ij}^{n+1}-U_{ij}^n-\Delta t\Delta_h\frac{U_{ij}^{n+1}+U_{ij}^n}{2}=\Delta t\frac{q_{ij}^{n+1}+q_{ij}^n}{2}.$$

We get the inner product $(\cdot,\cdot)_h$ and its two members above in relation to $U^{n+1}+U^n$.

$$(U^{n+1}-U^n,U^{n+1}+U^n)_h-\frac{\Delta t}{2}(\Delta_h(U^{n+1}+U^n),U^{n+1}+U^n)_h=\Delta t(\frac{q_{ij}^{n+1}+q_{ij}^n}{2},U^{n+1}+U^n)_h$$

Using the Lemma 2.2 we have

$$\|U^{n+1}\|_h^2-\|U^n\|_h^2+\frac{\Delta t}{2}|U^{n+1}+U^n|_{1,h}^2=\frac{\Delta t}{2}(q^{n+1}+q_{ij}^n,U^{n+1}+U^n)_h$$
$$\|U^{n+1}\|_h^2-\|U^n\|_h^2+\frac{\Delta t}{2}|U^{n+1}+U^n|_{1,h}^2\le\frac{\Delta t}{2}\|q^{n+1}+q_{ij}^n\|_h\|U^{n+1}+U^n\|_h.$$

Even from the condition Lipschitz for q we have

$$\|U^{n+1}\|_h^2-\|U^n\|_h^2\le\Delta tC(\|\Phi^{n+1}\|_h+\|\Phi^n\|_h)(\|U^{n+1}\|_h+\|U^n\|_h)$$

That is,

$$\|U^{n+1}\|_h-\|U^n\|_h\le C\Delta t(\|\Phi^{n+1}\|_h+\|\Phi^n\|_h)$$

Adding we have

$$\|U^n\|_h-\|U^0\|_h\le C\Delta t\sum_{k=0}^{n}\|\Phi^k\|_h.(*2)$$

So, from inequality Cauchy-Schwarz on IR^{n+1}

$$\|U^n\|_h^2\le\|U^0\|_h^2+C\Delta t^2(\sum_{k=0}^{n}\|\Phi^k\|_h)^2\le\|U^0\|_h^2+C\Delta t\sum_{k=0}^{n}\|\Phi^k\|_h^2.$$

Replacing the above relation in $(*1)$ we have

$$\| \Phi^M \|_h^2 \leq \Delta t C \sum_{n=0}^{M} \| \Phi^n \|_h^2 + \Delta t C \sum_{n=0}^{M-1} [\| U^0 \|_h^2 + C \Delta t \sum_{k=0}^{n} \| \Phi^k \|_h^2] + \| \Phi^0 \|_h^2,$$

that is

$$\| \Phi^M \|_h^2 \leq \Delta t C \sum_{n=0}^{M} \| \Phi^n \|_h^2 + C(\| U^0 \|_h^2 + \| \Phi^0 \|_h^2)$$

Using the Gronwall Lemma: If $\delta^m \leq a + c\Delta t \sum_{n=0}^{m} \delta^n \quad \forall\ m \geq 1$ with $c \geq 0, a > 0, \delta^0 \leq a,$ then for $\Delta t \leq \Delta t_0 < \frac{1}{c}$,

$$\max_{0 \leq n \leq N} \delta^n \leq a e^{CT} \ with\ T \geq N \Delta t.$$

Therefore

$$\max_{0 \leq n \leq N} \| \Phi^n \|_h \leq C(\| U^0 \|_h + \| \Phi^0 \|_h), \tag{17}$$

where $C = C(T)$.

Replacing the $(*2)$ we have

$$\max_{0 \leq n \leq N} \| U^n \|_h \leq C(\| U^0 \|_h + \| \Phi^0 \|_h), \tag{18}$$

where $C = C(T)$.

The proof of the lemma is complete ((16))

2.6 Error estimation with the energy method

2.6.1 Local error estimation

We define the local errors of the method ((8)). Here $u_{ij}^n := u(x_i, y_j, t^n), \phi_{ij}^n := \phi(x_i, y_j, t^n)$.

For $0 \leq n \leq N-1$:

$$\varepsilon_{ij}^n := \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} - \Delta_h \frac{u_{ij}^{n+1} + u_{ij}^n}{2} - \frac{1}{2}[q(\phi_{ij}^{n+1}) + q(\phi_{ij}^n)] \tag{19}$$

For $1 \leq n \leq N-1$:

$$\begin{aligned}
\zeta_{ij}^n :&= \frac{\phi_{ij}^{n+1} - \phi_{ij}^n}{\Delta t} \\
&- \frac{1}{h^2}\Bigg[a\Big(\frac{\phi_{i+1,j}^{n+1} + \phi_{i,j}^{n+1}}{2}, \frac{\phi_{i+1,j}^{n+1} - \phi_{i,j}^{n+1}}{h}, \frac{(\phi_{i+1,j+1}^{n+1} + \phi_{i,j+1}^{n+1}) - (\phi_{i+1,j-1}^{n+1} + \phi_{i,j-1}^{n+1})}{4h}\Big)(\phi_{i+1,j}^{n+1} \\
&- \phi_{ij}^{n+1}) - a\Big(\frac{\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{2}, \frac{\phi_{i,j}^{n+1} - \phi_{i-1,j}^{n+1}}{h}, \frac{(\phi_{i,j+1}^{n+1} + \phi_{i-1,j+1}^{n+1}) - (\phi_{i,j-1}^{n+1} + \phi_{i-1,j-1}^{n+1})}{4h}\Big)(\phi_{ij}^{n+1} - \phi_{i-1,j}^{n+1})\Bigg] \\
&- \frac{1}{h^2}\Bigg[b\Big(\frac{\phi_{i,j+1}^{n+1} + \phi_{i,j}^{n+1}}{2}, \frac{(\phi_{i+1,j+1}^{n+1} + \phi_{i+1,j}^{n+1}) - (\phi_{i-1,j+1}^{n+1} + \phi_{i-1,j}^{n+1})}{4h}, \frac{\phi_{i,j+1}^{n+1} - \phi_{i,j}^{n+1}}{h}\Big)(\phi(_{i,j+1}^{n+1} \\
&- \phi_{ij}^{n+1}) - b\Big(\frac{\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{2}, \frac{(\phi_{i+1,j}^{n+1} + \phi_{i+1,j-1}^{n+1}) - (\phi_{i-1,j}^{n+1} + \phi_{i-1,j-1}^{n+1})}{4h}, \frac{\phi_{i,j}^{n+1} - \phi_{i,j-1}^{n+1}}{h}\Big)(\phi_{ij}^{n+1} - \phi_{i,j-1}^{n+1})\Bigg] \\
&- \frac{1}{h^2}\Bigg[g\Big(\frac{(\phi_{i+1,j}^{n+1} + \phi_{i,j}^{n+1}) + (\phi_{i+1,j-1}^{n+1} + \phi_{i,j-1}^{n+1})}{4}, \frac{(\phi_{i+1,j}^{n+1} + \phi_{i+1,j-1}^{n+1}) - (\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1})}{2h}, \frac{(\phi_{i+1,j}^{n+1} + \phi_{i,j}^{n+1}) - (\phi_{i+1,j-1}^{n+1} + \phi_{i,j-1}^{n+1})}{2h}\Big)(\phi_{i+1,j}^{n+1} \\
&- \phi_{i+1,j-1}^{n+1}) - g\left(\frac{(\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}) + (\phi_{i,j-1}^{n+1} + \phi_{i-1,j-1}^{n+1})}{4}, \frac{(\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}) - (\phi_{i-1,j}^{n+1} + \phi_{i-1,j-1}^{n+1})}{2h}, \frac{(\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}) - (\phi_{i,j-1}^{n+1} + \phi_{i-1,j-1}^{n+1})}{2h}\right) \\
&\qquad (\phi_{ij}^{n+1} - \phi_{i,j-1}^{n+1})\Bigg] \\
&+ \frac{1}{h^2}\Bigg[g\Big(\frac{(\phi_{i,j+1}^{n+1} + \phi_{i-1,j+1}^{n+1}) + (\phi_{ij}^{n+1} + \phi_{i-1,j}^{n+1})}{4}, \frac{(\phi_{i,j+1}^{n+1} + \phi_{i,j}^{n+1}) - (\phi_{i-1,j+1}^{n+1} + \phi_{i-1,j}^{n+1})}{2h}, \frac{(\phi_{i,j+1}^{n+1} + \phi_{i-1,j+1}^{n+1}) - (\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1})}{2h}\Big)(\phi_{i,j+1}^{n+1} \\
&- \phi_{i-1,j+1}^{n+1}) - g\left(\frac{(\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}) + (\phi_{i,j-1}^{n+1} + \phi_{i-1,j-1}^{n+1})}{4}, \frac{(\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}) - (\phi_{i-1,j}^{n+1} + \phi_{i-1,j-1}^{n+1})}{2h}, \frac{(\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}) - (\phi_{i,j-1}^{n+1} + \phi_{i-1,j-1}^{n+1})}{2h}\right) \\
&\qquad (\phi_{ij}^{n+1} - \phi_{i-1,j}^{n+1})\Bigg] \\
&- f(\phi_{ij}^n, u_{ij}^n).
\end{aligned} \tag{20}$$

Define

$$A_{i,j}^{n+1} := \left[\phi_{i+1,j}^{n+1} - \phi_{ij}^{n+1}\right], (*\,A)$$

$$B_{i,j}^{n+1} := \left[\phi_{i,j+1}^{n+1} - \phi_{ij}^{n+1}\right], (*\,B)$$

$$D_{ij}^{n+1}(A) := \frac{a\left(\frac{\phi_{i+1,j}^{n+1}+\phi_{i,j}^{n+1}}{2}, \frac{\phi_{i+1,j}^{n+1}-\phi_{i,j}^{n+1}}{h}, \frac{(\phi_{i+1,j+1}^{n+1}+\phi_{i,j+1}^{n+1})-(\phi_{i+1,j-1}^{n+1}+\phi_{i,j-1}^{n+1})}{4h}\right)A_{i,j}^{n+1}}{h^2}$$

$$-\frac{a\left(\frac{\phi_{i,j}^{n+1}+\phi_{i-1,j}^{n+1}}{2}, \frac{\phi_{i,j}^{n+1}-\phi_{i-1,j}^{n+1}}{h}, \frac{(\phi_{i,j+1}^{n+1}+\phi_{i-1,j+1}^{n+1})-(\phi_{i,j-1}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h}\right)A_{i-1,j}^{n+1}}{h^2}, (*\,DA)$$

$$E_{ij}^{n+1}(B) := \frac{\frac{\phi_{i,j+1}^{n+1}+\phi_{i,j}^{n+1}}{2}, \frac{(\phi_{i+1,j+1}^{n+1}+\phi_{i+1,j}^{n+1})-(\phi_{i-1,j+1}^{n+1}+\phi_{i-1,j}^{n+1})}{4h}, \frac{\phi_{i,j+1}^{n+1}-\phi_{i,j}^{n+1}}{h})B_{i,j}^{n+1}}{h^2}$$

$$-\frac{b\left(\frac{\phi_{i,j}^{n+1}+\phi_{i,j-1}^{n+1}}{2}, \frac{(\phi_{i+1,j}^{n+1}+\phi_{i+1,j-1}^{n+1})-(\phi_{i-1,j}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h}, \frac{\phi_{i,j}^{n+1}-\phi_{i,j-1}^{n+1}}{h}\right)B_{i,j-1}^{n+1}}{h^2}. (*\,EB)$$

$$G_{ij}^{n+1} = \frac{g\left(\frac{(\phi_{i+1,j}^{n+1}+\phi_{i,j}^{n+1})+(\phi_{i+1,j-1}^{n+1}+\phi_{i,j-1}^{n+1})}{4}, \frac{(\phi_{i+1,j}^{n+1}+\phi_{i+1,j-1}^{n+1})-(\phi_{i,j}^{n+1}+\phi_{i,j-1}^{n+1})}{2h}, \frac{(\phi_{i+1,j}^{n+1}+\phi_{i,j}^{n+1})-(\phi_{i+1,j-1}^{n+1}+\phi_{i,j-1}^{n+1})}{2h}\right)(\phi_{i+1,j}^{n+1}-\phi_{i+1,j-1}^{n+1})}{h^2}$$

$$-\frac{g\left(\frac{(\phi_{i,j}^{n+1}+\phi_{i-1,j}^{n+1})+(\phi_{i,j-1}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4}, \frac{(\phi_{i,j}^{n+1}+\phi_{i,j-1}^{n+1})-(\phi_{i-1,j}^{n+1}+\phi_{i-1,j-1}^{n+1})}{2h}, \frac{(\phi_{i,j}^{n+1}+\phi_{i-1,j}^{n+1})-(\phi_{i,j-1}^{n+1}+\phi_{i-1,j-1}^{n+1})}{2h}\right)(\phi_{ij}^{n+1}-\phi_{i,j-1}^{n+1})}{h^2} (*\,G)$$

The following local error estimate then applies ("consequence" of the method).

Lemma 2.6 *Let* $(u,\phi)(u,\phi\ quite\ smooth)$ *the system solution* ((8)). *Then there are constant* C_ε, C_ζ *independent the* $\Delta t, h$ *and* u,ϕ, *such that*

$$\max_{\substack{0\le n\le N\\(x_i,y_j)\in\Omega_h}} |\varepsilon_{ij}^n| \le C_\varepsilon(\Delta t^2 + h^2) \tag{21}$$

$$\max_{\substack{0\le n\le N\\(x_i,y_j)\in\Omega_h}} |\zeta_{ij}^n| \le C_\zeta(\Delta t + h) \tag{22}$$

Proof.

For a fairly smooth function $v(x,y,t)$ we have

$$v(x_i,y_j,t^n) = v(x_i,y_j,t^{n+1}) - \Delta t v_t(x_i,y_j,t^{n+1}) + \frac{\Delta t^2}{2} v_{tt}(x_i,y_j,\tau_1^n)(*\,1)$$

where $\tau_1^n \in [t^n, t^{n+1}]$, depends on i,j,n *and* v.

that is

$$v(x_i,y_j,t^{n+1}) - v(x_i,y_j,t^n) = \Delta t v_t(x_i,y_j,t^{n+1}) - \frac{\Delta t^2}{2} v_{tt}(x_i,y_j,\tau_1^n)(*\,2)$$

Moreover,

$$v(x_{i+1},y_j,t) = v(x_i,y_j,t) + h v_x(x_i,y_j,t) + \frac{h^2}{2} v_{xx}(x_i,y_j,t) + \frac{h^3}{6} v_{xxx}(x_i,y_j,t) + \frac{h^4}{24} v_{xxxx}(\xi_1,y_j,t)(*\,3)$$

where $\xi_1^n \in [x_i, x_{i+1}]$, depends on i,j,t *and* v.

Still,

$$v(x_{i-1},y_j,t) = v(x_i,y_j,t) - h v_x(x_i,y_j,t) + \frac{h^2}{2} v_{xx}(x_i,y_j,t) - \frac{h^3}{6} v_{xxx}(x_i,y_j,t) + \frac{h^4}{24} v_{xxxx}(\xi_2,y_j,t)(*\,4)$$

where $\xi_2^n \in [x_{i-1}, x_i]$, depends on i,j,t *and* v.

Subtracting it $(*\,4)$ from $(*\,3)$ we have

$$v(x_{i+1},y_j,t) - v(x_{i-1},y_j,t) = 2h v_x(x_i,y_j,t) + O(h^3)(*\,5)$$

Respectively for y we have

$$v(x_i,y_{j+1},t) - v(x_i,y_{j-1},t) = 2h v_y(x_i,y_j,t) + O(h^3)(*\,6)$$

Also

$$v(x_{i+1},y_j,t)=v(x_{i+\frac{1}{2}},y_j,t)+\frac{h}{2}v_x(x_{i+\frac{1}{2}},y_j,t)+\frac{h^2}{8}v_{xx}(x_{i+\frac{1}{2}},y_j,t)$$
$$+\frac{h^3}{48}v_{xxx}\left(x_{i+\frac{1}{2}},y_j,t\right)+\frac{h^4}{384}v_{xxxx}(\xi_3,y_j,t)(*7)$$

where $\xi_3^n\in[x_{i+\frac{1}{2}},x_{i+1}]$, depends on $i,j,t\ and\ v.$

$$v(x_i,y_j,t)=v(x_{i+\frac{1}{2}},y_j,t)-\frac{h}{2}v_x(x_{i+\frac{1}{2}},y_j,t)+\frac{h^2}{8}v_{xx}(x_{i+\frac{1}{2}},y_j,t)$$
$$-\frac{h^3}{48}v_{xxx}(x_{i+\frac{1}{2}},y_j,t)+\frac{h^4}{384}v_{xxxx}(\xi_4,y_j,t)(*8)$$

where $\xi_4^n\in[x_i,x_{i+\frac{1}{2}}]$, depends on $i,j,t\ and\ v.$

Combining $(*7)$ and $(*8)$ we have

$$v(x_{i+1},y_j,t)+v(x_i,y_j,t)=2v(x_{i+\frac{1}{2}},y_j,t)++\frac{h^2}{4}v_{xx}\left(x_{i+\frac{1}{2}},y_j,t\right)+\frac{h^4}{384}\left[v_{xxxx}(\xi_3,y_j,t)+v_{xxxx}(\xi_4,y_j,t)\right](*9)$$

and

$$v(x_i,y_j,t)+v(x_{i-1},y_j,t)=2v(x_{i-\frac{1}{2}},y_j,t)++\frac{h^2}{4}v_{xx}\left(x_{i-\frac{1}{2}},y_j,t\right)+\frac{h^4}{384}\left[v_{xxxx}(\xi_3,y_j,t)+v_{xxxx}(\xi_4,y_j,t)\right](*10)$$

Subtracting $(*8)$ from $(*7)$ we have

$$v(x_{i+1},y_j,t)-v(x_i,y_j,t)=hv_x\left(x_{i+\frac{1}{2}},y_j,t\right)+\frac{h^3}{24}v_{xxx}\left(x_{i+\frac{1}{2}},y_j,t\right)+O(h^5)\partial_x^5v \qquad (*11)$$

and

$$v(x_i,y_j,t)-v(x_{i-1},y_j,t)=hv_x\left(x_{i-\frac{1}{2}},y_j,t\right)+\frac{h^3}{24}v_{xxx}\left(x_{i-\frac{1}{2}},y_j,t\right)+O(h^5)\partial_x^5v \qquad (*12)$$

Directly from $(*A)$, we have

$$A_{i,j}^{n+1}:=\left[\phi(x_{i+1},y_j,t^{n+1})-\phi(x_i,y_j,t^{n+1})\right]=h\phi_x(x_{i+\frac{1}{2}},y_j,t^{n+1})+O(h^3),(A)$$

Similar to $(*B)$, we get

$$B_{i,j}^{n+1}:=\left[\phi(x_i,y_{j+1},t^{n+1})-\phi(x_i,y_j,t^{n+1})\right]=h\phi_y\left(x_i,y_{j+\frac{1}{2}},t^{n+1}\right)+O(h^3)(B)$$

Still

$$v\left(x_{i+\frac{1}{2}},y_j,t\right)-v\left(x_{i-\frac{1}{2}},y_j,t\right)=hv_x(x_i,y_j,t)+\frac{h^3}{24}v_{xxx}(x_i,y_j,t)+O(h^5)\partial_x^5v \qquad (*13)$$

From the $(*6)$ we have

$$\frac{v(x_{i+\frac{1}{2}},y_{j+1},t)-v(x_{i+\frac{1}{2}},y_{j-1},t)}{2h}=v_y(x_{i+\frac{1}{2}},y_j,t)+O(h^2)$$

Replacing by (*9) we have

$$\frac{\frac{v(x_{i+1},y_{j+1},t)+v(x_i,y_{j+1},t)}{2}-\frac{v(x_{i+1},y_{j-1},t)+v(x_i,y_{j-1},t)}{2}}{2h}=v_y\left(x_{i+\frac{1}{2}},y_j,t\right)+O(h^2)(*14)$$

Now, we assume that the function $a(\phi,\phi_x,\phi_y)$ is smooth for $x,y\in\Omega,\ t\in[0,T],\ \phi,\phi_x,\phi_y\in$ IR. Then

$$a(u,u_x,u_y)=a(v,v_x,v_y)+O((u-v))+O((u_x-v_x))+O\left((u_y-v_y)\right). \qquad (*15)$$

Therefore from $(*9),(*11),(*14)$,

$$a\left(\frac{\phi_{i+1,j}^{n+1}+\phi_{i,j}^{n+1}}{2},\frac{\phi_{i+1,j}^{n+1}-\phi_{i,j}^{n+1}}{h},\frac{(\phi_{i+1,j+1}^{n+1}+\phi_{i,j+1}^{n+1})-(\phi_{i+1,j-1}^{n+1}+\phi_{i,j-1}^{n+1})}{4h}\right)=$$
$$=a(\phi(x_{i+\frac{1}{2}},y_j,t^{n+1}),\phi_x(x_{i+\frac{1}{2}},y_j,t^{n+1}),\phi_y(x_{i+\frac{1}{2}},y_j,t^{n+1}))+O(h^2)$$

Replacing $(*DA)$ with (A) and above, we have

$$D_{ij}^{n+1}(A):=\frac{a\left(\frac{\phi_{i+1,j}^{n+1}+\phi_{i,j}^{n+1}}{2},\frac{\phi_{i+1,j}^{n+1}-\phi_{i,j}^{n+1}}{h},\frac{(\phi_{i+1,j+1}^{n+1}+\phi_{i,j+1}^{n+1})-(\phi_{i+1,j-1}^{n+1}+\phi_{i,j-1}^{n+1})}{4h}\right)A_{i,j}^{n+1}}{h^2}$$
$$-\frac{a\left(\frac{\phi_{i,j}^{n+1}+\phi_{i-1,j}^{n+1}}{2},\frac{\phi_{i,j}^{n+1}-\phi_{i-1,j}^{n+1}}{h},\frac{(\phi_{i,j+1}^{n+1}+\phi_{i-1,j+1}^{n+1})-(\phi_{i,j-1}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h}\right)A_{i-1,j}^{n+1}}{h^2}=$$

$$= [a(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_x(x_i,y_j,t^{n+1})]_x + O(h^2).$$

Therefore

$$D_{ij}^{n+1}(A) = [a(x_i,y_j,t^{n+1},\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_x(x_i,y_j,t^{n+1})]_x + O(h^2). (DA)$$

Similar to above

$$E_{ij}^{n+1}(B) = \frac{b(\frac{\phi_{i,j+1}^{n+1}+\phi_{i,j}^{n+1}}{2},\frac{(\phi_{i+1,j+1}^{n+1}+\phi_{i+1,j}^{n+1})-(\phi_{i-1,j+1}^{n+1}+\phi_{i-1,j}^{n+1})}{4h},\frac{\phi_{i,j+1}^{n+1}-\phi_{i,j}^{n+1}}{h})B_{i,j}^{n+1}}{h^2}$$

$$-\frac{b(\frac{\phi_{i,j}^{n+1}+\phi_{i,j-1}^{n+1}}{2},\frac{(\phi_{i+1,j}^{n+1}+\phi_{i+1,j-1}^{n+1})-(\phi_{i-1,j}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h},\frac{\phi_{i,j}^{n+1}-\phi_{i,j-1}^{n+1}}{h})B_{i,j-1}^{n+1}}{h^2} =$$

$$= [b(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_y(x_i,y_j,t^{n+1})]_y + O(h^2).$$

$$E_{ij}^{n+1}(B) = [b(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_y(x_i,y_j,t^{n+1})]_y + O(h^2). (EB)$$

Similarly replacing $(*G)$ from $(*4),(*15)$ that is

$$G_{ij}^{n+1} = [g(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_y(x_i,y_j,t^{n+1})]_x + O(h). (G)$$

Still

$$f(\phi(x_i,y_j,t^n),u(x_i,y_j,t^n)$$
$$= f(\phi(x_i,y_j,t^{n+1}),u(x_i,y_j,t^{n+1})) + O(\phi(x_i,y_j,t^{n+1}) - \phi(x_i,y_j,t^n)) + O(u(x_i,y_j,t^{n+1}) - u(x_i,y_j,t^n))$$

that is

$$f(\phi(x_i,y_j,t^n),u(x_i,y_j,t^n) = f(\phi(x_i,y_j,t^{n+1}),u(x_i,y_j,t^{n+1})) + O(\Delta t)(f)$$

Since ((20)) and from $(*2),(DA),(EB)$ and $(G),(f)$ we get for $0 \le n \le N-1$:

$$|\zeta_{ij}^n| = \Big|\phi_t(x_i,y_j,t^{n+1}) + O(\Delta t) +$$
$$-[a(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_x(x_i,y_j,t^{n+1})]_x + O(h^2)$$
$$-[b(t^{n+1},\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_y(x_i,y_j,t^{n+1})]_y + O(h^2)$$
$$+[g(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_y(x_i,y_j,t^{n+1})]_x + O(h)$$
$$-[g(\phi(x_i,y_j,t^{n+1}),\phi_x(x_i,y_j,t^{n+1}),\phi_y(x_i,y_j,t^{n+1}))\phi_x(x_i,y_j,t^{n+1})]_y + O(h)$$
$$-f(\phi(x_i,y_j,t^{n+1}),u(x_i,y_j,t^{n+1})) + O(\Delta t)\Big| \le C_\zeta(\Delta t + h)$$

where using ((8)).Therefore we have

$$\max_{\substack{1\le n\le N\\(x_i,y_j)\in\Omega_h}} |\zeta_{ij}^n| \le C_\zeta(\Delta t + h) \qquad (23)$$

Also similar to ((19)) we have for $0 \le n \le N-1$

$$\varepsilon_{ij}^n := \frac{u(x_i,y_j,t^{n+1}) - u(x_i,y_j,t^n)}{\Delta t} - \Delta_h\frac{u(x_i,y_j,t^{n+1}) + u(x_i,y_j,t^n)}{2} - \frac{q(\phi(x_i,y_j,t^{n+1})) + q(\phi(x_i,y_j,t^n))}{2}$$

So from $(*19),(*29)$ and $(*33)$ we get for $0 \le n \le N-1$:

$$|\varepsilon_{ij}^n| = \Bigg|\frac{u(x_i,y_j,t^{n+1}) - u(x_i,y_j,t^n)}{\Delta t} - \Delta_h\left(\frac{u(x_i,y_j,t^{n+1}) + u(x_i,y_j,t^n)}{2}\right)$$
$$- \frac{q(\phi(x_i,y_j,t^{n+1})) + q(\phi(x_i,y_j,t^n))}{2}\Bigg|$$

$$= \Big|u_t(x_i,y_j,t^{n+\frac{1}{2}}) + O(\Delta t^2) - \Delta u(x_i,y_j,t^{n+\frac{1}{2}}) + O(\Delta t^2 + h^2) - q(\phi(x_i,y_j,t^{n+\frac{1}{2}})) + O(\Delta t^2)\Big| \le C_\varepsilon(\Delta t^2 + h^2) (24)$$

where using the ((8)). Therefore, we have the desired

$$\max_{\substack{0\le n\le N\\(x_i,y_j)\in\Omega_h}} |\varepsilon_{ij}^n| \le C_\varepsilon(\Delta t^2 + h^2)$$

2.7 Error estimate

Theorem 2.7 *Let that be the solution u,ϕ the problem of ((2))) is sufficient smooth. Let U^n,Φ^n the solution distinct problem ((8)). Let $\frac{\Delta t}{h} \le a$, where a small enough and also that $\| z^1 \|_0 = O(\Delta t^2 + h^2)$, where $z^1 = \phi^1 - \Phi^1$. Then exists const C, C' independent the Δt and h such that*

$$\max_{0\le n\le N} \| u^n - U^n \|_h \le C(\Delta t + h), \qquad (25)$$

$$\max_{0\le n\le N} |\phi^n - \Phi^n|_{1,h} \le C'(\Delta t + h), \qquad (26)$$

Proof.

Let $e^n := u^n - U^n, z^n := \phi^n - \Phi^n$. Then $e^n, z^n \in X_h^0$.

From relationships ((20)) and ((8)) we have, subtracting for $1 \le n \le N-1$:

$$\left.\begin{aligned}
&\frac{z_{ij}^{n+1}-z_{ij}^n}{\Delta t}\\
&-\delta_x\Big(a\Big(\frac{\phi_{i,j}^{n+1}+\phi_{i-1,j}^{n+1}}{2},\frac{\phi_{i,j}^{n+1}-\phi_{i-1,j}^{n+1}}{h},\frac{(\phi_{i,j+1}^{n+1}+\phi_{i-1,j+1}^{n+1})-(\phi_{i,j-1}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h}\Big)\delta_{\bar{x}}\phi_{ij}^{n+1}\Big)\\
&-\delta_y\Big(b\Big(\frac{\phi_{i,j}^{n+1}+\phi_{i,j-1}^{n+1}}{2},\frac{(\phi_{i+1,j}^{n+1}+\phi_{i+1,j-1}^{n+1})-(\phi_{i-1,j}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h},\frac{\phi_{i,j}^{n+1}-\phi_{i,j-1}^{n+1}}{h}\Big)\delta_{\bar{y}}\phi_{ij}^{n+1}\Big)\\
&+\delta_x\Big(g\Big(\frac{(\phi_{i,j}^n+\phi_{i-1,j}^n)+(\phi_{i,j-1}^n+\phi_{i-1,j-1}^n)}{4},\frac{(\phi_{i,j}^n+\phi_{i,j-1}^n)-(\phi_{i-1,j}^n+\phi_{i-1,j-1}^n)}{2h},\frac{(\phi_{i,j}^n+\phi_{i-1,j}^n)-(\phi_{i,j-1}^n+\phi_{i-1,j-1}^n)}{2h}\Big)\delta_{\bar{y}}\phi_{ij}^{n+1}\Big)\\
&-\delta_y\Big(g\Big(\frac{(\phi_{i,j}^n+\phi_{i-1,j}^n)+(\phi_{i,j-1}^n+\phi_{i-1,j-1}^n)}{4},\frac{(\phi_{i,j}^n+\phi_{i,j-1}^n)-(\phi_{i-1,j}^n+\phi_{i-1,j-1}^n)}{2h},\frac{(\phi_{i,j}^n+\phi_{i-1,j}^n)-(\phi_{i,j-1}^n+\phi_{i-1,j-1}^n)}{2h}\Big)\delta_{\bar{x}}\phi_{ij}^{n+1}\Big)\\
&+\delta_x\Big(a\Big(\frac{\Phi_{i,j}^{n+1}+\Phi_{i-1,j}^{n+1}}{2},\frac{\Phi_{i,j}^{n+1}-\Phi_{i-1,j}^{n+1}}{h},\frac{(\Phi_{i,j+1}^{n+1}+\Phi_{i-1,j+1}^{n+1})-(\Phi_{i,j-1}^{n+1}+\Phi_{i-1,j-1}^{n+1})}{4h}\Big)\delta_{\bar{x}}\Phi_{ij}^{n+1}\Big)\\
&+\delta_y\Big(b\Big(\frac{\Phi_{i,j}^{n+1}+\Phi_{i,j-1}^{n+1}}{2},\frac{(\Phi_{i+1,j}^{n+1}+\Phi_{i+1,j-1}^{n+1})-(\Phi_{i-1,j}^{n+1}+\Phi_{i-1,j-1}^{n+1})}{4h},\frac{\Phi_{i,j}^{n+1}-\Phi_{i,j-1}^{n+1}}{h}\Big)\delta_{\bar{y}}\Phi_{ij}^{n+1}\Big)\\
&-\delta_x\Big(g\Big(\frac{(\Phi_{i,j}^n+\Phi_{i-1,j}^n)+(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{4},\frac{(\Phi_{i,j}^n+\Phi_{i,j-1}^n)-(\Phi_{i-1,j}^n+\Phi_{i-1,j-1}^n)}{2h},\frac{(\Phi_{i,j}^n+\Phi_{i-1,j}^n)-(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{2h}\Big)\delta_{\bar{y}}\Phi_{ij}^{n+1}\Big)\\
&+\delta_y\Big(g\Big(\frac{(\Phi_{i,j}^n+\Phi_{i-1,j}^n)+(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{4},\frac{(\Phi_{i,j}^n+\Phi_{i,j-1}^n)-(\Phi_{i-1,j}^n+\Phi_{i-1,j-1}^n)}{2h},\frac{(\Phi_{i,j}^n+\Phi_{i-1,j}^n)-(\Phi_{i,j-1}^n+\Phi_{i-1,j-1}^n)}{2h}\Big)\delta_{\bar{x}}\Phi_{ij}^{n+1}\Big)\\
&= f(\phi_{ij}^n,u_{ij}^n)-f(\Phi_{ij}^n,U_{ij}^n)+\zeta_{ij}^n \quad i,j=1,\dots,J
\end{aligned}\right\} \tag{27}$$

We define, for $n \ge 1$ (similarly defined and the corresponding quantities with Φ)

$\phi_{i,j}^n := \phi(x_i, y_j, t^n)$,

$$a_{i-\frac{1}{2},j}^{n+1}(\phi) := a\Big(\frac{\phi_{i,j}^{n+1}+\phi_{i-1,j}^{n+1}}{2},\frac{\phi_{i,j}^{n+1}-\phi_{i-1,j}^{n+1}}{h},\frac{(\phi_{i,j+1}^{n+1}+\phi_{i-1,j+1}^{n+1})-(\phi_{i,j-1}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h}\Big), \qquad i=1,\dots,J+1,$$
$$j=0,\dots,J+1.$$

$$b_{i,j-\frac{1}{2}}^{n+1}(\phi) := b\Big(\frac{\phi_{i,j}^{n+1}+\phi_{i,j-1}^{n+1}}{2},\frac{(\phi_{i+1,j}^{n+1}+\phi_{i+1,j-1}^{n+1})-(\phi_{i-1,j}^{n+1}+\phi_{i-1,j-1}^{n+1})}{4h},\frac{\phi_{i,j}^{n+1}-\phi_{i,j-1}^{n+1}}{h}\Big) \; i=0,\dots,J+1, \qquad j=1,\dots,J+1.$$

$$g_{i-\frac{1}{2},j-\frac{1}{2}}^{n+1}(\phi) = g\Big(\frac{(\phi_{i,j}^n+\phi_{i-1,j}^n)+(\phi_{i,j-1}^n+\phi_{i-1,j-1}^n)}{4},\frac{(\phi_{i,j}^n+\phi_{i,j-1}^n)-(\phi_{i-1,j}^n+\phi_{i-1,j-1}^n)}{2h},\frac{(\phi_{i,j}^n+\phi_{i-1,j}^n)-(\phi_{i,j-1}^n+\phi_{i-1,j-1}^n)}{2h}\Big) \; i=0,\dots,J+1, \qquad j=0,\dots,J+1.$$

Taking it ℓ^2 distinct internal product $(\cdot,\cdot)_h$ and of its two members ((27)), with the function z^{n+1} we have for $n \ge 2$

$$\begin{aligned}
&(z^{n+1}-z^n,z^{n+1})_h -\\
&-\Delta t\big(\delta_x(a^{n+1}(\phi)\delta_{\bar{x}}\phi^{n+1})+\delta_y\big(b^{n+1}(\phi)\delta_{\bar{y}}\phi^{n+1}\big),z^{n+1}\big)_h\\
&+\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}\Phi^{n+1})+\delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}\Phi^{n+1}\big),z^{n+1}\big)_h\\
&+\Delta t\big(\delta_x\big(g^{n+1}(\phi)\delta_{\bar{y}}\phi^{n+1}\big)-\delta_y(g^{n+1}(\phi)\delta_{\bar{x}}\phi^{n+1}),z^{n+1}\big)_h\\
&-\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}\Phi^{n+1}\big)-\delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}\Phi^{n+1}),z^{n+1}\big)_h\\
&=\Delta t(f(\phi^n,u^n)-f(\Phi^n,U^n),z^{n+1})_h+\Delta t(\zeta^n,z^{n+1})_h
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\| z^{n+1} \|_h^2 - (z^{n+1}, z^n)_h - \\
&-\Delta t\big(\delta_x(a^{n+1}(\phi)\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big(b^{n+1}(\phi)\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&-\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}\Phi^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}\Phi^{n+1}\big), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x\big(g^{n+1}(\phi)\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y(g^{n+1}(\phi)\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h \\
&-\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h \\
&-\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}\Phi^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}\Phi^{n+1}), z^{n+1}\big)_h \\
&= \Delta t(f(\phi^n, u^n) - f(\Phi^n, U^n), z^{n+1})_h + \Delta t(\zeta^n, z^{n+1})_h
\end{aligned}
$$

$$
\begin{aligned}
&\| z^{n+1} \|_h^2 - (z^{n+1}, z^n)_h - \\
&-\Delta t\big(\delta_x(a^{n+1}(\phi)\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big(b^{n+1}(\phi)\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&-\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}z^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}z^{n+1}\big), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x\big(g^{n+1}(\phi)\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y(g^{n+1}(\phi)\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h \\
&-\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}z^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}z^{n+1}), z^{n+1}\big)_h \\
&= \Delta t(f(\phi^n, u^n) - f(\Phi^n, U^n), z^{n+1})_h + \Delta t(\zeta^n, z^{n+1})_h
\end{aligned}
$$

$$
\left.
\begin{aligned}
&\| z^{n+1} \|_h^2 - (z^{n+1}, z^n)_h - \\
&-\Delta t\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}z^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}z^{n+1}\big), z^{n+1}\big)_h \\
&+\Delta t\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}z^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}z^{n+1}), z^{n+1}\big)_h \\
&= \Delta t(f(\phi^n, u^n) - f(\Phi^n, U^n), z^{n+1})_h + \Delta t(\zeta^n, z^{n+1})_h \\
&+\Delta t\big(\delta_x([a^{n+1}(\phi) - a^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big([b^{n+1}(\phi) - b^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&-\Delta t\big(\delta_x\big([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h
\end{aligned}
\right\} \quad (28)
$$

From the Lemma 2.3

$$
\begin{aligned}
&-\big(\delta_x(a^{n+1}(\Phi)\delta_{\bar{x}}z^{n+1}) + \delta_y\big(b^{n+1}(\Phi)\delta_{\bar{y}}z^{n+1}\big), z^{n+1}\big)_h \\
&+\big(\delta_x\big(g^{n+1}(\Phi)\delta_{\bar{y}}z^{n+1}\big) - \delta_y(g^{n+1}(\Phi)\delta_{\bar{x}}z^{n+1}), z^{n+1}\big)_h \\
&= \frac{h^2}{2}\sum_{i=1}^{J+1}\sum_{j=1}^{J+1}\Big(a^{n+1}_{i-\frac{1}{2},j}(\Phi^{n+1})\big(\delta_{\bar{x}}z^{n+1}_{ij}\big)^2 + b^{n+1}_{i,j-\frac{1}{2}}(\Phi^{n+1})\big(\delta_{\bar{y}}z^{n+1}_{ij}\big)^2\Big) \geq C|z^{n+1}|^2_{1,h}
\end{aligned}
$$

where $C = c_a^0 + c_b^0$.

We also define the following quantity

$$
\begin{aligned}
I := &\big(\delta_x([a^{n+1}(\phi) - a^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big([b^{n+1}(\phi) - b^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
&-\big(\delta_x\big([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h
\end{aligned}
$$

or

$$
\begin{aligned}
I_1 &:= \big(\delta_x([a^{n+1}(\phi) - a^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}) + \delta_y\big([b^{n+1}(\phi) - b^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big), z^{n+1}\big)_h \\
I_2 &:= -\big(\delta_x\big([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h
\end{aligned}
$$

Simplifying it I_2 and then using that $z = \phi - \Phi$ we have

$$
\begin{aligned}
I_2 &:= -\big(\delta_x\big([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\big) - \delta_y([g^{n+1}(\phi) - g^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}), z^{n+1}\big)_h \\
&= h^2\sum_{i=1}^{J}\sum_{j=1}^{J}[g^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}(\phi - z) - g^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}(\phi)][\delta_{\bar{y}}z^{n+1}_{i,j}\delta_{\bar{x}}\phi^{n+1}_{i,j} - \delta_{\bar{x}}z^{n+1}_{i,j}\delta_{\bar{y}}\phi^{n+1}_{i,j}]
\end{aligned}
$$

Using the conditions Lipschitz and ϕ smooth in $\overline{\Omega}$ we have

$$|I_2| \le h^2 \sum_{i=1}^{J} \sum_{j=1}^{J} |g_{i-\frac{1}{2},j-\frac{1}{2}}^{n+1}(\phi - z) - g_{i-\frac{1}{2},j-\frac{1}{2}}^{n+1}(\phi)|(|\delta_{\bar{y}} z_{i,j}^{n+1}| + |\delta_{\bar{x}} z_{i,j}^{n+1}|)$$

$$|I_2| \le C' L_{g,\nabla} |z|_{1,h}^2$$

Similar for I_1 we have

$$I_1 := \left(\delta_x([a^{n+1}(\phi) - a^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}) + \delta_y\left([b^{n+1}(\phi) - b^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}\right), z^{n+1}\right)_h$$

$$= ([a^{n+1}(\phi) - a^{n+1}(\Phi)]\delta_{\bar{x}}\phi^{n+1}, \delta_{\bar{x}} z^{n+1})_h + \left([b^{n+1}(\phi) - b^{n+1}(\Phi)]\delta_{\bar{y}}\phi^{n+1}, \delta_y z^{n+1}\right)_h$$

$$= ([a^{n+1}(\phi) - a^{n+1}(\phi - z)]\delta_{\bar{x}}\phi^{n+1}, \delta_{\bar{x}} z^{n+1})_h + \left([b^{n+1}(\phi) - b^{n+1}(\phi - z)]\delta_{\bar{y}}\phi^{n+1}, \delta_y z^{n+1}\right)_h$$

Using the conditions Lipschitz and ϕ smooth in$\overline{\Omega}$ we have

$$|I_1| \le C''(L_{a,\nabla} + L_{b,\nabla})|z|_{1,h}^2$$

Finally

$$\Delta t|I| \le \Delta t C_*(L_{a,\nabla} + L_{b,\nabla} + L_{g,\nabla})|z|_{1,h}^2$$

Replacing the ((28)) the above we have

$$\| z^{n+1} \|_h^2 - (z^{n+1}, z^n)_h + \Delta t C |z^{n+1}|_{1,h}^2 \le$$
$$\le \Delta t(f(\phi^n, u^n) - f(\Phi^n, U^n), z^{n+1})_h + \Delta t(\zeta^n, z^{n+1})_h + C_* \Delta t(L_{a,\nabla} + L_{b,\nabla} + L_{g,\nabla})|z^{n+1}|_{1,h}^2$$

$$\| z^{n+1} \|_h^2 - (z^{n+1}, z^n)_h + \Delta t(C - C_*(L_{a,\nabla} + L_{b,\nabla} + L_{g,\nabla}))|z^{n+1}|_h^2$$
$$\le \Delta t(f(\phi^n, u^n) - f(\Phi^n, U^n), z^{n+1})_h + \Delta t(\zeta^n, z^{n+1})_h$$

Assuming that $C = c_a^0 + c_b^0 \ge C_*(L_{a,\nabla} + L_{b,\nabla} + L_{g,\nabla})$ we get

$$\| z^{n+1} \|_h^2 \le c\Delta t(\| z^n \|_h + \| e^n \|_h) \| z^{n+1} \|_h + \Delta t \| \zeta^n \|_h \| z^{n+1} \|_h + \| z^{n+1} \|_h \| z^n \|_h$$
$$\| z^{n+1} \|_h^2 \le c\Delta t(\| z^n \|_h + \| e^n \|_h) \| z^{n+1} \|_h + \Delta t \| \zeta^n \|_h \| z^{n+1} \|_h + \| z^{n+1} \|_h \| z^n \|_h$$

Finally

$$\| z^{n+1} \|_h \le c\Delta t \| e^n \|_h + \Delta t \| \zeta^n \|_h + C'''\Delta t \| z^n \|_h \tag{29}$$

Adding by $n = 1$ to m we have

$$\sum_{n=1}^{m} \| z^{n+1} \|_h^2 \le c_1 \Delta t \sum_{n=1}^{m} \| e^n \|_h + c_2 \Delta t \sum_{n=1}^{m} \| \zeta^n \|_h + c_3 \Delta t \sum_{n=1}^{m} \| z^n \|_h$$

$$\| z^{m+1} \|_h \le c_1 \Delta t \sum_{n=1}^{m} \| e^n \|_h + c_2 \Delta t \sum_{n=1}^{m} \| \zeta^n \|_h + c_{3'} \Delta t \sum_{n=1}^{m} \| z^n \|_h$$

Using it Lemma the Gronwall: If $\delta^{m'} \le a + c\Delta t \sum_{n=0}^{m'} \delta^n \quad \forall\ m' \ge 1$ with $c \ge 0,\ a > 0, \delta^0 \le a,$ then for $\Delta t \le \Delta t_0 < \frac{1}{c}$,

$$\max_{0 \le n \le N} \delta^n \le a e^{CT} \ with\ T = N\Delta t.$$

Therefore

$$\max_{0 \le n \le m'} \| z^n \|_h \le c_{1'} \Delta t \sum_{n=0}^{m-1} \| e^n \|_h + c_{2'} \Delta t \sum_{n=0}^{m-1} \| \zeta^n \|_h$$

Finally

$$\max_{0 \le n \le m'} \| z^n \|_h \le c_{1'} \Delta t \textstyle\sum_{n=0}^{m-1} \| e^n \|_h + c_{2'}(\Delta t + h) 1 \le m \le M. \tag{30}$$

Similar with above relationships ((19)) and ((8)) we have, subtracting, for $0 \le n \le N - 1$:

$$\frac{e_{ij}^{n+1} - e_{ij}^n}{\Delta t} - \Delta_h \left(\frac{e_{ij}^{n+1} + e_{ij}^n}{2}\right) = q(\phi(x_i, y_j, t^{n+1})) - q(\Phi(x_i, y_j, t^{n+1})) + \varepsilon_{ij}^n \ \ i, j = 1, \dots, J$$

Taking it ℓ^2 distinct internal product $(\cdot,\cdot)_h$ and of the two members of the above relationship, with the function $e^{n+1} + e^n$ we have

$$(e^{n+1} - e^n, e^{n+1} + e^n)_h - \Delta t \left(\Delta_h(\frac{e^{n+1} + e^n}{2}), e^{n+1} + e^n\right)_h = \Delta t(q(\phi^{n+1}) - q(\Phi^{n+1}), e^{n+1} + e^n)_h + \Delta t(\varepsilon^n, e^{n+1} + e^n)_h \tag{31}$$

From the Lemma 2.2 we have

$$(e^{n+1} - e^n, e^{n+1} + e^n)_h + \frac{\Delta t}{2} |e^{n+1} + e^n|_{1,h}^2$$

$$\leq \Delta t(q(\phi^{n+1}) - q(\Phi^{n+1}), e^{n+1} + e^n)_h + \Delta t(\varepsilon^n, e^{n+1} + e^n)_h$$

Therefore,

$$\| e^{n+1} \|_h^2 - \| e^n \|_h^2 + \frac{\Delta t}{2} |e^{n+1} + e^n|_{1,h}^2$$
$$\leq \Delta t(q(\phi^{n+1}) - q(\Phi^{n+1}), e^{n+1} + e^n)_h + \Delta t(\varepsilon^n, e^{n+1} + e^n)_h$$

Adding by members the above relation from $n = 0$ until $m' - 1$ we have

$$\sum_{n=0}^{m'-1} (\| e^{n+1} \|_h^2 - \| e^n \|_h^2) + \frac{\Delta t}{2} \sum_{n=0}^{m'-1} |e^{n+1} + e^n|_{1,h}^2$$
$$\leq \sum_{n=0}^{m'-1} \{\Delta t(q(\phi^{n+1}) - q(\Phi^{n+1}), e^{n+1} + e^n)_h + \Delta t(\varepsilon^n, e^{n+1} + e^n)_h\}.$$

So, after the sum $\sum_{n=0}^{m'-1} (\| e^{n+1} \|_h^2 - \| e^n \|_h^2)$, we have

$$\| e^{m'} \|_h^2 - \| e^0 \|_h^2 + \frac{\Delta t}{2} \sum_{n=0}^{m'-1} |e^{n+1} + e^n|_{1,h}^2$$
$$\leq \sum_{n=0}^{m'-1} \{\Delta t(q(\phi^{n+1}) - q(\Phi^{n+1}), e^{n+1} + e^n)_h + \Delta t(\varepsilon^n, e^{n+1} + e^n)_h\}.$$

Because $e^0 = 0$ we have

$$\| e^{m'} \|_h^2 + \frac{\Delta t}{2} \sum_{n=0}^{m'-1} |e^{n+1} + e^n|_{1,h}^2 \leq \sum_{n=0}^{m'-1} \{\Delta t(q(\phi^{n+1}) - q(\Phi^{n+1}), e^{n+1} + e^n)_h + \Delta t(\varepsilon^n, e^{n+1} + e^n)_h\}.$$

$$\overset{Cauchy-Schwarz}{\leq} \Delta t \sum_{n=0}^{m'-1} \| q(\phi^{n+1}) - q(\Phi^{n+1}) \|_h \| e^{n+1} + e^n \|_h + \Delta t \sum_{n=0}^{m'-1} \| \varepsilon^n \|_h \| e^{n+1} + e^n \|_h$$

$$\overset{ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2}{\leq} \frac{\Delta t}{4\varepsilon} \sum_{n=0}^{m'-1} \| q(\phi^{n+1}) - q(\Phi^{n+1}) \|_h^2 + \varepsilon \Delta t \sum_{n=0}^{m'-1} \| e^{n+1} + e^n \|_h^2 + \frac{\Delta t}{4\varepsilon} \sum_{n=0}^{m'-1} \| \varepsilon^n \|_h^2 + \Delta t \varepsilon \sum_{n=0}^{m'-1} \| e^{n+1} + e^n \|_h^2$$

$$\overset{\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)}{\leq} \frac{\Delta t}{4\varepsilon} \sum_{n=0}^{m'-1} \| q(\phi^{n+1}) - q(\Phi^{n+1}) \|_h^2 + C_\varepsilon \Delta t \sum_{n=0}^{m'-1} \| \varepsilon^n \|_h^2 + \Delta t \varepsilon \sum_{n=0}^{m'-1} \| e^n \|_h^2$$

From (7) we have

$$\| q(\phi^{n+1}) - q(\Phi^{n+1}) \|_h \leq L_q \| z^{n+1} |_h$$

Therefore,

$$\| e^{m'} \|_h^2 \leq C \Delta t \sum_{n=0}^{m'-1} \| z^{n+1} \|_h^2 + C_\varepsilon \Delta t \sum_{n=0}^{m'-1} \| \varepsilon^n \|_h^2 + \Delta t \varepsilon \sum_{n=0}^{m'-1} \| e^n \|_h^2$$

With her help ((30)) we get

$$\| e^{m'} \|_h^2 \leq C \Delta t \sum_{n=0}^{m'-1} (\Delta t + h)^2 + C_\varepsilon \Delta t \sum_{n=0}^{m'-1} \| \varepsilon^n \|_h^2 + \Delta t C' \sum_{n=0}^{m'-1} \| e^n \|_h^2$$

Using his Gronwall Lemma : If $\delta^{m'} \leq a + c \Delta t \sum_{n=0}^{m'} \delta^n \quad \forall\ m' \geq 1$ with $c \geq 0$, $a > 0, \delta^0 \leq a$, then for $\Delta t \leq \Delta t_0 < \frac{1}{c}$,

$$\max_{0 \leq n \leq N} \delta^n \leq a e^{CT} \ with\ T = N \Delta t.$$

Therefore,

$$\max_{0 \leq n \leq m} \| e^n \|_h \leq C(\Delta t + h). \tag{32}$$

Replacing it ((32)) on ((30)) we have

$$\max_{0 \leq n \leq m} \| z^n \|_h \leq C(\Delta t + h). \tag{33}$$

3. Numerical Results

Let $T > 0$. We consider the following problem of initial and conditions. Required ϕ, u fairly smooth functions in $(x, y, t) \in \bar{\Omega} \times [0, T]$, where $\Omega = (\alpha, \beta) \times (\alpha, \beta)$, such that:

$$\begin{cases} q(\theta)\frac{\partial}{\partial t}\phi - div(A(\nabla\phi)\nabla\phi) = f(\phi,u), & on\ \overline{\Omega}\times[0,T] \\ \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial y^2}u = \frac{\partial}{\partial t}p(\phi), & on\ \overline{\Omega}\times[0,T] \end{cases} \quad (34)$$

The $\phi = \phi(x,y,t)$ is the phase function, $\theta = \arctan\frac{\phi_y}{\phi_x}$, $u = u(x,y,t)$ is a function of temperature, defined in Ω. The q, p, and f are known gradient functions with $f(\phi,u)$, $p(\phi)$, and A is 2×2 anisotropic array of dependent functions θ.

$$A(\theta) = \begin{pmatrix} r^2(\theta) & -r(\theta)r'(\theta) \\ r(\theta)r'(\theta) & r^2(\theta) \end{pmatrix}.$$

Here, $r(\theta) = 1 + \delta_\gamma\cos(k\theta)$, where δ_γ is the positive constant (less than 1) that characterizes the magnitude of the anisotropy and $k > 1$ is an integer that defines the number of dendrites. We still have $q(\theta) = (1 + \delta_\gamma\cos(k\theta))/m(1 + \delta_\mu\cos(k\theta))$, where m are stable and δ_μ is the positive constant (less than 1) that characterizes the magnitude of the motion of the anisotropy.

- If $\delta_\gamma = \delta_\mu = 0$ our model is called *isotropic*.
- If $\delta_\gamma = 0$ and $\delta_\mu \neq 0$ $(A(\theta) = I)$, then we will call our model *semi-anisotropic*.
- If $\delta_\gamma \neq 0$ and $\delta_\mu \neq 0$, then we will call our model *anisotropic*.

The above problem is completed with the given initial conditions $t = 0\phi(x,y,0) = \phi_0(x,y)$, $u(x,y,0) = u_0(x,y)$, $(x,y) \in \Omega$, and the corresponding boundary conditions or Neumann or Dirichlet for ϕ and u at the border of the $\partial\Omega$ for $t \geq 0$.

3.1 The direct Euler-ADI method

We write in the rectangle $\Omega = [\alpha,\beta]\times[\alpha,\beta]$ his general problem ((34)) for the anisotropic case: For $(x,y,t) \in \Omega\times[0,T]$ we consider the system

$$\begin{aligned} q\phi_t &= \partial_x(a\partial_x\phi) + \partial_y(b\partial_y\phi) - \partial_x(g\partial_y\phi) + \partial_y(g\partial_x\phi) + f, \\ u_t &= \Delta u + \partial_t p, \end{aligned} \quad (35)$$

where $q := q(x,y,t,\phi,\partial_x\phi,\partial_y\phi), a := a(x,y,t,\phi,\partial_x\phi,\partial_y\phi), b := b(x,y,t,\phi,\partial_x\phi,\partial_y\phi), g := g(x,y,t,\phi,\partial_x\phi,\partial_y\phi), f := f(x,y,t,\phi,u), p := p(x,y,t,\phi)$ gradient functions are known. The system ((35)) supplemented by the initial conditions

$\phi(x,y,0) = \phi_0(x,y)$, $u(x,y,0) = u_0(x,y), (x,y) \in \Omega$(36)

and, in the examples we use homogeneous Dirichlet boundary conditions at its boundary Ω:

$\phi(x,y,t) = 0$, $u(x,y,t) = 0, (x,y,t) \in \partial\Omega\times[0,T]$. (37)

We distinguish it((35))-((37)) as follows. With $h = (\beta-\alpha)/(J+1)$, $J \in \mathrm{IN}$ and $x_i = \alpha + ih, y_j = \alpha + jh, 0 \leq i,j \leq J+1$.

We are still defining $\Omega_h := \{(x_i,y_j), i,j = 1,\ldots,J\}$ and $\partial\Omega_h := \{(x_i,y_j), i = 0\ or\ i = J+1\ or\ j = 0\ or\ j = J+1$. Also, $tn=n\Delta t,n=0,\ldots,N$, where $\Delta t=T/N$, and we define $tn+12=tn+\Delta t/2,xi\pm12=xi\pm h/2,yj\pm12=yj\pm h/2$, and

$$S_h := \{U = (U_{00},\ldots,U_{J+1,J+1})^T \in \mathrm{IR}^{(J+2)\times(J+2)} : U_{ij} = 0\ on\ \partial\Omega_h\}.$$

We are approaching their solution ((35))-((37)) from a grid of functions $U^n, \Phi^n \in S_h$ using the direct Euler-ADI scheme as defined below:

$$
\begin{array}{lll}
 & \Phi_{ij}^{0} = \phi_0(x_i,y_j), U_{ij}^{0} = u_0(x_i,y_j), & (x_i,y_j) \in \Omega_h \cup \partial\Omega_h \\
 & For\ \ n = 0,1,\dots,N-1: & \\
(i) & q_{ij}^{n} \frac{\Phi_{ij}^{n+1} - \Phi_{ij}^{n}}{\Delta t} - (L_h^n \Phi)_{ij} = f_{ij}^{n}, & (x_i,y_j) \in \Omega_h \\
 & \Phi_{ij}^{n+1} = 0, & (x_i,y_j) \in \partial\Omega_h \\
(ii) & \frac{2(U_{ij}^{n+\frac{1}{2}} - U_{ij}^{n})}{\Delta t} - \delta_{x\bar{x}} U_{ij}^{n+\frac{1}{2}} - \delta_{y\bar{y}} U_{ij}^{n} = \frac{P_{ij}^{n+1} - P_{ij}^{n}}{\Delta t}, & (x_i,y_j) \in \Omega_h \\
(iii) & \frac{2(U_{ij}^{n+1} - U_{ij}^{n+\frac{1}{2}})}{\Delta t} - \delta_{x\bar{x}} U_{ij}^{n+\frac{1}{2}} - \delta_{y\bar{y}} U_{ij}^{n+1} = \frac{P_{ij}^{n+1} - P_{ij}^{n}}{\Delta t}, & (x_i,y_j) \in \Omega_h \\
 & U_{ij}^{n+\frac{1}{2}} = U_{ij}^{n+1} = 0, & (x_i,y_j) \in \partial\Omega_h
\end{array}
\tag{38}
$$

where $(L_h^n v)_{i,j} := [\delta_x(a_{i-\frac{1}{2},j}^n \delta_{\bar{x}} v_{ij}^n) + \delta_y(b_{i,j-\frac{1}{2}}^n \delta_{\bar{y}} v_{ij}^n) - \delta_x(g_{ij}^n \delta_{\hat{y}} v_{ij}^n) + \delta_y(g_{ij}^n \delta_{\hat{x}} v_{ij}^n)]/h^2$, if $(x_i,y_j) \in \Omega_h$, and $(L_h^n v)_{i,j} := 0$, if $(x_i,y_j) \in \partial\Omega_h$, $P_{i,j}^n := p(x_i,y_j,t^n,\Phi_{i,j}^n)$, $\delta_x v_{ij} = (v_{i+1,j} - v_{i,j})/h$, $\delta_{\bar{x}} v_{ij} = (v_{i,j} - v_{i-1,j})/h$, $\delta_y v_{ij} = (v_{i,j+1} - v_{i,j})/h$, $\delta_{\bar{y}} v_{ij} = (v_{i,j} - v_{i,j-1})/h$, $\delta_{x\bar{x}} v_{ij} = (v_{i+1,j} - 2v_{i,j} + v_{i-1,j})/h^2$, $\delta_{y\bar{y}} v_{ij} = (v_{i,j+1} - 2v_{i,j} + v_{i,j-1})/h^2$, $\Delta_h v_{ij} := \delta_{x\bar{x}} v_{ij} + \delta_{y\bar{y}} v_{ij}$. Still

$$a_{i+\frac{1}{2},j}^n := a(x_{i+\frac{1}{2}}, y_j, t^n, \frac{\Phi_{i+1,j}^n + \Phi_{i,j}^n}{2}, \frac{\Phi_{i+1,j}^n - \Phi_{i,j}^n}{h}, \frac{(\Phi_{i+1,j+1}^n + \Phi_{i,j+1}^n) - (\Phi_{i+1,j-1}^n + \Phi_{i,j-1}^n)}{4h}),$$

$$a_{i-\frac{1}{2},j}^n := a(x_{i-\frac{1}{2}}, y_j, t^n, \frac{\Phi_{i,j}^n + \Phi_{i-1,j}^n}{2}, \frac{\Phi_{i,j}^n - \Phi_{i-1,j}^n}{h}, \frac{(\Phi_{i,j+1}^n + \Phi_{i-1,j+1}^n) - (\Phi_{i,j-1}^n + \Phi_{i-1,j-1}^n)}{4h}),$$

$$b_{i,j+\frac{1}{2}}^n := b(x_i, y_{j+\frac{1}{2}}, t^n, \frac{\Phi_{i,j+1}^n + \Phi_{i,j}^n}{2}, \frac{(\Phi_{i+1,j+1}^n + \Phi_{i+1,j}^n) - (\Phi_{i-1,j+1}^n + \Phi_{i-1,j}^n)}{4h}, \frac{\Phi_{i,j+1}^n - \Phi_{i,j}^n}{h}),$$

$$b_{i,j-\frac{1}{2}}^n := b(x_i, y_{j-\frac{1}{2}}, t^n, \frac{\Phi_{i,j}^n + \Phi_{i,j-1}^n}{2}, \frac{(\Phi_{i+1,j}^n + \Phi_{i+1,j-1}^n) - (\Phi_{i-1,j}^n + \Phi_{i-1,j-1}^n)}{4h}, \frac{\Phi_{i,j}^n - \Phi_{i,j-1}^n}{h}),$$

$$g_{ij}^n := g(x_i, y_j, t^n, \Phi_{ij}^n, \frac{\Phi_{i+1,j}^n - \Phi_{i-1,j}^n}{2h}, \frac{\Phi_{i,j+1}^n - \Phi_{i,j-1}^n}{2h}), f_{ij}^n := f(x_i, y_j, t^n, \Phi_{ij}^n, U_{ij}^n),$$

$$q_{ij}^n := q(x_i, y_j, t^n, \Phi_{ij}^n, \frac{\Phi_{i+1,j}^n - \Phi_{i-1,j}^n}{2h}, \frac{\Phi_{i,j+1}^n - \Phi_{i,j-1}^n}{2h}).$$

The aforementioned direct Euler-ADI finite difference technique (38)is used precisely as follows:For $n \geq 1$, known Φ_{ij}^n, U_{ij}^n, we calculate Φ_{ij}^{n+1}, on the step (i). If it continues, we calculate $U_{ij}^{n+\frac{1}{2}}$ on the step (ii) solving for every j one $J \times J$ triangular linear system, and in step (iii) we calculate the U_{ij}^{n+1} solving for each i one $J \times J$ triangular linear system.

Thus, the final number of operations that must be performed to produce the solution at each time step is $O(J^2)$. Applying ((38)) the best time step we can use is based on stability and convergence $\Delta t = O(h^2)$ since to solve its first differential equation ((35).) we made a discretization with the direct Euler. But instead of solving the whole system ((38)) with time step $\Delta t = O(h^2)$, we practically worked as shown below: We believed $\Delta t = O(h)$ which we defined as (big step) in time and which we use in its shape ADI ((38).ii) and ((38).iii) with $t^n = n\Delta t$. Knowing Φ_{ij}^n and U_{ij}^n, for the initial calculation Φ_{ij}^{n+1} we use the direct-Euler scheme ((38).i) which we solve again with a step that we define as (small step) in time Δt_{Euler}, where $\Delta t = M\Delta t_{Euler}, M = O(N)$, and with this they are valued the functions q, a, b, g and f at $t^\nu = t^n + \nu\Delta t_{Euler}$, for $\nu = 0,1,\dots,M, \Phi_{ij}^\nu$, and U_{ij}^n (for f). We find that our shape is more accurate if we take their calculation $\phi, \partial_x\phi, \partial_y\phi$of its prices a, b, and g on $\frac{3}{2}\Phi_{ij}^\nu - \frac{1}{2}\Phi_{ij}^{\nu-1}$.

3.2 Arithmetic Experiments

In the first arithmetic experiment we will try to determine experimentally the convocation order of the above figure. We take $\Omega = [0,1] \times [0,1], T = 1, r(\theta) = 1 + \delta_\gamma \cos(4\theta), \theta = \arctan(\phi_x \phi_y)$, (to avoid peculiarity around zero in ϕ_x) $a = b = r^2(\theta), g = -r(\theta)r'(\theta), q = (1 + \delta_\gamma \cos(4\theta)/(m(1 + \delta_\mu \cos(4\theta)))f = \phi(1-\phi)u/(1 + 0.25u), p = \phi^3(10 - 15\phi + 6\phi^2)$, and const $m = 1, \delta_\gamma = 0.05, \delta_\mu = 0.05$. In addition, we select appropriate non-homogeneous terms so that they are the solution combined with initial conditions and corresponding Neumann border conditions $(\phi, u) = (e^{-t}\cos(x(x-1))\cos(y(y-1)), e^{-t}\cos(\pi x)\cos(\pi y))$.

Table 1. Errors and convocation order above direct Euler-ADI scheme.

J	$\|\cdot\|_\infty$		$\|\cdot\|_1$		$\|\cdot\|_2$	
	Error	Order	Error	Order	Error	Order
50	ϕ: 6.25614e − 04	− − −	6.50159e − 04	− − −	6.37656e − 04	− − −
	u: 5.36702e − 04	− − −	5.08066e − 04	− − −	4.98544e − 04	− − −
100	ϕ: 1.62660e − 04	1.943410	1.65872e − 04	1.970727	1.64245e − 04	1.956926
	u: 1.36654e − 04	1.973595	1.26969e − 04	2.000543	1.25784e − 04	1.986771
200	ϕ: 4.12925e − 05	1.977910	4.16990e − 05	1.991981	4.14926e − 05	1.984927
	u: 3.43531e − 05	1.992013	3.16395e − 05	2.004676	3.14978e − 05	1.997625
400	ϕ: 1.04275e − 05	1.985482	1.04563e − 05	1.995639	1.04303e − 05	1.992072
	u: 8.65979e − 06	1.988037	7.93134e − 06	1.996090	7.91527e − 06	1.992540

In Table 1 we see the experimental errors and the corresponding convocations of the numerical solution for $T = 1$ at the discrete maximum rate $\|\cdot\|_\infty$, norm $\|\cdot\|_1$and also in the discrete ℓ_2 norm $\|\cdot\|_2$. We count on $h = 1/(J + 1)$, where $J = 50,100,200,400$. The (maximum step) in time is Δt and we take it equal to h (with $N = J$), while the (small step) in time is Δt_{Euler} and we calculate it as a function $\Delta t_{Euler} = 0.1\Delta t/N$. (Here we get $\Delta t_{Euler} = 0.1h^2$). For two consecutive prices h_1, h_2 the h, we get the bugs e_1, e_2, respectively, and the convening order is calculated as $\log(e_2/e_1)/\log(h_2/h_1)$. This is the expected order of convergence and is calculated at 2 for functions ϕ and u.

Conclusions

To summarize, in this article, we developed a set of theoretical tools and shown that the finite difference approach exists a solution for this nonlinear system (1). We also proved some preliminary results, stability, on various finite difference approximations, and we state and prove the main result of the paper, establishing that the finite difference approximations to u and ϕ convergewith bounds of order $\Delta t + h$, in the discrete L_2 norms, where Δt is the time step and $h$is the (uniform) spatial discretization meshlength under a stability condition of the form form $\frac{\Delta t}{h} \leq \sigma$. We conclude with some basic numerical tests that validate order of convergence. In the future, efforts will be made to increase the size of these shipments.

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References

[1] J.S. Langer, "Models of pattern formation in first-order phase transitions", in *Directions in Condensed Matter Physics,* ed. By G. Grinsteil and G. Mazenko, World Scientific, Singapore 1986, pg. 164-186.

[2] S.L. Wang, R.F. Sekerka, A.A. Wheeler, B.T. Murray, S.R., Coriell, RJ. Braun and G.B. McFadden, "Thermodynamically-consistent phase-field models for solidification", *Physica D,* Vol. 69, (1993), pg. 189-200.

[3] C.B. McFadden, A.A. Wheeler, R.J. Braun, S.R. Coriell, and R.F. Sekerka. "Phase-field models for anisotropic interfaces", *Physical Review E,* Vol. 48, (1993), pg. 2016-2024.

[4] S. L.Wang, "Computation of dendritic growth at large supercoolings by using the phase field model", Ph.D. thesis, Department of Physics, Carnegie-Mellon University, Pittsburgh, 1995.

[5] S. L. Wang and R. F. Sekerka, "Algorithms for phase field computation of the dendritic operating state at large supercoolings", *Journal of Computational Physics,* Vol. 127 (1996), pg. 110-117.

[6] E. Burman and J. Rappaz, "Existence of solutions to an anisotropic phase field model", *Mathematical Methods in the Applied Sciences,* Vol. 26 (2003), pg. 1137-1160.

[7] E. Burman, M. Picasso. and J. Rappaz, "Analysis and computation of dendritic growth in binary alloys using a phase-field model", to appear in Proccedings of ENUMATH conference.

[8] E. Burman, D. Kessler, J.Rappaz, “Convergence of the finite element method applied to an anisotropic phase-field model”, *Annales Math. B. Pascal,* Vol. 11 (2004), pg. 69-95.

[9] Chr. A. Sfyrakis and V. A. Dougalis, “A fast numerical method for a simplified phase field model”, in *Mathematical Methods in Scattering Theory and Biomedical Engineering,* ed. by World Sci. Publ., Hackensack, NJ, 2006, pg. 208-215.

[10] Chr. A. Sfyrakis, “Mathematical and numerical models for materials phase change problems”, Ph.D. thesis, Department of Mathematics, University of Athens, (in Greek,) 2008.

[11] Chr. A. Sfyrakis, “A numerical method for a simplified anisotropic phase field model”, in *Bull. Greek Math. Soc.* Vol. 54, (2007), pg. 273-279.

[12] Chr. A. Sfyrakis, “Finite difference methods for an anisotropic phase field model”, *Recent Advances in Mathematics and Computers in Biology and Chemistry,* ed. by WSEAS Press, 2009, pg. 142-146.

[13] F.E. Browder, Existence and uniqueness theorems for solutions of nonlinear boundary value problems. 1965 Proc. Sympos. Appl. Math., Vol. XVII pp. 24-49 Amer. Math. Soc., Providence, R.I.

[14] V. Alexiades and A. D. Solomon, Mathematical modeling of melting and freezing processes, Hemisphere Publishing Co., Washington D.C., 1993.

[15] B. Billia and R. Trevedi, “Pattern formation in crystal growth”, in M. E. Glicksman and S. P. Marsh, “The dendrite”, in *Handbook of Crystal Growth,* Vol. 1, ed. by D.T.J. Hurle, North-Holland, Amsterdam, 1993.

[16] J.E. Dendy, “Alternating direction methods for nonlinear time-depedent problems”, *SIAM .I. Numer. Anal.,* Vol. 14 (1977), pg. 313-326.