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Study on Predator-Prey System with Impulsive Release of Predator Population in Polluted Water

Xiuxiu Wang, Meng Zhang*

School of Science, Beijing University of Civil Engineering and Architecture, Beijing, China.

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***Corresponding author:** Meng Zhang, School of Science, Beijing University of Civil Engineering and Architecture, Beijing, China.

Abstract

The phenomenon of destroying aquatic ecosystems caused by eutrophication of water occurs from time to time. This paper proposes a prey-predator system with impulsive release of predator populations in polluted water. By constructing Lyapunov function, a sufficient condition for the global asymptotic stability of the system without impulsive effect is obtained. Then, by using the geometric theory of semi-continuous dynamical systems and the successor function method, the existence and uniqueness of the order-1 periodic solution are analyzed, and the stability of the order-1 periodic solution is discussed by using the Analogue of the Poincaré criterion of impulsive differential equations. Finally, we illustrate our theoretical results and biological significance by numerical simulation.

Keywords

Polluted Water, Impulsive Release, Semicontinuous Dynamical System, Order-1 Periodic Solution, Successor Function

1. Differential mean value theorem

In recent years, with the rapid development of the city, industrial wastewater, agricultural pollution, urban domestic sewage and so on have aggravated the pollution of water resources, and the destruction of aquatic ecosystems caused by eutrophication of water has occurred from time to time. For example, Hubei Yaer Lake was polluted by organochlorine and organophosphorus pesticides and produced a water bloom that killed a large number of fish in the lake. It can be seen that eutrophication of water has a great impact on plankton and fish, which not only destroys the balance and stability of aquatic ecosystems, but also affects the economic benefits of aquaculture.

Impulsive state feedback control dynamic system is a powerful tool to solve this problem. By monitoring the plankton and fish populations in the aquatic environment, some control measures are taken: when the plankton bloom reaches above the monitoring threshold, some control measures are taken to suppress the plankton bloom and reduce its population density in order to avoid the production of water bloom. The corresponding control measures are executed according to the state of the target species, which is called impulsive state feedback control [1]. Mathematically, impulsive state feedback control of dynamic systems can accurately describe these behaviors. In recent years, great progress has been made in the study of impulsive state feedback control [1-18]. In [2], the basic theory of impulsive semicontinuous dynamical systems and the existence and stability of periodic solutions of impulsive semicontinuous dynamical systems are introduced. In [3-18], impulsive state feedback controlled dynamic systems were applied in different fields such as vegetation protection, algal fish systems, rare animal protection, and cyber security. Through state feedback pulse control, the interaction between various groups in the aquatic ecosystem can also be adjusted, which helps to maintain the balance and stability of the system. In the context of the destruction of aquatic ecosystems due to eutrophication in water, this paper presents a prey-predator system with impulsive releasing of predator population in polluted water.

The rest of the paper is organized as follows. In the second section, the free development model and the corresponding state feedback impulsive model of the predator-prey system are presented. In the third section, a qualitative analysis of the free development model is presented the dynamic performance of the state feedback impulsive model is discussed. In the last section, some numerical simulations are given and biological implications are illustrated.

2. Model Formulation

In this section, we introduce the initial idea of free developing model and state feedback impulsive model of predator-prey system.

2.1 Free developing system

First we consider the following model:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{rx}{k+\omega x} - \lambda xy, \\ \dfrac{dy}{dt} = hxy - fy. \end{cases} \tag{2.1}$$

Assume x and y represent the population density of prey (plankton) and predator (fish) at time t respectively; the growth rate of plankton is $f\left(x\right) = r/k + \omega x$ (where $r, k, \omega > 0$), $k + \omega x$ is the environmental capacity of the prey, and the prey is restricted by its own density, that is, the growth rate of the prey gradually slows down with the increase of the number of prey; λ is the predation rate of fish on plankton; h is the ratio of plankton absorbed by fish and transformed into viable fish offspring ; f is the mortality rate of fish under the influence of polluted water (here we assume that polluted water only affects fish and ignores the impact on plankton).

2.2 State feedback impulsive system

As the water is polluted by organic matter, it will make algae and other plankton abnormally proliferate, leading to a decrease in water transparency and dissolved oxygen, resulting in the deterioration of water quality, which will lead to the death of fish due to lack of oxygen.In order to achieve a certain amount of fish production, and also to avoid the production of water wars, we will pulse some drugs in the polluted water through the state feedback impulsive control method to inhibit the plankton bloom, however, these drugs will also have certain side effects on the survival of fish, let the effect coefficients of drugs on plankton and fish be p , q ($0 < p, q < 1$) respectively; At the same time, in order to achieve a certain amount of fish production, it is also necessary to pulse some fry to supplement the number of fish killed by water pollution (set the pulse delivery rate as u), so as to maintain the stability of the prey-predator system and generate economic benefits.Setting the threshold value as l , the state feedback pulse system is obtained as follows:

$$\begin{cases} \left.\begin{aligned} \dfrac{dx}{dt} &= \dfrac{rx}{k+\omega x} - \lambda xy, \\ \dfrac{dy}{dt} &= hxy - fy, \end{aligned}\right\} x < l \\ \left.\begin{aligned} \Delta x &= -px, \\ \Delta y &= -qy + u. \end{aligned}\right\} x = l \end{cases} \tag{2.2}$$

When the plankton population is less than l , no water bloom occurs and the predator-prey system develops according to the first two equations of system (2.2). When the plankton population reaches a threshold value l , a water bloom warning is issued and the predator-prey system in the water column is kept stable by means of state feedback pulse control.

Based on the practical biological significance, we consider only the case of system (2.2) in region $\left\{\left(x, y\right) \middle| x > 0, y > 0\right\}$.

3. Analysis of the system

3.1 Qualitative analysis of the free developing system (2.1).

In this section we will analyze the qualitative nature of the free development model within R_+^2. The equilibrium point of the system (2.1) satisfies the following equations:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{rx}{k+\omega x} - \lambda xy = P(x,y), \\ \dfrac{dy}{dt} = hxy - fy = Q(x,y). \end{cases} \tag{3.1}$$

Its vertical isoclines are $x = 0$ and $y = r/\lambda(k+\omega x)$, and its horizontal isocline is $y = u/f$ .Solving Equation (3.1), we can obtain that system (3.1) has a boundary equilibrium point $O(0,0)$ and a positive equilibrium point $E(x^*, y^*)$, where $x^* = f/h$, $y^* = r/\lambda(k+\omega x)$.

In the following we will analyze the stability of the equilibrium points $O(0,0)$ and $E(x^*, y^*)$ of the system (2.1).

The Jacobi matrix of the system (2.1) at point O is:

$$J(O) = \begin{pmatrix} \dfrac{r}{k} & 0 \\ 0 & -f \end{pmatrix}.$$

It is calculated that $Det(J(O)) = -rf/k < 0$, that is, point $O(0,0)$ is a saddle.

The Jacobi matrix of the system (2.1) at point E is:

$$J(E) = \begin{pmatrix} -\dfrac{rh\omega f}{(kh+\omega f)^2} & -\dfrac{\lambda f}{h} \\ \dfrac{rh^2}{\lambda(kh+\omega f)} & 0 \end{pmatrix}.$$

It is calculated that $Det(J(E)) = rhf/kh + \omega f > 0$, that is, point $E(x^*, y^*)$ is a locally asymptotically stable node or focus.

So we get the following conclusion.

Theorem 1 The boundary equilibrium point $O(0,0)$ of the system (2.1) is a saddle and the positive equilibrium point $E(x^*, y^*)$ is a locally asymptotically stable node or focus.

In the following we will analyze the global asymptotic stability of the system (2.1) at the positive equilibrium point $E(x^*, y^*)$.

First construct the Lyapunov function:

$$V(x,y) = \left[(x - x^*) - x^* \ln\left(\frac{x}{x^*}\right) \right] + \eta \left[(y - y^*) - y^* \ln\left(\frac{y}{y^*}\right) \right] \tag{3.2}$$

Then calculate the derivative of the Lyapunov function along the time, and get:

$$\begin{aligned}\frac{dV}{dt}&=\frac{x-x^*}{x}\frac{dx}{dt}+\eta\frac{y-y^*}{y}\frac{dy}{dt}\\&=\left(x-x^*\right)\left(\frac{r}{k+\omega x}-\lambda y\right)+\eta\left(y-y^*\right)\left(hx-f\right)\end{aligned} \tag{3.3}$$

Consider the following equations:

$$\begin{cases}\dfrac{r}{k+\omega x^*}-\lambda y^*=0\\ hx^*-f=0\end{cases} \tag{3.4}$$

Substituting Equation (3.4) into Equation (3.3):

$$\begin{aligned}\frac{dV}{dt}&=\left(x-x^*\right)\left(\frac{r}{k+\omega x}-\lambda y-\frac{r}{k+\omega x^*}+\lambda y^*\right)+\eta\left(y-y^*\right)\left(hx-f+\frac{u}{y}-hx^*+f-\frac{u}{y^*}\right)\\&=\left(x-x^*\right)\left[\frac{r\omega\left(x^*-x\right)}{\left(k+\omega x\right)\left(k+\omega x^*\right)}-\lambda\left(y-y^*\right)\right]+\eta h\left(x-x^*\right)\left(y-y^*\right)\\&=-\frac{r\omega\left(x-x^*\right)^2}{\left(k+\omega x\right)\left(k+\omega x^*\right)}+\left(\eta h-\lambda\right)\left(x-x^*\right)\left(y-y^*\right)\end{aligned} \tag{3.5}$$

When $\eta=\lambda/h$, there is $dV/dt=-r\omega\left(x-x^*\right)^2\Big/\left[\left(k+\omega x\right)\left(k+\omega x^*\right)\right]<0$, i.e., dV/dt is negative definite in the first quadrant. Therefore, we obtain the following conclusion.

Theorem 2 System (2.1) is globally asymptotically stable at the positive equilibrium point $E\left(x^*,y^*\right)$.

Next we analyze the existence of limit cycles.First take the Dulac function as $B\left(x,y\right)=x^{-1}y^{-1}$; there is:

$$\psi\left(x,y\right)=B\left(x,y\right)P\left(x,y\right)=\frac{r}{\left(k+\omega x\right)y}-\lambda,$$

$$\zeta\left(x,y\right)=B\left(x,y\right)Q\left(x,y\right)=h-\frac{f}{x},$$

Then

$$\frac{\partial\psi\left(x,y\right)}{\partial x}=-\frac{r\omega y}{\left(k+\omega x\right)^2y^2}<0,$$

$$\frac{\partial\zeta\left(x,y\right)}{\partial y}=0,$$

So there is

$$\frac{\partial\psi\left(x,y\right)}{\partial x}+\frac{\partial\zeta\left(x,y\right)}{\partial y}<0.$$

According to the Dulac theorem, there is no limit cycle in the first quadrant of system (2.1).So we get the following conclusions.

Theorem 3 System (2.1) has no limit cycle in the first quadrant.

3.2 Existence and uniqueness of order-1 periodic solution of system (2.2).

In this section we will study the uniqueness of the existence of order-1 periodic solutions of the system (2.2). From the discussion in Section 3.1, it is clear that the equilibrium point $E\left(x^*, y^*\right)$ is globally asymptotically stable.

For convenience, for any point G, let x_G denote its horizontal coordinate and y_G its vertical coordinate. We assume that the initial point of the trajectory is at phase set N, and the intersection of the vertical isoclinic $dx/dt = 0$ and the phase set N is $C\left((1-p)l, y_C\right)$. If $C_1 = \left(l, y_{C_1}\right) \in M$, then the pulse occurs at point C_1, and the pulse function maps C_1 to C_1^+. If both the impulse set M and the phase set N are on the right side of the equilibrium point $E\left(x^*, y^*\right)$, that is, $x^* < (1-p)l < l$, then all solutions of the system (2.2) tend to the equilibrium point $E\left(x^*, y^*\right)$ after finite impulses, so we mainly discuss the cases $l < x^*$ or $(1-p)l < x^* < l$.

Case.1 $l < x^*$: When $l < x^*$, both the impulse set M and the phase set N are to the left of the equilibrium point $E\left(x^*, y^*\right)$. Let the intersection of the vertical isoclinic $dx/dt = 0$ and the phase set N be $C\left((1-p)l, y_C\right)$, and let $L(C,t)$ be the trajectory through point C and tangent to point C with phase set N. Let the trajectory $L(C,t)$ intersects the pulse set at the point $C_1\left(l, y_{C_1}\right)$, and after the pulse effect point C_1 is mapped to the point $C_1^+\left((1-p)l, y_{C_1^+}\right)$ on the phase set N, where $y_{C_1^+} = (1-q)y_{C_1} + u$, then the position of C_1^+ has the following three cases.

Case 1.1 $y_{C_1^+} = y_C$: There must exist some parameter $u = u_1^*$ such that $y_{C_1^+} = (1-q)y_{C_1} + u_1^* = y_C$, where the point C_1^+ coincides with the point C and the successor function of the point C is $f(C) = y_{C_1^+} - y_C = 0$. Then the trajectory L_{CC_1} and the line segment $C_1C_1^+$ form an order-1 periodic solution of the system (2.2) (see Figure 1).

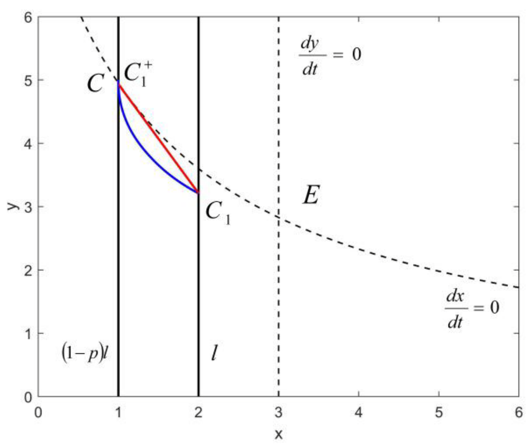


Figure 1. The existence of order-1 periodic solution for $u = u_1^*$, $l < x^*$.

Case 1.2 $y_{C_1^+} < y_C$: When the parameters $u < u_1^*$ and the parameters p, q remain the same as Case1.1, there is $y_{C_1^+} = (1-q)y_{C_1} + u < y_C$,and at this time the point C_1^+ is below the point C, then the successor function of the point C is $f(C) = y_{C_1^+} - y_C < 0$. Take another point $D\left((1-p)l, \varepsilon\right)$ on the phase set N, where ε is small enough and satisfies $0 < \varepsilon < u_2^*$.From the vector field, the track over the point D intersects the pulse set M at the point $D_1\left(l, y_{D_1}\right)$, after the pulse effect point D_1 is mapped to the point $D_1^+\left((1-p)l, y_{D_1^+}\right)$ on the phase set N, where $y_{D_1^+}=(1-q)y_{D_1}+u_2^*>\varepsilon$, so the successor function of the point D is $f(D)=y_{D_1^+}-y_D>0$. By the existence uniqueness theorem of the solution, there must be an order-1 periodic solution of the system (2.2) whose initial point is between points C and D on the phase set (see Figure 2).

Next we prove the uniqueness of the order-1 periodic solution. Since the point $E(x^*, y^*)$ is asymptotically stable, the trajectory starting from the sets $\{(x,y)|x=(1-p)l, y>y_C\}$ and $\{(x,y)|x=(1-p)l, 0<y<y_D\}$ enters the set $\{(x,y)|x=(1-p)l, y_D\le y\le y_C\}$ after at most a finite number of impulse effects, so the initial point of the order-1 periodic solution of the system (2.2) is considered to fall only on the straight line segment $\overline{CD}$. Obviously, $y_D<y_{D_1^+}<y_{C_1^+}<y_C$, so the lengths of line segments $\overline{CD}$ and $\overline{C_1^+D_1^+}$ satisfy $\left|\overline{C_1^+D_1^+}\right|<\left|\overline{CD}\right|$.

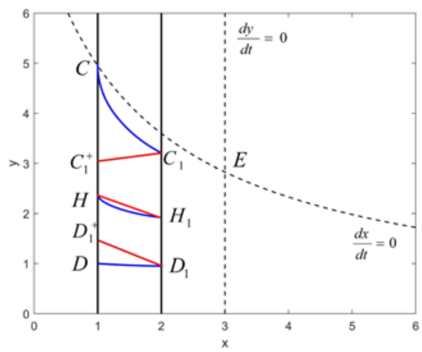


Figure 2. The existence of order-1 periodic solution for $u<u_1^*$, $l<x^*$.

Case 1.3 $y_{C_1^+}>y_C$: When the parameter $u>u_1^*$ and the parameters p, q remain the same as **Case1.1**, we have $y_{C_1^+}=(1-q)y_{C_1}+u>y_C$, and the point C_1^+ is above the point C. Let the trajectory through point C_1^+ intersect the pulse set at point $C_2\left(l, y_{C_2}\right)$, and after the pulse effect point C_2 is mapped to the point $C_2^+\left((1-p)l, y_{C_2^+}\right)$ on the phase set N. According to the disjointness of the vector field and the two trajectories, we know $y_{C_2}<y_{C_1}$, so there is $y_{C_2^+}<y_{C_1^+}$, so the successor function of point C_1^+ is $f\left(C_1^+\right)=y_{C_2^+}-y_{C_1^+}<0$. Take another point $D\left((1-p)l, y_C+\varepsilon\right)$ above point C, where $\varepsilon>0$ is small enough. Let the trajectory over the point D intersect the pulse set at the point $D_1\left(l, y_{D_1}\right)$, after the pulse effect point D_1 is mapped to the point $D_1^+\left((1-p)l, y_{D_1^+}\right)$ on the phase set N, due to the continuous dependence of the solution on the initial value and time, it is known that $y_{D_1}<y_{C_1}$, and the point D_1 is close enough to the point C_1, so that we have $y_{D_1^+}<y_{C_1^+}$, and the point D_1^+ is close enough to the point C_1^+. Since $y_C<y_{C_1^+}$, then $y_D<y_{D_1^+}$, the successor function of the point D is $f(D)=y_{D_1^+}-y_D>0$. From the existence uniqueness theorem of the solution, it follows that there exists an order-1 periodic solution of the system (2.2) whose initial point is between the point D and the point C_1^+ on the phase set (see Figure 3). The proof of the uniqueness of the order-1 periodic solution of the system (2.2) is similar to that of Case1.2.

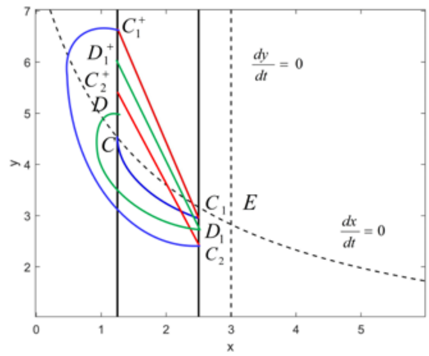


Figure 3. The existence of order-1 periodic solution for $u > u_1^*$, $l < x^*$.

Combining the above three cases, we obtain the following theorem.

Theorem 4 When $l < x^*$, the system (2.2) has a unique order-1 periodic solution.

Case.2 $(1-p)l < x^* < l$: When $(1-p)l < x^* < l$, the phase set N is to the left of the equilibrium point $E\left(x^*, y^*\right)$ and the pulse set M is to the right of the equilibrium point $E\left(x^*, y^*\right)$. The intersection point of the vertical isoclinic line $dx/dt = 0$ and the pulse set M is expressed as $F_1\left(l, y_{F_1}\right)$, and the trajectory passing through the point F_1 and tangent to the pulse set M at point F_1 is $L\left(F_1, t\right)$. The first intersection point of the trajectory $L\left(F_1, t\right)$ and the vertical isoclinic line $dx/dt = 0$ is expressed as $F\left(x_F, y_F\right)$, and the two intersections of the trajectory $L\left(F_1, t\right)$ and the phase set N are point $A\left(\left(1-p\right)l, y_A\right)$ and point $B\left(\left(1-p\right)l, y_B\right)$, respectively, and satisfy $y_B < y_F < y_A$ (see Fig.4).Suppose the trajectory $L\left(F_1, t\right)$ is subjected to impulse effects at the point F_1 and then is mapped to the point $F_1^+\left(\left(1-p\right)l, y_{F_1^+}\right)$ on the phase set.

Case 2.1 $x_F < (1-p)l$: When $x_F < (1-p)l$, there must exist some parameter $u = u_2^*$ such that $y_{F_1^+} = (1-q)y_{F_1} + u_2^* = y_A$, or there exists some parameter $u = u_3^*$ such that $y_{F_1^+} = (1-q)y_{F_1} + u_3^* = y_B$, i.e., $y_{F_1^+} = y_A$ or $y_{F_1^+} = y_B$, at which point $F_1^+\left(\left(1-p\right)l, y_{F_1^+}\right)$ coincides with point A or point B, then the trajectory L_{ABF_1} and line $\overline{F_1A}$ constitute an order-1 periodic solution of system (2.2), or the trajectory L_{BF_1} and line $\overline{F_1B}$ constitute an order-1 periodic solution of system (2.2).

When the parameters $u < u_3^*$ and the parameters p, q remain constant, we have $y_{F_1^+} = (1-q)y_{F_1} + u < y_B$, and $F_1^+\left(\left(1-p\right)l, y_{F_1^+}\right)$ is below the point B, i.e., $y_{F_1^+} < y_B$, then the successor function of the point B is $f\left(B\right) = y_{F_1^+} - y_B < 0$.Take another point $G\left(\left(1-p\right)l, \varepsilon\right)$ on the phase set N, where ε is small enough and $0 < \varepsilon < u_6^*$.The trajectory past the point G intersects with the impulse set M at the point $G_1\left(l, y_{G_1}\right)$, after the impulse effect point G_1 is mapped to the point $G_1^+\left(\left(1-p\right)l, y_{G_1^+}\right)$ on the phase set N, where $y_{G_1^+} = (1-q)y_{G_1} + u_6^* > \varepsilon$, so the successor function of the point G is $f\left(G\right) = y_{G_1^+} - y_G > 0$.By the existence uniqueness theorem of the solution, we obtain that the system (2.2) has an order-1 periodic solution and its initial point is between points B and G on the phase set (see Figure 4). The proof of the uniqueness of the order-1 periodic solution in this case is similar to that of Case 1.2.

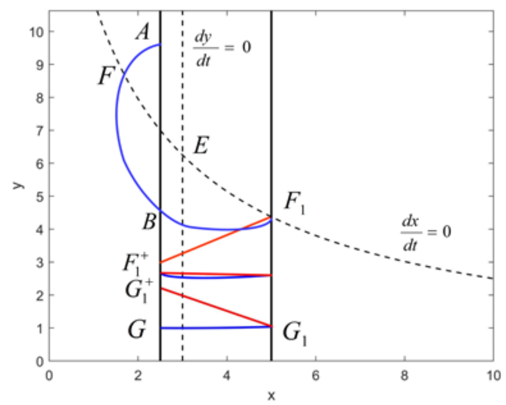


Figure 4. The existence of order-1 periodic solution for $u < u_3^*, x_F < (1-p)l,\ (1-p)l < x^* < l$.

When the parameters $u > u_2^*$ and the parameters p , q remain unchanged, so that $y_{F_1^+} = (1-q) y_{F_1} + u > y_A$, that is, $y_{F_1^+} > y_A$,at this time $F_1^+\left((1-p)l, y_{F_1^+}\right)$ is above the point A, then the successor function of the point A is $f(A) = y_{F_1^+} - y_A > 0$ .The track passing through the point F_1^+ intersects with the pulse set M at the point $F_2\left(l, y_{F_2}\right)$, which is mapped to the point $F_2^+\left((1-p)l, y_{F_2^+}\right)$ on the phase set N by the pulse effect .From the disjoint nature of vector fields and orbits we know that $y_{F_2} < y_{F_1}$, $y_{F_2^+} < y_{F_1^+}$, so the successor function of point F_1^+ is $f\left(F_1^+\right) = y_{F_2^+} - y_{F_1^+} < 0$. According to the existence uniqueness theorem of the solution, there exists an order-1 periodic solution of the system (2.2) and its initial point is between the point A and the point F_1^+ on the phase set (see Figure 5). The proof of the uniqueness of the order-1 periodic solution in this case is similar to that of Case 1.2.

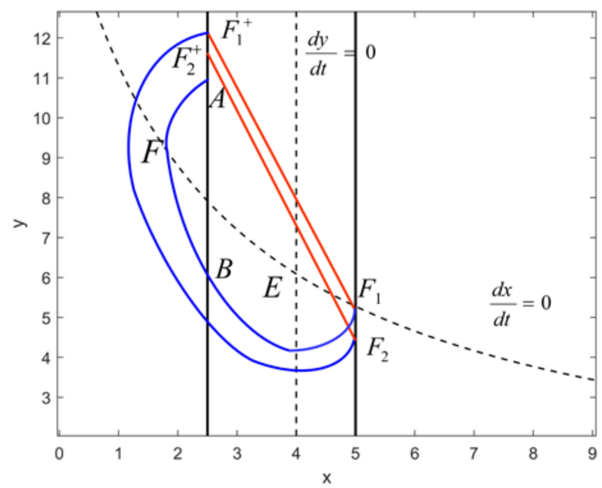


Figure 5. The existence of order-1 periodic solution for $u > u_2^*, x_F < (1-p)l,\ (1-p)l < x^* < l$.

If the point $F_1^+\left((1-p)l, y_{F_1^+}\right)$ is between points A and B i.e. $y_B < y_{F_1^+} < y_A$, then the trajectory from the initial point F_1^+ will be unaffected by the impulse action and eventually converge to the stable point E , thus failing to form an order-1 periodic solution.

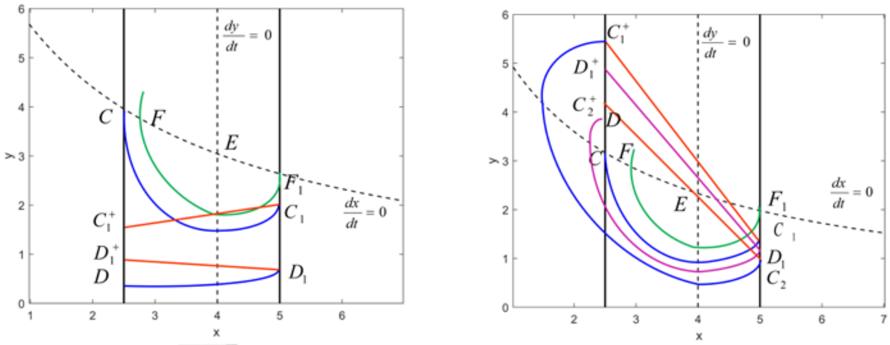
In summary, we conclude the following.

Theorem 5 When $(1-p)l < x^* < l$ and $x_F < (1-p)l$, if $y_{F_1^+} \geq y_A$ or $y_{F_1^+} \leq y_B$, then there is an order-1 periodic solution to the system (2.2); if $y_B < y_{F_1^+} < y_A$, then there is no order-1 periodic solution to the system (2.2).

Case 2.2 $x_F \geq (1-p)l$: When $x_F > (1-p)l$, take the intersection of the vertical isotropic $dx/dt = 0$ and the phase set N to be $C((1-p)l, y_C)$. Let the trajectory passing through the point C and tangent to the phase set N at the point C be $L(C,t)$. Then the trajectory $L(C,t)$ intersects with the pulse set at the point $C_1(l, y_{C_1})$, and after the pulse effect point C_1 is mapped to the point $C_1^+((1-p)l, y_{C_1^+})$ on the phase set N, here $y_{C_1^+} = (1-q)y_{C_1} + u$.

The next analysis is the same as in Case 1, and we can prove that in this case the system (2.2) has an order-1 periodic solution (as shown in Figures 6 (a), (b)). Therefore, we conclude the following.

Theorem 6 The system (2.2) has a unique order-1 periodic solution when $(1-p)l < x^* < l$ and $x_F \geq (1-p)l$.



(a) (b)

Figure 6. The existence of order-1 periodic solutionfor $x_F \geq (1-p)l$, $(1-p)l < x^* < l$.

3.3 Stability of the order-1 periodic solution of the system (2.2)

Theorem 7 Suppose that $(\xi(t), \eta(t))$ is an order-1 periodic solution of the system (2.2), then this solution is orbitally asymptotically stable if $\left| \dfrac{\xi_0 (k+\omega\xi_0)(\eta_0 - u)\left[r(1-q) - \lambda(\eta_0 - u)(k+\omega\xi_0)\right]}{\eta_0 (1-p)(1-q)\left[k(1-p)+\omega\xi_0\right]} \right| \leq 1$ holds.

Proof: Let $\Gamma(x = \xi(T), y = \eta(T))$ be an order-*1* periodic solution of the system (2.2) and have $\xi_0 = \xi(0)$,

$$\eta_0 = \eta(0),$$

$$\xi_1 = \xi(T), \eta_1 = \eta(T), \xi_1^+ = \xi(T+0), \eta_1^+ = \eta(T+0), \xi_1^+ = (1-p)\xi_1, \eta_1^+ = (1-q)\eta_1 + u.$$

From the system (2.2), it follows that

$$P(x,y) = \frac{rx}{k+\omega x} - \lambda xy, Q(x,y) = hxy - fy,$$

$$A(x,y) = -px, B(x,y) = -qy + u, \phi(x,y) = x - l.$$

The calculation gives that

$$\frac{\partial P}{\partial x}=\frac{r\left(k+\omega x\right)-r\omega x}{\left(k+\omega x\right)^{2}}-\lambda y=\frac{r}{k+\omega x}-\lambda y-\frac{r\omega x}{\left(k+\omega x\right)^{2}},\frac{\partial Q}{\partial y}=hx-f,$$

$$\frac{\partial A}{\partial x}=-p,\frac{\partial A}{\partial y}=0,\frac{\partial B}{\partial x}=0,\frac{\partial B}{\partial y}=-q,\frac{\partial \phi}{\partial x}=1,\frac{\partial \phi}{\partial y}=0.$$

Then

$$\begin{aligned}\Delta_1&=\frac{P_+\left(\frac{\partial B}{\partial y}\frac{\partial \phi}{\partial x}-\frac{\partial B}{\partial x}\frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial x}\right)+Q_+\left(\frac{\partial A}{\partial x}\frac{\partial \phi}{\partial y}-\frac{\partial A}{\partial y}\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\right)}{P\left(\frac{\partial \phi}{\partial x}\right)+Q\left(\frac{\partial \phi}{\partial y}\right)}\\&=\frac{(1-q)P\left(\xi\left(T^+\right),\eta\left(T^+\right)\right)}{P\left(\xi\left(T\right),\eta\left(T\right)\right)}\\&=\frac{\xi_0\left(k+\omega\xi_0\right)\left[r\left(1-q\right)-\lambda\left(\eta_0-u\right)\left(k+\omega\xi_0\right)\right]}{k\left(1-p\right)+\omega\xi_0},\end{aligned}$$

And

$$\begin{aligned}&\int_0^T\left[\frac{\partial P}{\partial x}\left(\xi\left(t\right),\eta\left(t\right)\right)+\frac{\partial Q}{\partial y}\left(\xi\left(t\right),\eta\left(t\right)\right)\right]d\\&=\int_0^T\left[\frac{r}{k+\omega x}-\lambda y-\frac{r\omega x}{\left(k+\omega x\right)^{2}}+hx-f\right]dt\\&=\ln\frac{\xi_1}{\xi_0}\frac{\eta_1}{\eta_0}-\int_0^T\frac{r\omega x}{\left(k+\omega x\right)^{2}}dt.\end{aligned}$$

Therefore

$$\begin{aligned}\mu_2&=\Delta_1\exp\left[\int_0^T\left(\frac{\partial P}{\partial x}\left(\xi\left(t\right),\eta\left(t\right)\right)+\frac{\partial Q}{\partial y}\left(\xi\left(t\right),\eta\left(t\right)\right)\right)d\quad\right]\quad t\\&=\frac{\xi_0\left(k+\omega\xi_0\right)\left[r\left(1-q\right)-\lambda\left(\eta_0-u\right)\left(k+\omega\xi_0\right)\right]}{k\left(1-p\right)+\omega\xi_0}\exp\left(\ln\frac{\xi_1\eta_1}{\xi_0\eta_0}-\int_0^T\frac{r\omega x}{\left(k+\omega x\right)^{2}}dt\right)\\&=\frac{\xi_0\left(k+\omega\xi_0\right)\left[r\left(1-q\right)-\lambda\left(\eta_0-u\right)\left(k+\omega\xi_0\right)\right]}{k\left(1-p\right)+\omega\xi_0}\frac{\eta_0-u}{\eta_0\left(1-p\right)\left(1-q\right)}\exp\left(-\int_0^T\frac{r\omega x}{\left(k+\omega x\right)^{2}}dt\right)\\&=\frac{\xi_0\left(k+\omega\xi_0\right)\left(\eta_0-u\right)\left[r\left(1-q\right)-\lambda\left(\eta_0-u\right)\left(k+\omega\xi_0\right)\right]}{\eta_0\left(1-p\right)\left(1-q\right)\left[k\left(1-p\right)+\omega\xi_0\right]}\exp\left(-\int_0^T\frac{r\omega x}{\left(k+\omega x\right)^{2}}dt\right).\end{aligned}$$

Since

$$\exp\left(-\int_0^T\frac{r\omega x}{\left(k+\omega x\right)^{2}}dt\right)<1.$$

Obviously, if $\left|\frac{\xi_0\left(k+\omega\xi_0\right)\left(\eta_0-u\right)\left[r\left(1-q\right)-\lambda\left(\eta_0-u\right)\left(k+\omega\xi_0\right)\right]}{\eta_0\left(1-p\right)\left(1-q\right)\left[k\left(1-p\right)+\omega\xi_0\right]}\right|\leq 1$, then $\left|\mu_2\right|<1$. According to the

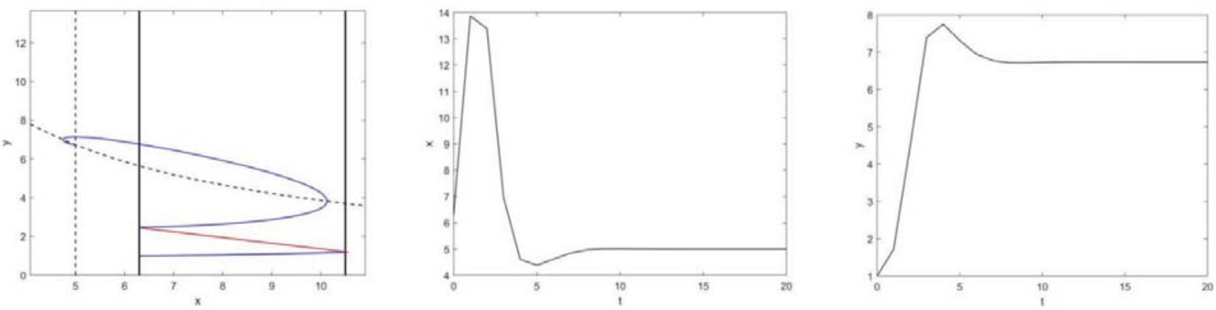
Analogue of the Poincaré criterion, the order-1 periodic solution of the system (2.2) is orbitally asymptotically stable.

4. Numerical analysis and biological conclusions

To verify the theoretical results of this paper, we consider the following examples.

$$(1)\begin{cases}\left.\begin{array}{l}\dfrac{dx}{dt}=\dfrac{9.5x}{1.2+0.7x}-0.3xy,\\ \dfrac{dy}{dt}=0.1xy-0.5y,\end{array}\right\} x<10.5,\\ \left.\begin{array}{l}\Delta x=-0.4x,\\ \Delta y=-0.3y+1.6,\end{array}\right\} x=10.5.\end{cases}$$

Let $r=9.5$, $k=1.2$, $\omega=0.7$, $\lambda=0.3$, $h=0.1$, $f=0.5$, $u=1.6$, the numerical calculation gives $x^*=5$, $y^*=6.738$ .If $l=10.5>x^*$, and $l(1-p)=6.3>x^*$, $p=0.4$, $q=0.3$, $l=10.5$, then the phase diagram and time series diagram at this point are shown in Figure 7, when the system (2.2) has no periodic solutions and all solutions converge to the equilibrium point E after a finite number of pulses .

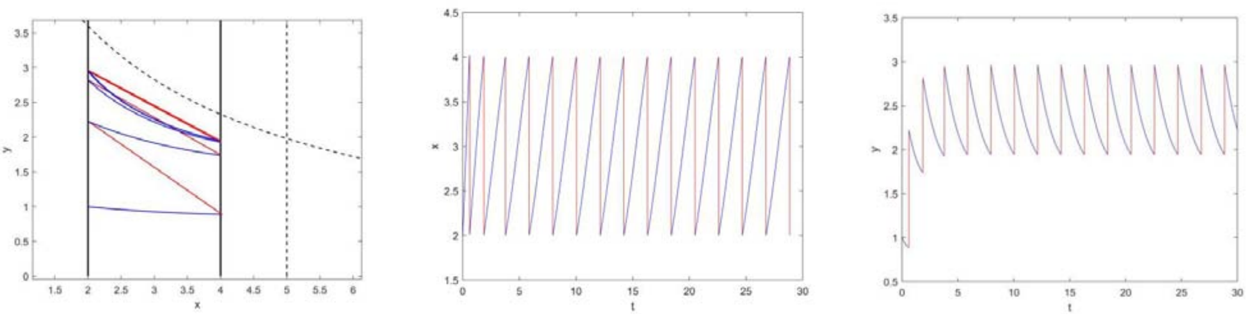


(a) (b) (c)

Figure 7. The phase diagram and time series diagram for $x^*<(1-p)l<l$.

$$(2)\begin{cases}\left.\begin{array}{l}\dfrac{dx}{dt}=\dfrac{9.5x}{2+1.2x}-0.3xy,\\ \dfrac{dy}{dt}=0.1xy-0.5y,\end{array}\right\} x<4,\\ \left.\begin{array}{l}\Delta x=0.5x,\\ \Delta y=0.3y+1.6,\end{array}\right\} x=4.\end{cases}$$

Let $r=9.5$, $k=2$, $\omega=1.2$, $\lambda=0.3$, $h=0.1$, $f=0.5$, $u=1.6$, by numerical calculation we can get $x^*=5$, $y^*=1.979$. If $l=4<x^*$, and $p=0.5$, $q=0.3$,then the phase diagram and time series diagram at this point are shown in Figure 8, combined with the analysis of Case1, we can see that the system (2.2) has an order-1 periodic solution.

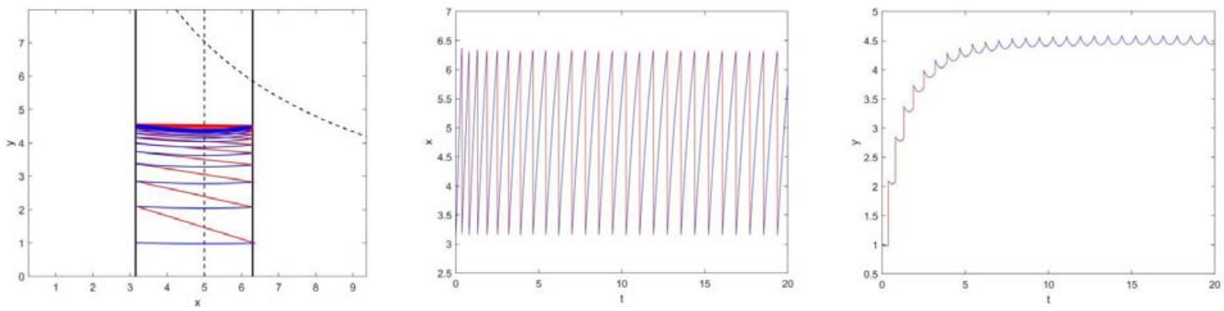


(a) (b) (c)

Figure 8. The phase diagram and time series diagram for $(1-p)l < l < x^*$.

$$(3)\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= \frac{9.5x}{1+0.7x} - 0.3xy, \\ \frac{dy}{dt} &= 0.1xy - 0.5y, \end{aligned}\right\} x < 6.3, \\ \left.\begin{aligned} \Delta x &= 0.5x, \\ \Delta y &= 0.3y + 1.4, \end{aligned}\right\} x = 6.3. \end{cases}$$

Let $r = 9.5$, $k = 1$, $\omega = 0.7$, $\lambda = 0.3$, $h = 0.1$, $f = 0.5$, $u = 1.4$, the numerical calculation gives $x^* = 5$, $y^* = 7.037$. If $l = 6.3 > x^*$, $l(1-p) = 3.15 < x^*$, and $p = 0.5$, $q = 0.3$, then the phase diagram and time series diagram at this point are shown in Figure 9, combined with the analysis of Case2, we can see that the system (2.2) has an order-1 periodic solution.

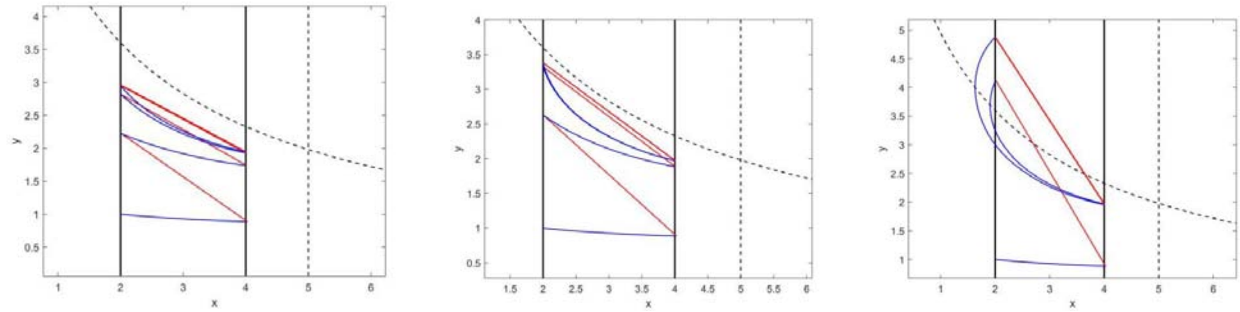


(a) (b) (c)

Figure 9. The phase diagram and time series diagram for $(1-p)l < x^* < l$.

From the above three examples, it can be seen that in polluted water, the predator-prey system with impulsive release of predator population has periodicity and tends to be stable, that is, the number of prey is in a controllable state.

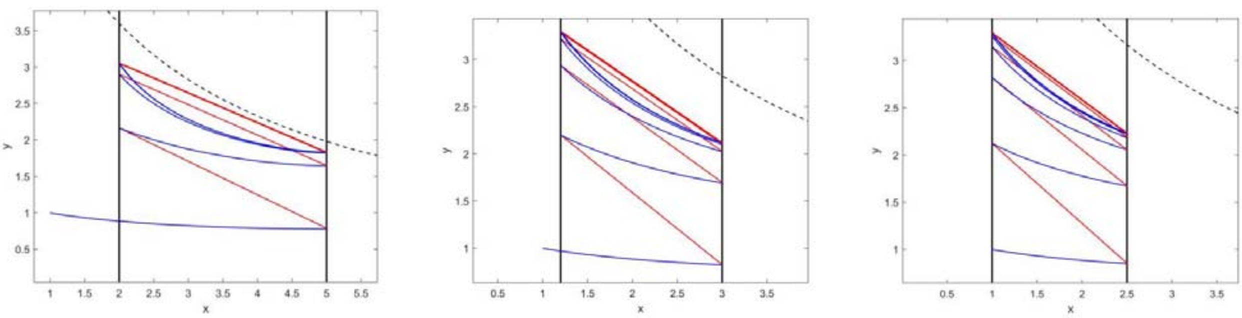
The phase diagram of the system (2.2) is shown in Fig.10, where the impulsive release amount u is the control parameter. From the figure, it can be seen that when the number of prey reaches the economic threshold, the number of prey can be suppressed below a certain level by using pulse state feedback control, and it is easier to control the number of prey below the economic threshold as the impulsive release amount u increases.



(a) $u = 1.6$ (b) $u = 2$ (c) $u = 3.5$

Figure 10. The impulsive release amount u.

If the economic threshold l is chosen as the control parameter, the phase diagram of the system (2.2) is shown in Figure 11. From the figure, it is clear that as the economic threshold l decreases, more pulses are needed to make the system converge to a stable periodic solution, which makes it more difficult to control the number of prey below the economic threshold.



(a) $l = 5$ (b) $l = 3$ (c) $l = 2.5$

Figure 11. The economic threshold l.

From the above analysis, it can be seen that the system can obtain a stable periodic solution by giving a suitable control parameter u and economic threshold l. The pulse state feedback control method can suppress the prey population to below a certain level, which can solve the eutrophication problem of water bodies well.

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