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Treatment of the Unsteady Heat Equation Subject to Heat Flux Boundary Conditions: The Method of Discretization in Time Complemented With Regression Analysis

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Abstract

The Method of Discretization in Time (MDT) is a general hybrid technique intended to alleviate partial differential equations of parabolic type. The MDT engenders a sequence of adjoint second order ordinary differential equations, wherein the space coordinate is the independent variable and the time appears as an embedded parameter. Essentially, the resulting adjoint second order ordinary differential equations are considered of "quasi-stationary" nature. In this work, the MDT is applied to the unsteady heat equation in simple bodies (large plate, long cylinder and sphere) with temperature-invariant thermophysical properties, constant initial temperature and surface heat flux boundary conditions. In engineering applications, the surface heat flux is customarily provided by electrical heating, radiative heating and pool fire heating. Using a single time jump and the first adjoint "quasi-stationary" heat equation, it is demonstrated that the approximate, semi-analytical MDT temperature solutions expressed in terms of the space coordinate and excluding time are easily obtainable for each simple body. As a direct consequence, usage of the second adjoint "quasi-stationary" heat equation engaging two time jumps come to be unnecessary. As a sound replacement, regression analysis is applied to the deviations of the dimensionless surface temperature as a function of the dimensionless time. Thereafter, the outcomes are articulated with the approximate, semi-analytical MDT temperature solutions.

Keywords

Unsteady Heat Conduction Equation, Simple Bodies, Uniform Heat Flux Boundary Conditions, Method of Discretization in Time (Mdt), Adjoint "quasi-stationary" Heat Equation

1. Introduction

The accurate analysis of unsteady heat conduction in simple bodies (large plate, long cylinder and sphere) with various heating/cooling conditions is of remarkable importance in scientific applications as cited in Arpaci[1], Luikov [2], Özişik [3], Poulikakos [4]. For each simple body, the exact analytical solutions of the unsteady heat equation subject to surface heat flux have led to well-known infinite series in rectangular, cylindrical and spherical coordinates. Herein, the temperatures are double-valued functions of the space coordinate and time. Beneficially, the infinite series possess intrinsic good convergence and recede to "one term" series when the time is large and the temperatures increase linearly with time [1-4].

Regrettably, the infinite series suffer from severe convergence at small time, necessitating many terms to secure adequate accuracy. In fact, the prevalent behavior becomes so abnormal for very short time that the evaluated temperatures consistently over predict the initial condition in contraposition with the physics of the problem.

Moreover, the exact analytical solutions of the unsteady heat equation subject to surface heat flux in simple bodies (large plate, long cylinder and sphere) have been considered a complex operation because of two factors: 1) the boundary condition at the body surface is non-homogeneous and 2) no steady-state asymptotic solution exists [1-4]. For scientific applications, the surface heat flux is commonly provided by electrical heating, radiative heating and pool fire heating.

The present study utilizes the Method of Discretization in Time (MDT) to transform the unsteady heat equations into a sequence of "quasi-stationary" heat equations having an embedded time parameter specified at the time where the temperature results are needed. For enhanced accuracy the temperature results based on the MDT are complemented with regression analysis of the temperature deviations at small time.

The technical paper is divided in six parts. The first part describes the physical system and the problem formulation for the continuous heating of three simple bodies (large plate, long cylinder and sphere) with uniform surface heat flux. A brief description of the ideas underlying the Method of Discretization in Time (MDT) is delineated in the second part. In the third part, the MDT is applied to the generalized unsteady heat equation for the simple bodies (large plate, long cylinder and sphere) under study. The all-time, exact analytical temperature distributions for the three simple bodies as taken from Luikov [2] are disclosed in the fourth part. The semi-analytical temperature results produced by the MDT with/without regression analysis of the deviations of the surface temperatures along with the baseline temperature results are reported and compared in the fifth part for the three simple bodies.

2. Physical System and Mathematical Formulation

Consideration is given to the unsteady heat conduction in a set of simple bodies (large plate of half-thickness R, long cylinder of radius R and sphere of radius R) with uniform initial temperature T_i throughout. For time $t \geq 0$, a uniform heat flux q_s is applied at the surface of each simple body. The thermal conductivity k and the thermal diffusivity α of the material are assumed nearly independent of temperature.

The generalized unsteady heat equation in one dimension is

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{n}{x}\frac{\partial T}{\partial x}\right) \qquad \text{in } 0 \leq x \leq R, \text{for } t > 0 \tag{1}$$

where the geometric parameter "n" equates to 0 for rectangular, 1 for cylindrical and 2 for spherical coordinate systems. The initial condition is

$$T(x,0) = T_{in} \tag{2}$$

and the applicable boundary conditions are

$$\frac{\partial T(0,t)}{\partial x} = 0 \tag{3a}$$

$$k\frac{\partial T(R,t)}{\partial x} = q_s \tag{3b}$$

In eq. (3b), the uniform surface heat flux q_s in scientific applications is provided by three heating sources: electric heating (Houston [5]), radiative heating (Modest [6]) and pool fire heating (Gorbett et al. [7]).

The pair of eqs. (3a) and (3b) is classified as Neumann boundary conditions after Carl Neumann [1-4]. In addition, eq. (3a) is homogeneous and implies zero heat flux (thermal symmetry) on the inner boundary $x = 0$ and, whereas eq. (3b) stipulates a finite heat flux on the outer boundary $x = R$ and is non-homogeneous.

Because of the Neumann boundary conditions, the appropriate dimensionless variables for space coordinate x, time t and temperature difference $T - T_{in}$ are

$$X = \frac{x}{R}, \qquad \tau = \frac{t}{t_{eq}}, \qquad \Phi = \frac{T - T_{in}}{T_{eq}} \tag{4}$$

with scales radius R, equivalent time $t_{eq} = R^2/\alpha$ and equivalent temperature $T_{eq} = q_s R/k$, respectively. Correspondingly, equation (1) is rewritten as

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial X^2} + \frac{n}{X}\frac{\partial \Phi}{\partial X} \qquad \text{in } 0 \leq X \leq 1, \text{for } \tau > 0 \tag{5}$$

In the same way, the initial condition becomes

$$\Phi\,(X,0) = 0 \tag{6}$$

and the applicable boundary conditions turn into

$$\frac{\partial \Phi\ (0,\tau)}{\partial X} = 0 \tag{7a}$$

$$\frac{\partial \Phi\ (1,\tau)}{\partial X} = 1 \tag{7b}$$

The set of eqs. (5) to (7) constitutes an initial/boundary value problem (IBVP).

3. Conventional Finite Difference Methods

The three popular finite difference methods for solving partial differential equations of parabolic type are the explicit method, the implicit method and Crank–Nicolson method (Larsson and Thomee [8]). The explicit method is the easiest to implement and the least numerically intensive, but requires the compliance of a stability criterion. The implicit method is unconditionally stable and is first order accurate in time and second order accurate in space, but it requires the solution of a large system of algebraic equations at each time step. The Crank-Nicolson method is the most accurate, because is unconditionally stable and is second order accurate in space and time. The implicit and the Crank-Nicolson methods require the solution of large systems of algebraic equations at each time step.

4. Basic Principle of Applied Mathematics

One of the basic principles of applied mathematics is to split a complex problem into a reduced number of less complex sub-problems, which either have been already solved or are easier to treat. Inspired in this basic principle is the goal of the present study within the umbrella of an initial boundary value problem representative of the unsteady heat equation in simple bodies (large plate, long cylinder and sphere) affected by uniform surface heat flux and framed in rectangular, cylindrical and spherical coordinates, respectively.

5. Outline of the Method of Discretization in Time (Mdt)

The Method of Discretization in Time (Rektorys [9]) also called the Transversal Method of Lines (Rothe [10]) embodies a powerful mathematical procedure for transforming partial differential equations of parabolic type in one dimension into ordinary differential equations of second order. In the case of equation (5), the basic idea underlying the MDT is to discretize the partial derivative in time $\frac{\partial \Phi}{\partial \tau}$, while the two partial derivatives in space $\frac{\partial \Phi}{\partial X}$ and $\frac{\partial^2 \Phi}{\partial X^2}$ remain continuous. The two-point backward finite difference approximation for the partial derivative in time $\frac{\partial \Phi}{\partial \tau}$ is

$$\left.\frac{\partial \Phi_p}{\partial \tau}\right|_{\tau_p} \approx \frac{\Phi_p - \Phi_{p-1}}{\Delta \tau} + O(\Delta\tau) \tag{8}$$

where the appended term $O\ (\Delta\tau)$ indicates truncation error of first order Chapra and Canale [11]. To be more specific, the MDT seeks to replace the parabolic partial differential equation of parabolic type eq. (5) in two independent variables X and τ by a sequence of adjoint ordinary differential equations of second order in X, while τ becomes an embedded parameter by way of a dimensionless time interval $\Delta\tau$. The required computational domain is constructed with a set of lines parallel to the X–coordinate and perpendicular to the τ–coordinate as illustrated in tehv sketch of Figure 1. For easiness, the lines are separated by dimensionless time intervals of equal size $\Delta\tau$.

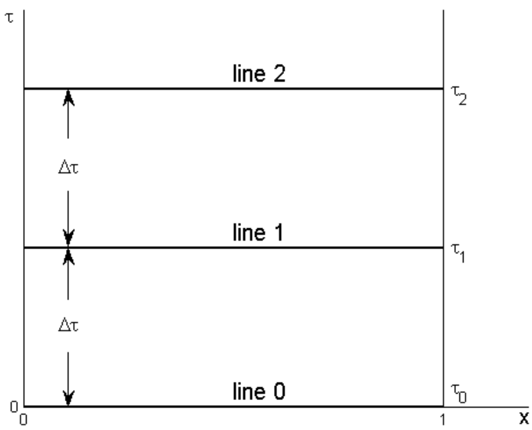

Figure 1. Computational domain for the MDTcontaining a set of lines

In view of the foregoing, the substitution of eq. (8) into eq. (5) generates a sequence of adjoint second order ordinary differential equations at time levels τ_p (p = 1, 2,..., P). That is:

$$\frac{d^2\Phi_p}{dX^2} + \frac{n}{X}\frac{d\Phi_p}{dX} = \frac{\Phi_p}{\Delta\tau} - \frac{\Phi_{p-1}}{\Delta\tau} \quad \text{in } 0 \le X \le 1 \text{ for fixed } \Delta\tau \text{ and } p = 1,2,... \tag{9}$$

From a physical perspective, each equation (9) for the dimensionless temperature variable $\Phi_p, p = 1,2, \ldots$ may be viewed as a "quasi-stationary" heat conduction equation attached to a fixed dimensionless time $\tau_{p,}\ p = 1,2, \ldots$ and the line p = 1,2,... Since the ordinary differential equation of second order (9) is linear, it can be integrated with analytical methods.

Likewise, the boundary conditions from eq. (7a) and (7b) are converted into

$$\frac{d\Phi_p(0)}{dX} = 0 \tag{10a}$$

$$\frac{d\Phi_p(1)}{dX} = 1 \tag{10b}$$

for $p = 1, 2$ ....

The main advantage of the hybrid TMOL over the standard finite difference methods (explicit, implicit and Crank-Nicolson) is that the truncation errors are quantified in terms of the

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* Recall that the MDT is unaffected by the dimensionless space coordinate interval ΔX.

Let us begin the implementation of the MDT in eq. (9) with the first time jump $p = 1$, which is linked to a dimensionless time $\tau_1 = \Delta\tau$ placed at line 1 in Figure 1. Correspondingly, the first adjoint ordinary differential equation of second order is

$$\frac{d^2\Phi_1}{dX^2} + \frac{n}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = -\frac{\Phi_0}{\Delta\tau} \tag{11}$$

Next, the initial condition $\Phi\,(X, 0) = 0 = \Phi_0$ taken from eq. (6) is introduced into eq. (11) gives way to the homogeneous, "quasi-stationary" heat equation:

$$\frac{d^2\Phi_1}{dX^2} + \frac{n}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{12}$$

along with the applicable boundary conditions

$$\frac{d\Phi_1(0)}{dX} = 0 \tag{13a}$$

$$\frac{d\Phi_1(1)}{dX} = 1 \tag{13b}$$

Herein, the subscript 1 in Φ designates the first dimensionless time jump at the line 1.

6. Quasi–Stationary" Heat Equations

The ensuing "quasi-stationary" heat equations at the dimensionless time τ_1 placed at line 1 are listed next

1) Large plate

For n = 0, eq. (12) with constant coefficients

$$\frac{d^2\Phi_1}{dX^2} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{14}$$

From here, the dimensionless semi-analytical surface temperature distribution is subject to the boundary conditions given in eqs. (13a) and (13b).

The particular solution of eq. (14) delivers the dimensionless semi–analytical temperature distribution

$$\Phi_1(X, \Delta\tau) = \left[\frac{\sqrt{\tau}}{\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right]\cosh\left(\frac{X}{\sqrt{\Delta\tau}}\right) \tag{15}$$

From here, the dimensionless semi-analytical surface temperature distribution is

$$\Phi_{1,s}(\Delta\tau) = \Phi_1(1, \Delta\tau) = \frac{\sqrt{\Delta\tau}}{\tanh\left(\frac{1}{\sqrt{\Delta\tau}}\right)} \tag{16}$$

and the dimensionless semi-analytical center temperature distribution is

$$\Phi_{1,C}(\Delta\tau) = \Phi_1(0, \Delta\tau) = \frac{\sqrt{\Delta\tau}}{\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)} \tag{17}$$

In addition, the dimensionless mean temperature distribution is defined as

$$\Phi_{1,m}(\Delta\tau) = \int_0^1 \Phi_1(X,\Delta\tau)\ dX \tag{18}$$

where $\Phi_1(X,\Delta\tau)$ comes from eq. (15). Alternatively, the dimensionless mean temperature distribution $\Phi_{1,m}(\Delta\tau)$ can be obtained exactly rom the so-called zero-dimensional heat equation. At the end, the resulting algebraic equation with slope one is

$$\Phi_{1,m}(\tau) = \Delta\tau \tag{19}$$

which is valid for the entire time domain $0 < \tau < \infty$ in conformity to Thermodynamics.

2) Long cylinder

For $n = 1$, eq. (12) having variable coefficients is called the modified Bessel equation

$$\frac{d^2\Phi_1}{dX^2} + \frac{1}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{20}$$

which is subject to the boundary conditions given in eqs. (13a) and (13b).

The particular solution of eq. (20) delivers the dimensionless semi-analytical temperature distribution

$$\Phi_1(X,\Delta\tau) = \left[\frac{\sqrt{\tau}}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right] I_0\left(\frac{X}{\sqrt{\Delta\tau}}\right) \tag{21}$$

where I_0 is the modified Bessel functions of second kind and order zero and I_1 is the modified Bessel functions of second kind and order one.

From here, the dimensionless semi–analytical surface temperature distribution is

$$\Phi_{1,S}(\Delta\tau) = \Phi_1(1,\Delta\tau) = \frac{\sqrt{\Delta\tau}\, I_0\left(\frac{1}{\sqrt{\Delta\tau}}\right)}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)} \tag{22}$$

and the dimensionless semi-analytical center temperature distribution is

$$\Phi_{1,C}(\Delta\tau) = \Phi_1(0,\Delta\tau) = \frac{\sqrt{\Delta\tau}}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)} \tag{23}$$

In addition, the dimensionless mean temperature distribution is defined by

$$\Phi_{1,m}(\Delta\tau) = 2\int_0^1 \Phi_1(X,\Delta\tau)\, X\, dX \tag{24}$$

where $\Phi_1(X,\Delta\tau)$ comes from eq. (21). Here again, the dimensionless mean temperature distribution $\Phi_{1,m}(\Delta\tau)$ can be obtained exactly from the so-called zero-dimensional heat conduction equation. At the end, the resulting algebraic equation with slope two is

$$\Phi_{1,m}(\tau) = 2\Delta\tau \tag{25}$$

which is valid for the entire time domain $0 < \tau < \infty$ in conformity to Thermodynamics.

3) Sphere

For $n = 2$, eq. (12) has variable coefficients

$$\frac{d^2\Phi_1}{dX^2} + \frac{2}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{26}$$

and is subject to the boundary conditions given in eqs. (13a) and (13b).

The particular solution of eq. (26) furnishes the dimensionless semi-analytical temperature distribution

$$\Phi_1(X,\Delta\tau) = \left[\frac{\sqrt{\Delta\tau}}{\cosh\left(\frac{1}{\sqrt{\Delta\tau}}\right) - \sqrt{\Delta\tau}\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right] \frac{\sinh\left(\frac{X}{\sqrt{\Delta\tau}}\right)}{X} \tag{27}$$

From here, the dimensionless semi-analytical surface temperature distribution is

$$\Phi_{1,S}(\Delta\tau) = \Phi_1(1,\Delta\tau) = \frac{\sqrt{\Delta\tau}}{\coth\left(\frac{1}{\sqrt{\Delta\tau}}\right) - \sqrt{\Delta\tau}} \tag{28}$$

and the dimensionless semi-analytical center temperature distribution is

$$\Phi_{1,C}(\Delta\tau) = \Phi_1(0,\Delta\tau) = \frac{1}{\cosh\left(\frac{1}{\sqrt{\Delta\tau}}\right) - \sqrt{\Delta\tau}\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)} \tag{29}$$

In addition, the dimensionless mean temperature distribution is defined by

$$\Phi_{1,m}(\Delta\tau) = 3\int_0^1 \Phi_1(X,\Delta\tau)\, X^2\, dX \quad (30)$$

where $\Phi_1(X,\Delta\tau)$ is taken from eq. (27). Here again, the analytical dimensionless mean temperature distribution $\Phi_{1,m}(\Delta\tau)$ can be obtained from the so-called zero-dimensional heat conduction equation. At the end, the resulting algebraic equation with slope three is

$$\Phi_{1,m}(\tau) = 3\Delta\tau \quad (31)$$

which is valid for the entire time domain $0 < \tau < \infty$ in conformity to Thermodynamics.

7. Presentation of the First Set of Temperature Results

Notwithstanding, the most important target temperature in the continuous heating of simple bodies (large plate, long cylinder and sphere) with prescribed uniform surface heat flux is the surface temperature, i.e., the highest temperatures at any given time, i.e., $\Phi_s(\tau) = \Phi(1,\tau)$.

1) Large plate

From eq. (A.1), the dimensionless exact, analytical surface temperature reduces to

$$\Phi_s(\tau) = \tau + \frac{1}{3} - 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(\mu_n)}{\mu_n^2} \exp(-\mu_n^2\,\tau) \quad (32)$$

and from eq. (A.4), the dimensionless asymptotic surface temperature becomes

$$\Phi_{s,asy}(\tau) = \tau + \frac{1}{3} \quad (32a)$$

2) Long cylinder

From eq. (A.2), the dimensionless exact, analytical surface temperature reduces to

$$\Phi_s(\tau) = 2\tau + \frac{1}{4} - 2\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \exp(-\mu_n^2\,\tau) \quad (33)$$

and from eq. (A.5), the dimensionless asymptotic surface temperature becomes

$$\Phi_{s,asy}(\tau) = 2\tau + \frac{1}{4} \quad (33a)$$

3) Sphere

From eq. (A.3), the dimensionless exact, analytical surface temperature reduces to

$$\Phi_s(\tau) = 3\tau + \frac{1}{5} - 2\sum_{n=1}^{\infty} \frac{\tan(\mu_n)}{\mu_n^3} \exp(-\mu_n^2\,\tau) \quad (34)$$

and from eq. (A.6), the dimensionless asymptotic surface temperature becomes

$$\Phi_{s,asy}(\tau) = 3\tau + \frac{1}{5} \quad (34a)$$

The infinite series in eqs. (32)-(34) are evaluated numerically with the symbolic algebra software Maple 15 [13] assigning a term-by-term convergence criterion of 0.01%.

The dimensionless threshold time between eqs. (32) and (32a) is set at τ_{th} = 0.3 for the large plate, between eqs. (33) and (33a) is set at τ_{th} = 0.2 for the long cylinder and between eqs. (34) and (34a) is set at τ_{th} = 0.1 for the sphere.

Tables 1, 3 and 5 present the evaluated dimensionless surface temperatures varying with the dimensionless time $\Phi_s(\tau)$ associated with the MDT. The root mean square errors are RMSE = 9.86% in the sub−domain $0 < \tau \le 0.3$ for the large plate, RMSE = 7.44% in the sub-domain $0 < \tau \le 0.2$ for the long cylinder and RMSE = 6.69% in the sub-domain $0 < \tau \le 0.1$ for the sphere.

8. Presentation of the Second Set of Temperature Results

To reduce the numerical errors and to enhance the accuracy of the surface temperature values provided by the MDT, there are two options.

Option 1:

The adjoint ordinary differential equation of second order and inhomogeneous at the dimensionless time $\tau_2 = \tau_1 + \Delta\tau$ should go to two time jumps. Again, usage is made of the two-point backward finite-difference approximation

$$\left.\frac{\partial z}{\partial t}\right|_{t_2} \approx \frac{z_2 - z_1}{\Delta t} + O(\Delta t) \quad (32)$$

where the term $O(\Delta\tau)$ denotes truncation error of first order (Chapra and Canale [14]). Accordingly, this operation leads to the inhomogeneous, "quasi-stationary" heat equation

$$\frac{d^2\Phi_2}{dX^2}+\frac{n}{X}\frac{d\Phi_2}{dX}-\frac{\Phi_2}{\Delta\tau}=-\frac{\Phi_1\,(X,\Delta\tau)}{\Delta\tau} \tag{33}$$

subject to the boundary conditions

$$\frac{d\Phi_2(0)}{dX}=0 \tag{34a}$$

$$\frac{d\Phi_2(1)}{dX}=1 \tag{34b}$$

The term $\frac{\Phi_1\,(X,\Delta\tau)}{\Delta\tau}$ on the RHS of eq. (33) stands for the dimensionless temperature distribution associated with the first time jump $\Delta\tau$. The corresponding equations are

1) Large plate

$$\frac{d^2\Phi_2}{dX^2}-\frac{\Phi_2}{\Delta\tau}=-\left[\frac{1}{\sqrt{\tau}\,\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right]\cosh\left(\frac{X}{\sqrt{\Delta\tau}}\right) \tag{35}$$

2) Long cylinder

$$\frac{d^2\Phi_2}{dX^2}+\frac{1}{X}\frac{d\Phi_2}{dX}-\frac{\Phi_2}{\Delta\tau}=-\left[\frac{1}{\sqrt{\tau}\,I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right]I_0\left(\frac{X}{\sqrt{\Delta\tau}}\right) \tag{36}$$

3) Sphere

$$\frac{d^2\Phi_2}{dX^2}+\frac{2}{X}\frac{d\Phi_2}{dX}-\frac{\Phi_2}{\Delta\tau}=-\left[\frac{1}{\sqrt{\Delta\tau}\cosh\left(\frac{1}{\sqrt{\Delta\tau}}\right)-\Delta\tau\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right]\frac{\sinh\left(\frac{X}{\sqrt{\Delta\tau}}\right)}{X} \tag{37}$$

Inspection of eqs. (35), (36) and (37) demonstrates that Option 1 is very laborious and time consuming.

Option 2:

A short cut approach is to apply regression analysis to the dimensionless surface temperature deviations $\Delta\,\Phi_s$ related to the MDT which are listed in the fourth columns of Table 1, 3, and 5. The outcome delivers the following set of two-part equations

1) Large plate

$$\Phi_{s,ra}\,(\Delta\tau)=\frac{\sqrt{\Delta\tau}}{\tanh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}+\;0.085\;(\Delta\tau)^{0.34} \tag{38}$$

with correlation coefficient R = 0.99.

2) Long cylinder

$$\Phi_{s,ra}\,(\Delta\tau)=\frac{\sqrt{\Delta\tau}\;I_0\left(\frac{1}{\sqrt{\Delta\tau}}\right)}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)}+\;0.053\;(\Delta\tau)^{0.218} \tag{39}$$

with correlation coefficient R = 0.968

3) Sphere

$$\Phi_{s,ra}\,(\Delta\tau)=\frac{\sqrt{\Delta\tau}}{\coth\left(\frac{1}{\sqrt{\Delta\tau}}\right)-\sqrt{\Delta\tau}}+0.04\;(\Delta\tau)^{0.17} \tag{40}$$

with correlation coefficient R = 1.0

The root mean square errors associated with the MDT complemented with regression analysis are RMSE = 0.21% (47 times less than MDT alone) for the large plate in the sub-domain $0<\tau\leq 0.3$, RMSE = 0.08% (93 times less than MDTalone) for the long cylinder in the sub-domain $0<\tau\leq 0.2$ and RMSE = 0.07% (97 times less than MDT alone) for the sphere in the sub-domain $0<\tau\leq 0.1$.

Table 1. Values of the dimensionless surface temperature $\Phi_s(\tau)$ in the sub-domain $0<\tau\leq 0.3$ obtained with MDT for a large plate

Dimensionless time τ	Exact analytical Φ_s, eq. (32)	Semi-analytical MDT Φ_s, eq. (16)	Deviation $\Delta\Phi_s$ (Relative error)	Asymptotic $\Phi_{s,asy}$, eq. (32a)	Deviation $\Delta\Phi_s$ (Relative error)
0	0	0			
0.1	0.35683	0.31736	−0.03947 (−11.06%)		
0.2	0.50517	0.45755	−0.04762 (−9.43%)	0.5333	0.02816 (5.58%)
0.3 threshold	0.63379	0.57691	−0.5688 (−8.97%)	0.63333	−0.00046 (−0.07%)
			RMSE..= 9.86%		

Table 2. Values of the dimensionless surface temperature $\Phi_s(\tau)$ in the sub-domain $0 < \tau \leq 0.3$ for a large plate obtained with MDT complemented with regression analysis

Dimensionless time τ	Exact, analytical Φ_s, eq. (32)	Semi analytical MDT complemented with regression analysis $\Phi_{s,wra}$, eq. (38)	Deviation $\Delta\Phi_s$ (Relative error)
0	0	0	
0.1	0.35683	0.35621	−0.00062 (−0.17%)
0.2	0.50517	0.50673	0.00156 (0.31%)
0.3 threshold	0.63379	0.63336	−0.00043 (−0.07%)
			RMSE = 0.21%

Table 3. Values of the dimensionless surface temperature $\Phi_s(\tau)$ in the sub-domain $0 < \tau \leq 0.2$ obtained with MDT for a long cylinder

Dimensionless time τ	Exact analytical Φ_s, eq. (33)	Semi-analytical MDT Φ_s, eq. (22)	Deviation $\Delta\Phi_s$ (Relative error)	Asymptotic $\Phi_{s,asy}$, eq. (33a)	Deviation $\Delta\Phi_s$ (Relative error)
0	0	0			
0.05	0.28104	0.25424	−0.0268 (−9.54%)		
0.1	0.41833	0.38503	−0.0333 (−7.96%)	0.53333	
0.15	0.53492	0.50063	−0.0343 (−6.41%)	0.55000	0.01508 (2.82%)
0.2 threshold	0.64277	0.61014	−0.0326 (−5.08%)	0.65000	0.00723 (1.12%)
			RMSE..= 7.44%		

Table 4. Values of the dimensionless surface temperature $\Phi_s(\tau)$ in the sub-domain $0 < \tau \leq 0.2$ obtained with MDT complemented with regression analysis for a long cylinder

Dimensionless time τ	Exact Analytical Φ_s, eq. (33)	Semi analytical MDT complemented with regression analysis $\Phi_{s,wra}$, eq. (39)	Deviation $\Delta\Phi_s$ (Relative error)
0	0	0	
0.05	0.28104	0.53548	0.00056 (0.10%)
0.1	0.41833	0.53548	0.00056 (0.10%)
0.15	0.53492	0.53548	0.00056 (0.10%)
0.20 threshold	0.64277	0.64274	− 0.00003 (−0.005%)
			RMSE = 0.08%

Table 5. Values of the dimensionless surface temperature $\Phi_s(\tau)$ in the sub-domain $0 < \tau \leq 0.1$ obtained with MDT for a sphere

Dimensionless time τ	Exact analytical Φ_s, eq. (34)	Semi-analytical MDT..Φ_s, eq.(28)	Deviation $\Delta\Phi_s$ (Relative error)	Asymptotic $\Phi_{s,asy}$, eq. (34a)	Deviation $\Delta\Phi_s$ (Relative error)
0	0	0			
0.05	0.31217	0.28791	−0.024 (7.7%)		
0.1 threshold	0.48676	0.46006	−0.027 (5.5%)	0.50	0.01324 (2.72%)
			RMSE..= 6.69%		

Table 6. Values of the dimensionless surface temperature $\Phi_s(\tau)$ for a sphere in the sub-domain $0 < \tau \leq 0.1$ obtained with MDT complemented with regression analysis

Dimensionless time τ	Exact, analytical Φ_s, eq. (34)	Semi analytical MDT complemented with regression analysis ,$\Phi_{s,wra}$, eq. (40)	Deviation $\Delta\Phi_s$ (Relative error)
0	0	0	
0.05	0.31217	0.31195	−0.00022 (0.07%)
0.1 threshold	0.48676	0.48710	0.00034 (0.07%)
			RMSE = 0.07%

9. Conclusions

The present study dealt with the analysis of the unsteady heat equation influenced by heat flux boundary conditions in large plates, long cylinders and spheres. The main conclusion that may drawn from the study is that the prediction of surface temperatures provided by the Method of Discretized Time (DMT) complemented with regression analysis of the surface temperature deviations produce levels of exceptional accuracy in the problematic "small time" sub-domain $0 < \tau \leq 0.3$ in the large plate, the "small time" sub-domain $0 < \tau \leq 0.2$ in the long cylinder and the"small time" sub-domain $0 < \tau \leq 0.1$ in the sphere. Conversely, the surface temperatures in the "large time" sub-domains $\tau \geq 0.3$ in the large plate, $\tau \geq 0.2$ in the long cylinder and $\tau \geq 0.1$ in the sphere can be properly handled with the three standard asymptotic equations (32a), (33a) and (34a) for the large plate, the long cylinder and the sphere, respectively.

Nomenclature

c_v specific heat capacity at constant volume

k hermal conductivity

n geometric parameter: 0 for large plate, 1 for long cylinder and 2 for sphere

q_S rface heat flux

R characteristic length: half-thickness of large plate, radius of long cylinder, radius of sphere

t time

t_{eq} equivalent time, R^2/α

t_{th} threshold time

T temperature

T_c center temperature

T_{eq} equivalent temperature, $\frac{q_s R}{k}$

T_{in} initial temperature

T_m mean temperature

T_s surface temperature

x space coordinate

X dimensionless space coordinate, $\frac{x}{R}$

Greek letters

α thermal diffusivity, $\frac{k}{\rho c_v}$

τ dimensionless time or Fourier number, $\frac{t}{t_{eq}}$

τ_{th} dimensionless threshold time, $\frac{t_{th}}{t_{eq}}$

Φ dimensionless temperature, $\frac{T-T_{in}}{T_{eq}}$

ρ density

Subscripts

asy asymptotic

wra with regression analysis

1 first time jump, line 1

2 second time jump, line 2

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APPENDIX: THE EXACT, ANALYTICAL TEMPERATURE DISTRIBUTIONS

The exact, analytical temperature distributions in dimensionless form for the three simple bodies are given by the following infinite series in Luikov [2]:

1) Large plate

$$\Phi(X,\tau) = \tau + \left(\frac{X^2}{2} - \frac{1}{6}\right) - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\mu_n^2} \cos(\mu_n X)\ \exp(-\mu_n^2\ \tau) \tag{A.1}$$

where the eigenvalues are the sequence

$$\mu_n = n\pi, \quad n = 1, 2, 3, \ldots. \tag{A.1a}$$

2) Long cylinder

$$\Phi(X,\tau) = 2\tau + \left(\frac{X^2}{2} - \frac{1}{4}\right) - 2\sum_{n=1}^{\infty} \frac{1}{\mu_n^2 J_0(\mu_n)} J_0(\mu_n X) \exp(-\mu_n^2\ \tau) \tag{A.2}$$

in which the eigenvalues μ_n are the positive roots of the transcendental equation

$$J_1(\mu_n) = 0, \quad n = 1, 2, 3, \ldots. \tag{A.2a}$$

where $J_1(*)$ is the Bessel function of the first kind and order one.

3) Sphere

$$\Phi(X,\tau) = 3\tau + \left(\frac{X^2}{2} - \frac{3}{10}\right) - 2\sum_{n=1}^{\infty} \frac{1}{\mu_n^2 \cos(\mu_n)} \frac{\sin(\mu_n X)}{\mu_n X} \exp(-\mu_n^2\ \tau) \tag{A.3}$$

where the eigenvalues μ_n are the positive roots of the transcendental equation

$$\mu_n = \tan \mu_n, \quad n = 1, 2, 3, \ldots. \tag{A.3a}$$

Approximate analytical solutions for "very large time"

The common exponential function $\exp(-\mu_n^2\ \tau); n = 1,2,\ldots$ appearing in eqs. (A.1), (A.2) and (A.3) decays rapidly with the dimensionless time τ. Thre particular situation is tied up to very large dimensionless time $\tau \to \infty$ wherein the subsequent $\exp(-\mu_n^2\ \tau); n = 1,2,\ldots$ vanishes. This signifies that the infinite series in eqs. (A.1), (A.2) and (A.3) reduce to the corresponding asymptotic algebraic equations

1) Large plate

$$\Phi_{asy}(X,\tau) = \tau + \left(\frac{X^2}{2} - \frac{1}{6}\right) \tag{A.4}$$

2) Long cylinder

$$\Phi_{asy}(X,\tau) = 2\tau + \left(\frac{X^2}{2} - \frac{1}{4}\right) \tag{A.5}$$

3) Sphere

$$\Phi_{asy}(X,\tau) = 3\tau + \left(\frac{X^2}{2} - \frac{3}{10}\right) \tag{A.6}$$