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Some New Integer Sequences of Transitive Relations

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Abstract

Enumerative Combinatorics is the study of methods and problems related to enumeration or counting objects of various finite sets. Among several open problems in enumerative combinatorics is the problem of counting transitive relations on a set. In this paper, we discuss three problems closely related to the open problem of counting transitive relations on a finite set. These are the problems of counting the number of transitive but not symmetric relations on a set, that of counting transitive relations involving all the elements of a finite set, and that of counting transitive relations that involve a specific element of a set. We highlight the inclusion of three new sequences to the Online Encyclopedia of Integer Sequences (OEIS) that correspond to these special kinds of transitive relations. We also tabulate the first seventeen terms of each of these three sequences. The paper can be viewed as a demonstration also. The ideas demonstrated in this paper can be used as instances for giving rise to more related combinatorial problems from a given problem.

Keywords

Transitive Relations, Symmetric Relations, Equivalence Relations, Enumeration, Counting

1. Introduction

Enumerative Combinatorics has been thoroughly written on [1-3]. In fact, good literature continues to be added to the already-rich database of the available resources [4-5]. One of the most exciting parts of this subject of study is its deceptive look. Problems in Enumerative Combinatorics often appear to be simple and do not call for any special or sophisticated expertise to understand them. However, solutions of these problems often require involved combinatorial techniques and considerable expertise.

No explicit formula or a recursive relation is known to exist for the count of all relations on a set that are transitive. But some efforts [6-12] have been made in this direction.

Throughout the upcoming discussion, $t(n)$ denotes the number of transitive relations on a finite set with n elements.

2. Number of transitive but not symmetric relations on an n-set

A beautiful result concerning the number of relations that are both symmetric and transitive was discussed in [10]. This result says that the number of relations that are both symmetric and transitive on a set with n elements equals the number of equivalence relations (that is relations that are reflexive, symmetric, and transitive) on a set with $n+1$ elements. Consequently, the open problem of counting transitive relations on a finite set can safely or equivalently be targeted by attempting to count all relations on a finite set that are transitive but not symmetric.

The sequence counting relations that are transitive but not symmetric on an n-set has now been included in the OEIS by the author of this paper. It has been assigned the A-number A345317. The first seventeen terms of the sequence are tabulated as under (Table 1):

Table 1. Table for the number of transitive-but-not-symmetric relations of a set

n	A345317(n)
0	0
1	0
2	8
3	156
4	3942
5	154100
6	9414312
7	878218390
8	122207682476
9	24890747805972
10	7307450298831718
11	3053521546328889460
12	1797003559223742679800
13	1476062693867018935173990
14	1679239558149570227773844452
15	2628225174143857306613215434524
16	5626175867513779058706923151723150

3. Number of transitive relations involving all the elements of an n-set

Not all transitive relations involve all the elements of a set. For example, on the set $\{1,2,3\}$, the relation $\{(1,2),(3,3)\}$ is transitive and involves at least one presence of each element of the set, that is it involves all the elements of the set. However, the relation $\{(1,2)\}$ is transitive but does not involve all the elements of the set.

Is it possible to count all the transitive relations on a set that involve all the elements of an n-set?

It turns out that this problem is closely knitted to the problem of counting all the transitive relations on a set. As a matter of fact, this number is expressible as

$$\sum_{k=0}^{n}(-1)^k\, C(n,k)t(n-k)$$

The sequence corresponding to it has also now been included in the OEIS by the author of this paper. It has been assigned the A-number A348137. The first seventeen terms of the sequence are tabulated as under (Table 2):

Table 2. Table for the number of transitive relations involving all the elements of a set

n	A348137(n)
0	1
1	1
2	10
3	137
4	3381
5	135922
6	8546045
7	815422505
8	115437178060
9	23821722677391
10	7063938719374373
11	2974488705436714248
12	1760838176228838354751
13	1452937749988032952760937
14	1658737103542768935354921618
15	2603190753864086778265813466485
16	5584324950136613655245377359839793

4. Number of transitive relations involving a particular element of an n-set

Consider two transitive relations on the set $\{1,2,3\}$. The first of these transitive relations $\{(1,2),(3,3)\}$ involves the element 3, but the other $\{(1,2)\}$ does not.

The enumeration of all such relations that contain or involve a specific element of the set is a problem very closely related to the problem of counting all the transitive relations.

It turns out that if $t(n)$ is the number of all the transitive relations on an n-set, then $t(n+1)$ counts transitive relations on $n+1$ elements, and consequently, $t(n+1)-t(n)$ counts all the transitive relations on a set with $n+1$ elements that involve a specific element out of these $n+1$ elements of the set.

The sequence corresponding to it stands now included in the OEIS by the author of this paper. It has been assigned the A-number A348151. The first seventeen terms of the sequence are tabulated as under (Table 3):

Table 3. Table for the number of transitive relations involving a particular element of a set

n	A348151(n)
0	0
1	1
2	11
3	158
4	3823
5	150309
6	9260886
7	868807341
8	121329481093
9	24768540218324
10	7282559551588341
11	3046214096033592769
12	1793950037677437221180
13	1474265690307795355749075
14	1677763495455703210030729685
15	2626545934585707736394538773674
16	5623547642339635201400382321016283

5. Conclusion

Mathematics is about problems and their solutions. As is often the case, the pursuit of the solution of one problem introduces us to a plethora of other related problems. In this paper, it has been shown/demonstrated how the age-old problem of counting all transitive relations on a finite set has opened a gateway to many other closely related problems, three of which have been discussed here. A deeper understanding of any sequence from the Online Encyclopedia of Integer Sequences (OEIS) can lead to a manipulation providing both new sequences related to the problem at hand and new insights into the nature and a possible solution of the problem. And it is with this thing in mind that an invitation is sent, in this paper, to all the readers. This invitation is the invitation to manipulate terms of an integer sequence to get wind of new facts and varied perspectives from which an open problem, such as that of counting transitive relations, could be viewed and attacked.

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