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M-ideals of a Uniquely Representable De Morgan Quasiring

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Abstract

The concept of a (Boolean) quasiring was introduced by Dorninger, Langer and Maczynski in order to get a ring-like counterpart of an orthomodular lattice. Various types of such quasirings were described and compared by the authors in “A note on orthopseudorings and Boolean quasirings”. A similar way was applied when ring-like structures were assigned to De Morgan algebras in Chajda I, Eigenthaler G. (2008). The resulting ring-like structures were called De Morgan quasirings. De Morgan quasirings are used as algebraic models in the foundations of Lukasiewicz logic, constructive logic with strong negation. Chajda and Eigenthaler established a common axiom system for both De Morgan quasirings and De Morgan algebras and showed how an interval of a De Morgan algebra (or De Morgan quasiring) can be viewed as a De Morgan algebra (or De Morgan quasiring, respectively). In this paper, we give a characterization of the M-ideals within a uniquely representable De Morgan quasiring R and show that all the M-ideals of R form a distributive lattice under set inclusion.

Keywords

Uniquely representable, normal De Morgan quasiring, M-ideal, Boolean M-ideal

1. Introduction and notation

A study of ring-like structures which are generalizations of Boolean rings has been initiated by Dorninger, Langer and Maczynski in [1] and later developed in [2] and [3]. These ring-like structures are called Boolean quasirings, generalized Boolean quasirings and orthopseudorings in [4-8] and their mutual relations were investigated in [5, 9].

A De Morgan algebra is a bounded distributive lattice with an antitone involution, which is reduct of the algebras of various nonclassical logics such as Lukasiewicz algebras, Nelson algebras (For concepts concerning lattice theory we refer the reader to the monographs ([10-12]). The correspondence between De Morgan algebras and so-called De Morgan quasirings is obtained in [13, 14].

In the present paper, we introduce the notion of M-ideal within a uniquely representable De Morgan quasiring R. We show that all the M-ideals of R form a distributive lattice under set inclusion for a uniquely representable normal De Morgan quasiring R. We pay special attention to Boolean M-ideals and prove that if a uniquely representable De Morgan quasiring has more than one Boolean M-ideals then they are ring isomorphic to each other.

Recall that a Boolean ring with unit is a unitary ring $R=(R,+,\cdot,0,1)$ that is idempotent, *i.e.* the identity $xx=x$ holds. It is well-known that every Boolean ring is commutative and of characteristic 2, *i.e.*, the identity $x+x=0$ holds.

In [13], a De Morgan quasiring was introduced as an algebra $R=(R,+,\cdot,0,1)$ of type $(2,2,0,0)$ which contains two elements 0 and 1 such that the following laws (1)-(7) hold:

$$xx = x \tag{1}$$

$$xy = yx \tag{2}$$

$$(xy)z = x(yz) \tag{3}$$

$$1x = x \tag{4}$$

$$0x = 0 \tag{5}$$

$$1+0=1, 1+1=0 \tag{6}$$

$$x\left[1+(1+y)(1+z)\right]=1+(1+xy)(1+xz) \tag{7}$$

A De Morgan quasiring R is uniquely representable (cf. [1]) if the following identity holds in R.

$$x+y=(1+(1+x)(1+y))(1+xy). \tag{8}$$

As in [13], if one defines in a uniquely representable De Morgan quasiring R $x \vee y := 1+(1+x)(1+y), x \wedge y := xy$ and $x' := 1+x$, the algebra $\mathbf{L}(R) := (R;\vee,\wedge,',0,1)$ is a De Morgan algebra. On the other hand, if one starts with a De Morgan algebra $(L;\vee,\wedge,',0,1)$ and defines $x+y := (x' \vee y') \wedge (x \vee y)$ and $xy := x \wedge y$, then $\mathbf{R}(L) := (L;+,\cdot,0,1)$ is a uniquely representable De Morgan quasiring. Furthermore, $\mathbf{R}(\mathbf{L}(R)) = R$ and $\mathbf{L}(\mathbf{R}(L)) = L$.

2. Characterization of M-ideals within a uniquely representable De Morgan quasiring

In order to characterize the M-ideals of a uniquely representable De Morgan quasiring, we require the following results and definitions.

Lemma 2.1 Let R be a uniquely representable De Morgan quasiring, then R satisfies the following:

$$1+(1+x)=x \tag{9}$$

$$1+(1+x)(1+xy)=x \tag{10}$$

$$x+0=x \tag{11}$$

$$x+y=y+x \tag{12}$$

$$x[1+(1+x)(1+y)]=x \tag{13}$$

$$x+1=1+x \tag{14}$$

Proof (9)-(12) follow from Lemma 1 and Corollary 1 in [13].
(13) follows from (1), (2), (7) and(10).
(14) follows from (5), (6) and (8).

Definition 2.1 Let R be a uniquely representable De Morgan quasiring. The nonempty subset $I \subseteq R$ is said to be an L-ideal of R if

(i) $(a+1)(b+1)+1 \in I$ for all $a,b \in I$;

(ii) $ra \in I$ for all $r \in R, a \in I$.

The nonempty subset $F \subseteq R$ is said to be an L-filter of R if

(i) $ab \in F$ for all $a,b \in F$;

(ii) $(a+1)(r+1)+1 \in F$ for all $r \in R, a \in F$.

An L-filter F of R is proper if $F \neq R$.

Obviously, if $I \subseteq R$ is an L-ideal of R then $a+b \in I$ for all $a,b \in I$.

Definition 2.2 Let R be a uniquely representable De Morgan quasiring. A nonempty subset $M \subseteq R$ is said to be an M-ideal of R if

(i) $a+1 \in M$ for all $a \in M$;

(ii) $ab \in M$ for all $a,b \in M$;

(iii) $a(a+1)+r[a(a+1)+1] \in M$ for all $r \in R, a \in M$.

An M-ideal M of R is proper if $M \neq R$.

Remark If R is a Boolean ring, then R has no proper M-ideal.

The following Proposition is easily seen.

Proposition 2.1 Let R be a uniquely representable De Morgan quasiring and E be an L-ideal (or L-filter). Then $E^d := \{x+1 \mid x \in E\}$ is an L-filter (or L-ideal). Moreover, $E \cap E^d$ is an M-ideal.

Proposition 2.2 Let R be a uniquely representable De Morgan quasiring and $a \in R$. Then the set

$$M_a := \{a(a+1) + r[a(a+1)+1] \mid r \in R\}$$

is the smallest M-ideal of R which contains a. The M-ideal M_a is called a principal M-ideal generated by a.

Proof If $x \in M_a$ then there exists $r \in R$ such that $x = a(a+1) + r[a(a+1)+1]$. Letting $p = a(a+1)$. Then $p(p+1) = p$ by (9) and (13). Identities (7)-(9) and (13)-(14) together imply

$$\begin{aligned} x = p + r(p+1) &= [1+(1+p)(1+r(p+1))](1+p(p+1)r) = [1+(1+p)(1+r)](1+p(p+1))(1+pr) \\ &= [1+(1+p)(1+r)](1+p)[1+(1+(1+p))(1+(1+r))] = (1+p)[1+(1+p)(1+r)]. \end{aligned}$$

Thus

$$\begin{aligned} x+1 = 1+(1+p)[1+(1+p)(1+r)] &= (1+p(1+p))[1+(1+p)r] \\ &= (1+p)[1+(1+p)r] = (1+p)[1+(1+p)(1+(1+r))] = p + (r+1)(p+1). \end{aligned}$$

Hence $x+1 \in M_a$.

If $x, y \in M_a$ then there exist $r_1, r_2 \in R$ such that $x = p + r_1(p+1), y = p + r_2(p+1)$. Identities (7), (8), (9) and (13) together imply

$$\begin{aligned} xy &= (1+p)[(1+p)(1+r_1)+1](1+p)[(1+p)(1+r_2)+1] \\ &= (1+p)[1+(1+p)(r_1 r_2 + 1)] \\ &= p + (r_1 r_2)(p+1). \end{aligned}$$

Thus $xy \in M_a$. Likely, for all $s \in R, x \in M_a$, we have $x(x+1) + s[x(x+1)+1] \in M_a$. Moreover, $a(a+1) + a[a(a+1)+1] = a(a+1) + a = a$ Thus $a \in M_a$.

Now suppose that M is an M-ideal of R which contains a. Then, by Definition 2.2, $a(a+1) + r[a(a+1)+1] \in M$ for all $r \in R$. Hence $M_a \subseteq M$.

In what follows, we shall be concerned with the structure of all the M-ideals in a uniquely representable De Morgan quasiring.

Proposition 2.3 Let R be a uniquely representable De Morgan quasiring and $a, b \in R$. Then

$$\{ab(a+1)(b+1) + r[ab(a+1)(b+1)+1] \mid r \in R\}$$

is an M-ideal of R generated by a and b. We denote the M-ideal by $M_a \vee M_b$.

Proof By a similar argument as in the proof of Proposition 2.2, we can also see that

$$\{ab(a+1)(b+1) + r[ab(a+1)(b+1)+1] \mid r \in R\}$$

is an M-ideal of R. Noting that $a(1+a(1+a)) = a$. Identities (8), (9) and (13) we have

$$\begin{aligned} ab(a+1)(b+1) + a[ab(a+1)(b+1)+1] &= ab(a+1)(b+1) + a(1+a(1+a))[a(a+1)b(b+1)+1] \\ &= ab(a+1)(b+1) + a = \{1+[1+ab(a+1)(b+1)](1+a)\}\{1+ab(a+1)(b+1)\} \\ &= \{1+(1+a)[1+a(a+1)][1+ab(a+1)(b+1)]\}\{1+ab(a+1)(b+1)\} \\ &= a\{1+ab(a+1)(b+1)\} = a(1+a(1+a))\{1+ab(a+1)(b+1)\} = a. \end{aligned}$$

Hence $a \in M_a \vee M_b$. Likely, $b \in M_a \vee M_b$. Therefore, $M_a, M_b \subseteq M_a \vee M_b$.

Suppose that M is an M-ideal such that $a, b \in M$. Then we obtain that $a(a+1)b(b+1) + r[a(a+1)b(b+1)+1] \in M$ for all $r \in R$. Consequently, $M_a \vee M_b \subseteq M$. Thus, $M_a \vee M_b$ is generated by a and b.

However, in a uniquely representable De Morgan quasiring R, the intersection of two M-ideals is probably empty.

Example 2.1 Consider the algebra R described as above. It is easy to see that $(R;+,\cdot,0,1)$ is a uniquely representable De Morgan quasiring. By a simple calculation we can see that
$M_a \cap M_b = \{a + ra \mid r \in R\} \cap \{b + rb \mid r \in R\} = \{a\} \cap \{b\} = \varnothing$.

+	0	a	b	1
0	0	a	b	1
a	a	a	1	a
b	b	1	b	b
1	1	a	b	0

.	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Definition 2.3 A uniquely representable De Morgan quasiring is called normal if $+$ is associative.

Proposition 2.4 Let R be a uniquely representable normal De Morgan quasiring. Then the intersection of any two M-ideals is not empty.

Proof Suppose that M_1 and M_2 are any two M-ideals of R and $a \in M_1, b \in M_2$. Then by Proposition 2.2, $M_a \subseteq M_1, M_b \subseteq M_2$. Letting $p = a(a+1)$ and $q = b(b+1)$. Then

$$p(p+q) = p[(p+1)(q+1)+1](pq+1) = p(pq+1) = p(p+1)(pq+1) = p(p+1) = p$$

and

$$\begin{aligned} p(p+(q+1)) &= p[(p+1)q+1][p(q+1)+1] = p(p+1)[(p+1)q+1][p(q+1)+1] \\ &= p(p+1) = p. \end{aligned}$$

Similarly, $q(p+q) = q$. Thus from the assumption of normality we have

$$(p+q+1)(q+1) = (p+q+1)(q(p+q)+1) = p+q+1.$$

Hence

$$p = p(p+q+1) = p(p+q+1)(q+1) = p(q+1).$$

Therefore, $q(p+1) = q(p(q+1)+1) = q$ and so

$$\begin{aligned} & p+[(p+1)(q+1)+1](p+1) \\ &= p+\{[p(p+1)+1][(p+1)q+1]+1\} \\ &= p+[(p+1)(q+1)+1] = (p+1)+(p+1)(q+1) \\ &= (p+1)[(p+1)(q+1)+1] = (1+p(p+1))(1+(p+1)q)+1 \\ &= (1+p)\{1+[1+p(q+1)]q\}+1 = (p+1)(q+1)+1. \end{aligned}$$

Consequently, $(p+1)(q+1)+1 \in M_a$. Similarly, we have $(p+1)(q+1)+1 \in M_b$. Thus $(p+1)(q+1)+1 \in M_a \cap M_b$.

Theorem 2.1 Let R be a uniquely representable normal De Morgan quasiring. Then all the principal M-ideals of R form a distributive lattice under set inclusion.

Proof Suppose that M_a, M_b are any two principal M-ideals of R, where $a, b \in R$. Then by Proposition 2.3 and 2.4, we have $M_a \vee M_b = M_{pq}$ and $(p+1)(q+1)+1 \in M_a \cap M_b$, where $p = a(a+1)$ and $q = b(b+1)$. It then follows from Proposition 2.2 that $M_{(p+1)(q+1)+1} \subseteq M_a \cap M_b$.

Conversely, suppose $x \in M_a \cap M_b$. Then $px = p, qx = q$ and $(p+1)x = x, (q+1)x = x$. It follows that

$$x[(p+1)(q+1)+1] = (xp+1)(xq+1)+1 = (p+1)(q+1)+1$$

and $x(p+1)(q+1)=x$. Hence $(x+1)(p+1)(q+1)=x+1$. Consequently,

$$\begin{aligned}&[(p+1)(q+1)+1]+x(p+1)(q+1)=[(p+1)(q+1)+1]+x\\&=[(p+1)(q+1)(x+1)+1]\{x[(p+1)(q+1)+1]+1\}\\&=[(x+1)+1]\{[(p+1)(q+1)+1]+1\}=x(p+1)(q+1)=x.\end{aligned}$$

Thus $x\in M_{(p+1)(q+1)+1}$, and so we have

$$M_a\cap M_b\subseteq M_{(p+1)(q+1)+1}.$$

Therefore,

$$M_a\cap M_b=M_{(p+1)(q+1)+1}.$$

Therefore all the principal M-ideals of R form a lattice under set inclusion.

Suppose that M_a,M_b,M_c are three principal M-ideals of R. Letting $r=c(c+1)$.

Then

$$M_c\cap(M_a\vee M_b)=M_c\cap M_{pq}=M_{(r+1)(pq+1)+1}$$

and

$$M_c\cap M_a=M_{(r+1)(p+1)+1},\qquad M_c\cap M_b=M_{(r+1)(q+1)+1}.$$

Hence we conclude that

$$(M_c\cap M_a)\vee(M_c\cap M_b)=M_{[(r+1)(p+1)+1][(r+1)(q+1)+1]}.$$

Since

$$[(r+1)(p+1)+1][(r+1)(q+1)+1]=(r+1)(pq+1)+1,$$

we have

$$M_c\cap(M_a\vee M_b)=(M_c\cap M_a)\vee(M_c\cap M_b).$$

Now we shall introduce a special class of the M-ideals in a uniquely representable De Morgan quasiring.

Definition 2.4 For a uniquely representable De Morgan quasiring R and $a\in R$, the M-ideal M_a is said to be a Boolean M-ideal if it is a Boolean ring under the two operations $+$ and $\cdot$ of R.

Remark For a uniquely representable De Morgan quasiring R, the Boolean M-ideals may not exist, even if they exist, they may not be unique. Moreover, a uniquely representable De Morgan quasiring R is a Boolean ring if and only if R R has unique Boolean M-ideal which is R.

Example 2.2 (1) The real interval $[0,1/2)\cup(1/2,1]$ is a uniquely representable De Morgan quasiring, which has no Boolean M-ideal under the following operations

$$x+y:=min\{max\{x,y\},max\{1-x,1-y\}\},\qquad x\cdot y:=min\{x,y\},$$

where max and min mean the binary operations of forming the maximum and the minimum, respectively.

(2) The algebra $(R;+,\cdot,0,1)$ described as follows is a uniquely representable De Morgan quasiring which has unique Boolean M-ideal $\{a\}$.

+	0	a	1
0	0	a	1
a	a	a	a
1	1	a	0

.	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

(3) The De Morgan quasiring R in Example 2.1 has two different Boolean M-ideals $\{a\}$ and $\{b\}$.

Theorem 2.2 If a uniquely representable De Morgan quasiring R has more than one Boolean M-ideals then they are ring isomorphic to each other.

Proof Suppose that M_a and M_b are Boolean M-ideals of R, where $a,b\in R$. Letting $p=a(a+1)$ and $q=b(b+1)$. Then we define the following mappings

$$\varphi: M_a \to M_b, x\to q+(q+1)x, \quad \psi: M_b\to M_a, y\to p+(p+1)y.$$

Now we first show that φ is a ring homomorphism from M_a to M_b. For $x_1,x_2\in M_a$, we have

$$\varphi(x_1+x_2)=[q+(q+1)x_1]+[q+(q+1)x_2]=q+(q+1)(x_1+x_2)=\varphi(x_1+x_2).$$

Similarly, $\varphi(x_1x_2)=\varphi(x_1)\varphi(x_2)$.

To show the mapping φ is a bijection, it suffices to verify that $\psi\varphi(x)=x$ and $\varphi\psi(y)=y$ for every $x\in M_a$ and $y\in M_b$. Note that $\varphi(x+1)=\varphi(x)+1$ and by assumption, we have $\varphi(p)=\varphi(p(p+1))=\varphi(p)\varphi(p+1)=\varphi(p)(\varphi(p)+1)=q$. So $\varphi(p+1)=q+1$. Similarly, $\psi(q)=p$ and $\psi(q+1)=p+1$. On the one hand, by assumption and Proposition 2.2, we have

$$\begin{aligned}
\psi\varphi(x) &= \psi(q+(q+1)x) = p+(p+1)(q+(q+1)x)\\
&=[p+(p+1)q]+[p+(p+1)(q+1)x]=p+[p+(p+1)(q+1)x]\\
&= p+(p+1)(q+1)x = p+x(q+1)\\
&=\{(p+1)[x(q+1)+1]+1\}[px(q+1)+1]\\
&=[(p+1)(x+1)+1][q(p+1)+1][p(q+1)+1]\\
&=[(x+1)+1][q(p+1)+1][p(q+1)+1]\\
&= x[q(p+1)+1]=xx[q(p+1)+1]=x\psi\varphi(x).
\end{aligned}$$

On the another hand,

$$\begin{aligned}
\psi\varphi(x) &= p+(p+1)(q+(q+1)x)\\
&= p+(p+1)(q+1)[(q+1)(x+1)+1]\\
&=[p+(p+1)(q+1)]\{p+(p+1)[(q+1)(x+1)+1]\}\\
&=(p+1)\{p+(p+1)[(q+1)(x+1)+1]\}\\
&=(p+1)\{p+[((p+1)q+1)(x(p+1)+1)+1]\}\\
&=(p+1)\{p+[((p+1)q+1)(x+1)+1]\}\\
&=(p+1)[(p+1)((p+1)q+1)(x+1)+1]\{p[((p+1)q+1)(x+1)+1]+1\}\\
&=(p+1)[(x+1)((p+1)q+1)+1]
\end{aligned}$$

and whence $x\psi\varphi(x)=x(p+1)[(x+1)((p+1)q+1)+1]=x.$ Thus, $\psi\varphi(x)=x$. Similarly, $\varphi\psi(y)=y$, exactly as required.

References

[1] Dorninger D, Langer H, Maczynski M. (1997). The logic induced by a system of homomorphisms and its various algebraic characterizations. Demonstr Math, 30:215-232.

[2] Dorninger D. (1998). Sublogics of ring-like quantum logics. Tatra Mt Math Publ, 15:75-83.

[3] Langer H. (1998). Generalizations of the correspondence between Boolean algebras and Boolean rings to orthomodular lattices. Tatra Mt Math Publ, 15:97-105.

[4] Chajda I. (1996). Pseudosemirings induced by ortholattices. Czechoslov Math J, 46:405-411.

[5] Chajda I, Eigenthaler G. (1998). A note on orthopseudorings and Boolean quasirings. Österr Akad Wiss Math Natur Kl Sitzungsber II, 207:83-94.

[6] Chajda I, Langer H. (2007). A common generalization of ortholattices and Boolean quasirings. Demonstr Math, 40:769-774.

[7] Chajda I, Langer H, Maczynski M. (2004). Ring-like structures corresponding to generalized orthomodular lattices. Math Slovaca, 54:143-150.

[8] Dorninger, Langer H, Maczynski M. (2000). Lattice properties of ring-like quantum logics. Int J Theor Phys, 39:1015-1026.

[9] Chajda I, Langer H. (2018). When does a generalized Boolean quasiring become a Boolean ring? Soft Computing, 22:6877-6879.

[10] Balbes R, Dwinger P. (1974). Distributive Lattices. University of Missouri Press, Missouri.

[11] Blyth T S, Varlet J C. (1994). Ockham Algebras. Oxford University Press, Oxford.

[12] Gratzer G. (2011). Lattice theory: foundation. Birkhauser, Basel.

[13] Chajda I, Eigenthaler G. (2008). De Morgan quasirings, Contributions to General Algebra 18. Verlag J.Heyn, 17-21.

[14] Chajda I, Eigenthaler G. (2009). Two constructions of de Morgan algebras and de Morgan quasirings. Discuss. Math. Gen. Algebra Appl, 29: 169-180.

[15] Dorninger D, Langer H. (2007). Term Extensions of Partial Ring-Like Quantum Logics. Int. J. Contemp. Math. Sciences, 21:999-1008.