Journal of Applied Mathematics and Computation, 2023, 7(1), 112-127
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

Asymptotic Behavior of Delayed SIR Epidemic Models of COVID-19 with Diffusion

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How to cite this paper: Kaori Saito, Toshiyuki Kohno, Yoshihiro Hamaya. (2023) Asymptotic Behavior of Delayed SIR Epidemic Models of COVID-19 with Diffusion. *Journal of Applied Mathematics and Computation*, 7(1), 112-127.
DOI: 10.26855/jamc.2023.03.012

Received: March 15, 2023
Accepted: April 12, 2023
Published: May 4, 2023

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Abstract

Some mathematical epidemic equation of SIR with diffusion, which appears as a model of COVID-19 in China for the spread of disease-causing, is treated. While many studies have investigated the COVID-19 models, local asymptotic stability of equilibrium points and bifurcation of periodic solutions, we have not come across a paper that deals with the asymptotic stability criteria of equilibrium points with time delay and space-diffusion. The asymptotic properties of the diffusive equation have studied by applying the technique of strong maximum principle, strong fading memory property and a luxury Lyapunov functional. Moreover, we feel that our paper is real interested original result for COVID-19 models.

Keywords

Delayed SIR epidemic model of COVID-19 with diffusion, Global asymptotic stability, Strong maximum principle

1. Introduction

The space and time dependent Susceptible, Exposed, Infectious and Removed (SEIR) model was proposed and applied to fit and then predict the space and time series of COVID-19 diffusive evolution observed in the last year till 4/30/2020 in various provinces and metropolises in China. The validated SEIR models which have responded differently to monitoring and mitigating COVID-19 so far, although these predictions contain high uncertainty due to the intrinsic change of the maximum infected population and infection/removed rates within the different countries. Mathematical models are among the necessary tools to quantify the COVID-19 dynamics and are the primary objective motivating this study. To address the questions mentioned above, this study is organized as follows. However, the SEIR model is 4-dimension system and it is unsuitable for analysis. As we canconsider that "E" implies almost equivalent to time delay of "I", we employ the SIR model with finite time delay to infections. Section 2 proposes an updated SIR model for COVID-19, where "S", "I" and "R" stand for Susceptible, Infections and Removed people, respectively [cf. 1, 2, 3, 4 and 5]. This model is then applied to fit and predict the COVID-19 spread in various provinces and major cities in China, resulting in abundant datasets to derive the core characteristics of the COVID-19 dynamics of transmission/infection and remove, because it change the term "E" stands for Exposed people to the term "I" with finite time delay.

In this paper, we shall consider the following diffusive system with boundary condition

$$\frac{\partial S}{\partial t}(t,x) = d\Delta S(t,x) - \beta\frac{S(t,x)I(t-h,x)}{N_1} - \mu_1 S(t,x) + b$$

$$t > 0, \quad x \in \Omega,$$

$$\frac{\partial I}{\partial t}(t,x) = d\Delta I(t,x) + \beta\frac{S(t,x)I(t-h,x)}{N_1} - (\mu_2 + \lambda)I(t,x)$$

$$\mathbf{t} > 0, \quad \mathbf{x} \in \mathbf{\Omega},$$

$$\frac{\partial \mathbf{R}}{\partial \mathbf{t}}(\mathbf{t},\mathbf{x}) = \mathbf{d}\Delta\mathbf{R}(\mathbf{t},\mathbf{x}) + \boldsymbol{\lambda}\mathbf{I}(\mathbf{t},\mathbf{x}) - \boldsymbol{\mu}_3\mathbf{R}(\mathbf{t},\mathbf{x}) \quad \mathbf{t} > 0, x \in \mathbf{\Omega}, \tag{1}$$

$$\frac{\partial \mathbf{S}}{\partial \mathbf{n}}(\mathbf{t},\mathbf{x}) = \frac{\partial \mathbf{I}}{\partial \mathbf{n}}(\mathbf{t},\mathbf{x}) = \frac{\partial \mathbf{R}}{\partial \mathbf{n}}(\mathbf{t},\mathbf{x}) = \mathbf{0} \quad \mathbf{t} > 0, x \in \partial\mathbf{\Omega},$$

Where $\mathbf{S}(\mathbf{t},\mathbf{x}) + \mathbf{I}(\mathbf{t},\mathbf{x}) + \mathbf{R}(\mathbf{t},\mathbf{x}) \equiv \mathbf{N_1}(\mathbf{t},\mathbf{x})$denotes the total number of a population at time t and space $\mathbf{x}$. Here $\Delta$is the Laplacian in $\mathbf{R^3}, \mathbf{\Omega} \subset \mathbf{R^3}$ is a bounded domain with smooth boundary $\partial\mathbf{\Omega}$ and $\partial/\partial\mathbf{n}$ is the outward normal derivative to $\partial\mathbf{\Omega}$. For the equation (1), $\mathbf{1} > d = mean\{\mathbf{d_S}, \mathbf{d_I}, \mathbf{d_R}\} > \mathbf{0}$ that the diffusion coefficients of each $\{\mathbf{d_S}, \mathbf{d_I}, \mathbf{d_R}\}$ for $\{\mathbf{S}, \mathbf{I}, \mathbf{R}\}$ is average constants, because $\mathbf{1} > \mathbf{d_S} > \mathbf{d_I} > \mathbf{d_R} \gg \mathbf{0}$. $\mathbf{S} \coloneqq \mathbf{S}(\mathbf{t},\mathbf{x})$ denotes the number of the population susceptible to the disease, $\mathbf{I} \coloneqq \mathbf{I}(\mathbf{t},\mathbf{x})$ denotes the number of infectious individual and $\mathbf{R} \coloneqq \mathbf{R}(\mathbf{t},\mathbf{x})$ denotes the number who have been removed from the possibility of infection through full immunity. It is assumed that all new-born is susceptible. The positive constants$\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 + \mathbf{p}, \boldsymbol{\mu}_3 = \boldsymbol{\mu}_1 + \mathbf{q}$, $\mathbf{p}$ and $\mathbf{q}$ are nonnegative constants, represent the death rates of susceptible, infective and removed, respectively. It is biologically natural from the dates of COVID-19 in China to assume that

$$\boldsymbol{\mu}_1 \leq \mathbf{min}\{\boldsymbol{\mu}_2, \boldsymbol{\mu}_3\}.$$

In addition, the positive constants $\mathbf{b}$ and $\boldsymbol{\lambda}$ represent the birth rate of the population and the remove rate of infective, respectively. The positive constant $\boldsymbol{\beta}$ is the average number of contacts per infective per day. The nonnegative constant $\mathbf{h}$ is the time delay of presented the symptoms of a disease. The term $\boldsymbol{\beta}\mathbf{S}(\mathbf{t},\mathbf{x})\mathbf{I}(\mathbf{t}-\mathbf{h},\mathbf{x})/\mathbf{N_1}(\mathbf{t},\mathbf{x})$ can be considered as the force of infection at time $\mathbf{t}$ and space $\mathbf{x}$, respectively. For the detailed biological meanings, refer to [1, 6, 7], [3, 8, 9] and [10, 11].

Historical Motivation. In 1979, for the ordinary differential equation (without time delay), Anderson and May [6] have studied the asymptotic stability of the following SIR epidemic differential equation

$$\frac{\mathbf{dS(t)}}{\mathbf{dt}} = -\boldsymbol{\beta}\mathbf{S(t)I(t)} - \boldsymbol{\mu}\mathbf{S(t)} + \boldsymbol{\mu},$$

$$\frac{\mathbf{dI(t)}}{\mathbf{dt}} = \boldsymbol{\beta}\mathbf{S(t)I(t)} - \boldsymbol{\mu}\mathbf{I(t)} - \boldsymbol{\lambda}\mathbf{I(t)}, \tag{2}$$

$$\frac{\mathbf{dR(t)}}{\mathbf{dt}} = \boldsymbol{\lambda}\mathbf{I(t)} - \boldsymbol{\mu}\mathbf{R(t)}, \quad \mathbf{t} \geq \mathbf{0}$$

where $\mathbf{b}$,$\boldsymbol{\beta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are positive constants. In (2), it assumes that the total number of the population $\mathbf{N(t)}$ is constant, that is $\mathbf{N(t)} = \mathbf{1}$ for all $\mathbf{t} \geq \mathbf{0}$, and that the birth and the death rates of population are the same value.

On the other hand, as the ordinary differential equation of SIR type with time delay, Takeuchi and Ma [10] have shown the global asymptotic stability of the solution $(\mathbf{S(t)}, \mathbf{I(t)}, \mathbf{R(t)})$ of

$$\frac{\mathbf{dS(t)}}{\mathbf{dt}} = -\boldsymbol{\beta}\mathbf{S(t)I(t-h)} - \boldsymbol{\mu}_1\mathbf{S(t)} + \mathbf{b},$$

$$\frac{\mathbf{dI(t)}}{\mathbf{dt}} = \boldsymbol{\beta}\mathbf{S(t)I(t-h)} - \boldsymbol{\mu}_2\mathbf{I(t)} - \boldsymbol{\lambda}\mathbf{I(t)}, \tag{3}$$

$$\frac{\mathbf{dR(t)}}{\mathbf{dt}} = \boldsymbol{\lambda}\mathbf{I(t)} - \boldsymbol{\mu}_3\mathbf{R(t)}, \quad \mathbf{t} \geq \mathbf{0},$$

which describe the spread within a population of infectious disease.

Recently, Hamaya and Arai [12] have studied the permanence of solution$(\mathbf{S}(\mathbf{t},\mathbf{x}), \mathbf{I}(\mathbf{t},\mathbf{x}), \mathbf{R}(\mathbf{t},\mathbf{x}))$ of the partial integrodifferential equation with diffusion for equation (3).

To allow for possible sensitive rate for COVID-19 evolution [5], we revise model (2)

$$\frac{\mathbf{dS(t)}}{\mathbf{dt}} = -\boldsymbol{\beta}\frac{\mathbf{S(t)I(t)}}{\mathbf{N(t)}} - \boldsymbol{\mu}\mathbf{S(t)},$$

$$\frac{\mathbf{dI(t)}}{\mathbf{dt}} = \boldsymbol{\beta}\frac{\mathbf{S(t)I(t)}}{\mathbf{N(t)}} - \boldsymbol{\lambda}\mathbf{I(t)},$$

$$\frac{\mathbf{dR(t)}}{\mathbf{dt}} = \boldsymbol{\lambda}\mathbf{I(t)} - \boldsymbol{\mu}\mathbf{R(t)},$$

$$\frac{d^{\beta}D(t)}{dt^{\beta}} = \lambda_{\beta} I(t) \quad t \geq 0,$$

where $\mathbf{D}$ represents the number of deaths which is one component in $\mathbf{I}$ and $\boldsymbol{\mu}$ is the rate of the removed individuals returning to the susceptible status. We add the fractional-order differential equation containing the death probability of $\boldsymbol{\lambda}$ while the other patients are cured, $\frac{d^{\beta}D}{dt^{\beta}} = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial D(s)}{\partial s}(t-s)^{-\beta}\, ds$, which is the Caputo fractional derivative [5] with order $\beta(0 < \beta \leq 1)$.

When the order $\beta = 1$, equation reduces to the classical integer-order differential equation or the death evolution.

In this article, we consider the permanence and global asymptotic properties of the solution of diffusive equation (1) with finite delay which based on [13 and cf. (12, 14-16)].

For equation (1), (S, I, R), (functions $S, I, R \in C([0,\infty)\times\bar{\Omega}, R)$), is called a (classical) solution of (1) if $\partial S/\partial t$, $\partial S/\partial x$, $\partial^2 S/\partial x^2$, $\partial I/\partial t$, $\partial I/\partial x$, $\partial^2 I/\partial x^2$, $\partial R/\partial t$, $\partial R/\partial x$ and $\partial^2 R/\partial x^2$, belong to the space $C\big((0,\infty)\times\Omega\big)$, $\partial S/\partial n$, $\partial I/\partial n$ and $\partial R/\partial n$ exist on $(0,\infty)\times\partial\Omega$ and (1) is identically satisfied. From [17, Chapter 6] and [18], we can show that the existence of solution is guaranteed for (1) whenever the initial function

$$\begin{aligned} S(\theta,x) &= \phi_1(\theta,x) \geq 0, \quad x\in\bar{\Omega}, \quad \theta\in C[-h,0],\\ I(\theta,x) &= \phi_2(\theta,x) \geq 0, \quad x\in\bar{\Omega}, \quad \theta\in C[-h,0],\\ R(\theta,x) &= \phi_3(\theta,x) \geq 0, \quad x\in\bar{\Omega}, \quad \theta\in C[-h,0], \end{aligned} \tag{4}$$

where $\phi_i(\theta,x) = \phi_i(0,x) > 0, \quad \theta\in C[-h,0],\ x\in C^1(\bar{\Omega}), \quad (i = 1,3)$,

and $\phi_2 := \phi_2(\theta,x) \geq 0, \quad \theta\in C[-h,0], x\in C^1(\bar{\Omega})$.

For any parameters h, β, b, λ and $\mu_i (i = 1,2,3)$ it is easy to check that the equilibrium solution$\big(S(t,x), I(t,x), R(t,x)\big)$ of (1) with the initial condition (4) exists and is a unique for all $t \geq 0$.

(i) If $b > 0$, then equation (1) always has a disease-free equilibrium $E_{S_0^*} = (S_0^*, 0, 0)$ where $S_0^* = b/\mu_1$.

(ii) Furthermore, if $\beta > \mu_2 + \lambda$, then there exists an $S_0^* > S^*$and that, equation (1) also has a unique positive endemic equilibrium $E^+ = (S^*, I^*, R^*)$. Here

$$S^* = \frac{b(\mu_3+\lambda)}{\mu_3\beta - \mu_3(\mu_2+\lambda) + \mu_1(\mu_3+\lambda)},$$

$$I^* = \frac{b-\mu_1 S^*}{\mu_2+\lambda} \quad \text{and}$$

$$R^* = \frac{\lambda}{\mu_3} I^* = \frac{\lambda(b-\mu_1 S^*)}{\mu_3(\mu_2+\lambda)}.$$

We next observe that $R(t,x)$ can be immediately obtained once $I(t,x)$ are known, so the system (1) can be reduced to

$$\begin{aligned} \frac{\partial S}{\partial t}(t,x) &= d\Delta S(t,x) - \beta\frac{S(t,x)I(t-h,x)}{N} - \mu_1 S(t,x) + b \\ & \qquad t > 0, x\in\Omega,\\ \frac{\partial I}{\partial t}(t,x) &= d\Delta I(t,x) + \beta\frac{S(t,x)I(t-h,x)}{N} - \mu_2 I(t,x) - \lambda I(t,x)\\ & \qquad t > 0, x\in\Omega,\\ \frac{\partial S}{\partial n}(t,x) &= \frac{\partial I}{\partial n}(t,x) = 0 \quad t > 0, x\in\partial\Omega, \end{aligned} \tag{5}$$

where $N := N(t,x) = S(t,x) + I(t,x)$.

Remark 1. It is clear for equation (5) that

(i') If $b > 0$, then equation (5) always has a disease-free equilibrium $E_{S_0^*} = (S_0^*, 0)$, where $S_0^* = b/\mu_1$.

(ii') Furthermore, if

$$(H_1) \qquad \beta > \mu_2 + \lambda,$$

then there exists an $S^* > 0$ such that $S_0^* > S^*$ and that, equation (5) also has a unique positive endemic equilibrium $E^+ = (S^*, I^*)$. Here

$$\mathbf{S}^* = \frac{\mathbf{b}}{\beta + \mu_1 - (\mu_2 + \lambda)},$$
$$\text{and} \quad \mathbf{I}^* = \frac{\mathbf{b} - \mu_1 \mathbf{S}^*}{\mu_2 + \lambda}. \qquad (5^*)$$

In particular, for parameter $\mathbf{b}$, we can only set from view of mathematical conditions as following, if $\mathbf{b} = \mathbf{0}$, then equation (1) always has a trivial equilibrium $\mathbf{E_0} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

In this paper, we do not need to treat this condition since our assumption is $\mathbf{b} > 0$.

We discuss the large time behaviour of the solution of equation (1) (cf.[19]).

Definition 1. The equation (1) is said to have the property of permanence if there are positive constants ν_i^* and $\mathbf{M}_i^* (\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3})$ such that

$$\nu_1^* \le \liminf_{t \to +\infty} \left[\inf_{x \in \bar{\Omega}} S(t,x) \right] \le \limsup_{t \to +\infty} \left[\sup_{x \in \bar{\Omega}} S(t,x) \right] \le M_1^*,$$

$$\nu_2^* \le \liminf_{t \to +\infty} \left[\inf_{x \in \bar{\Omega}} I(t,x) \right] \le \limsup_{t \to +\infty} \left[\sup_{x \in \bar{\Omega}} I(t,x) \right] \le M_2^*,$$

$$\nu_3^* \le \liminf_{t \to +\infty} \left[\inf_{x \in \bar{\Omega}} R(t,x) \right] \le \limsup_{t \to +\infty} \left[\sup_{x \in \bar{\Omega}} R(t,x) \right] \le M_3^*.$$

hold for any solution of (1) with the initial condition (4). Here ν_i^* and $\mathbf{M}_i^* (\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3})$ are independent of (4).

Remark 2. Similarly, for the equation (5), we can define the property of permanence if there are positive constants ν_i and $\mathbf{M}_i (\mathbf{i} = \mathbf{1}, \mathbf{2})$ such that

$$\nu_1 \le \liminf_{t \to +\infty} \left[\inf_{x \in \bar{\Omega}} S(t,x) \right] \le \limsup_{t \to +\infty} \left[\sup_{x \in \bar{\Omega}} S(t,x) \right] \le M_1,$$

$$\nu_2 \le \liminf_{t \to +\infty} \left[\inf_{x \in \bar{\Omega}} I(t,x) \right] \le \limsup_{t \to +\infty} \left[\sup_{x \in \bar{\Omega}} I(t,x) \right] \le M_2.$$

hold for any solution of (5) with the initial condition (4) except for $\mathbf{R}(\mathbf{0}, \mathbf{x})$. Here ν_i and $\mathbf{M}_i (\mathbf{i} = \mathbf{1}, \mathbf{2})$ are independent of (4) except for $\mathbf{R}(\mathbf{0}, \mathbf{x})$. We now employ above definition as the rest of this article.

2. Preliminary lemmas and permanence

Before main theorem, we mention the following theorem (the strong maximum principle in [20]), and then the main results of our paper are stated as follows.

Theorem A. Let $w \in C^{1,2}(D_T)$ and that

$$w_t - d\nabla^2 w + cw \ge 0 \quad in \quad D_T = (0,T] \times \Omega,$$
$$Bw = 0 \quad on \quad S_T = (0,T] \times \partial\Omega,$$
$$w(0,x) \ge 0 \quad in \quad \bar{\Omega},$$

where B is Neumann type boundary condition and $c \equiv c(t,x)$ is a bounded function in D_T. If $w$attains a maximum value$M$at some point in D_T, then $w \equiv M$ throughout D_T.

The following Theorem 1, 2 and 3 hold, if $I(t,x)$ has no influence from past time and that we say the strong fading memory property for time$t$and $h$as follows.

(H_2) $I(t-h,x) \le I(t,x) for t > 0, x \in \bar{\Omega}$, for any finite time-delay$h \ge 0$.

Theorem 1. Under the above assumptions of parameters and(H_2), for any time delay $h \ge 0$, if

$$S_0^* \equiv \frac{b}{\mu_1} > S^*, \qquad (6)$$

then the assumption (H_1) satisfies, and for each nonnegative continuous initial function, equation (1) has the property of permanence.

In the rest of this paper, we will report results only for system (5). Before the proof of Theorem 1, we prepare lemmas.

Lemma 1. The solution $(S(t,x), I(t,x))$ of equation (5) with (4) except for $R(0,x)$ satisfies for $t \ge 0$, the following inequality

$$0 < N(t,x) \le \max\{\sup_{x \in \bar{\Omega}} N(0,x), \frac{b}{\mu_1}\} := \mathrm{K}, \qquad t > 0, x \in \overline{\Omega}, \qquad (7)$$

where $N(t,x) = S(t,x) + I(t,x)$ and $N(0,x) = S_0(x) + I_0(x)$.

Proof. For the first inequality of (7), it is sufficient to prove that if for any small $\epsilon > 0, N(t_0, x) > \epsilon$ for some $t_0 > 0$ and $x \in \overline{\Omega}$, then $N(t,x) > \epsilon/2$ for $t > t_0$, $x \in \overline{\Omega}$. If it is not true, then

$$N(t,x) < \frac{\epsilon}{2} \text{ for } t > t_1, x \in \overline{\Omega} \ and \ N(t_1, x_1) = \frac{\epsilon}{2} \ for\ some\ t_1 > t_0, x_1 \in \overline{\Omega}$$

with t_1 being the smallest among all such points (t_1, x_1). If we set $w_0(t,x) = N(t,x) - \epsilon/2$, then $w_0(t,x) < 0 (t > t_1, x \in \overline{\Omega})$, $w_0(t_1, x_1) = 0$ and $\sup_{x\in\overline{\Omega}} w_0(t_0, x) > 0$, hence the function $w_0(t,x)$ takes a nonnegative minimum on $[t_0, t_1] \times \overline{\Omega}$. On the other hand, we have

$$\begin{aligned} \partial w_0/\partial t &= \partial N/\partial t \\ &= d\Delta N - \mu_1 S(t,x) - \mu_2 I(t,x) + b - \lambda I(t,x) \\ &= d\Delta w_0 - \mu_1 \left(w_0 + \frac{\epsilon}{2}\right) - pI + (b - \lambda I) \end{aligned}$$

and consequently

$$\begin{aligned} d\Delta w_0 - \partial w_0/\partial t - \mu_1 w_0 &= \lambda I + pI + \mu_1 \frac{\epsilon}{2} - b \\ &\le (\lambda + p)N + \mu_1 \frac{\epsilon}{2} - b \\ &\le (\lambda + \mu_1 + p)\frac{\epsilon}{2} - b < 0 \end{aligned}$$

on $(t_1, \infty) \times \Omega$.Then there arises a contradiction by the strong maximum principle (cf. [12, 14, 15, 20, 21]). Indeed, if $x_1 \in \Omega$, then $d\Delta w_0 - \partial w_0/\partial t - \mu_1 w_0$ must be nonnegative at(t_1, x_1). This is a contradiction. We thus obtain that $x_1 \in \partial\Omega$ and $w_0(t,x) > w_0(t_1, x_1)$ for all $(t,x) \in [t_0, t_1] \times \Omega$, and hence $\partial w_0/\partial n \le 0$at (t_1, x_1). This is a contradiction, again (cf.[20]). It is clear that, by the initial point $N(0,x) \ge 0$ and the reduction of the above, $N(t,x) > 0$ for $(t,x) \in (0, t_0) \times \overline{\Omega}$. Therefore, we must have the first inequality.

Let $K < K_0$ for some $K_0 > 0$. We claim that $N(t,x) \le K_0, [0,\infty) \times \overline{\Omega}$. If it is true, by letting $K_0 \to K$, we hold the second inequality of this lemma. If now this is not true, then there exists $(t_2, x_2) \in (0,\infty) \times \overline{\Omega}$ such that $N(t_2, x_2) > K_0$. If we set $w(t,x) = N(t,x) - K_0$, then $w(t_2, x_2) > 0$and $sup_{x\in\overline{\Omega}} w(0,x) \le 0, (t_2 > 0)$. Hence, the function $w(t,x)$ takes a positive maximum on $[0, t_2] \times \overline{\Omega}$. On the other hand, we have

$$\begin{aligned} \partial w/\partial t &= \partial N/\partial t \\ &= \Delta N - \mu_1 N + (b - (\lambda + p)I) \quad (by\ S = N - I, \mu_2 = \mu_1 + p) \\ &= d\Delta w - \mu_1 (w + K_0) + (b - (\lambda + p)I) \end{aligned}$$

and consequently

$$d\Delta w - \partial w/\partial t - \mu_1 w = (\lambda + p)I + (\mu_1 K_0 - b) > 0,$$

by $b/\mu_1 \le K < K_0$. Then there arises a contradiction by the strong maximum principle (cf.[12, 14,15, 20, 21]). Indeed, if $x_2 \in \Omega$, then $d\Delta w - \partial w/\partial t - \mu_1 w$ must be negative at (t_2, x_2). This is a contradiction. We thus obtain that $x_2 \in \partial\Omega$ and $w(t,x) < w(t_2, x_2)$ for all $(t,x) \in [0, t_2] \times \Omega$, and hence $\partial w/\partial n > 0$ at (t_2, x_2). This is a contradiction, again (cf.[20]). Therefore, we must have (7).

Lemma 2. Under the assumptions(H_1) and (H_2), the solution $(S(t,x), I(t,x))$ of equation (5) with (4) except for $R(0,x)$ satisfies the following inequality

$$\liminf_{t\to\infty} \left[\inf_{x\in\overline{\Omega}} S(t,x)\right] \ge \frac{b}{\mu_1 + \beta} \equiv \nu_1 > 0. \tag{8}$$

Proof. For some $t_3 > t_0$, we can show that

$$\hat{S}(t) \le S(t,x), \quad t > t_3, \quad x \in \overline{\Omega}, \tag{9}$$

where $\hat{S}(t)$ is the solution of ordinary differential equation

$$\frac{d}{dt}\hat{S}(t) = -(\mu_1 + \beta)\hat{S}(t) + b - \epsilon, \qquad for\ \epsilon > 0, \quad t > t_3, \tag{10}$$

$$\hat{S}(t_3) = \widehat{S_3} \ \ and \ \ \widehat{S_3} \ge \inf_{x\in\overline{\Omega}} S(t_2, x) \ge \frac{b}{\mu_1 + B}.$$

To see this, we consider the function $w_1(t,x) := S(t,x) - \hat{S}(t)$ on $[t_3, \infty) \times \overline{\Omega}$. Then $w_1(0,x) = S(0,x) - \hat{S}(0) \le$

0 for $x \in \overline{\Omega}$, and moreover, since $S + I = N$ is bounded by Lemma 1, and $\beta \frac{SI(t-h,x)}{N} \le \beta \frac{SI}{N} \le \beta \frac{SN}{N} = \beta S\ by\ (H_2)$,

$$\partial w_1/\partial t = \partial S/\partial t - d\hat{S}(t)/dt$$
$$\ge d\Delta S - \beta S - \mu_1 S + b - \left(-\mu_1 \hat{S} - \beta \hat{S} + b - \epsilon\right)$$
$$\ge d\Delta w_1 - \beta\left(w_1 + \hat{S}\right) - \mu_1\left(w_1 + \hat{S}\right) + b - \left(-\mu_1 \hat{S} - \beta \hat{S} + b - \epsilon\right)$$
$$\ge d\Delta w_1 - (\beta + \mu_1) w_1 + \epsilon.$$

Hence,

$$d\Delta w_1 - \partial w_1/\partial t - (\beta + \mu_1) w_1 \le -\epsilon < 0 \quad on \quad [t_3, \infty) \times \overline{\Omega}.$$

Thus, by the strong maximum principle, we have a contradiction. Therefore, by the same reasoning as the one for $w_0(t,x)$of Lemma 1, One can see that $w_1(t,x) \ge 0$ on $[t_3, \infty) \times \overline{\Omega}$. Thus, we must have (9). Moreover, by setting $M = \mu_1 + \beta$in (10), we have

$$\frac{d}{dt}\hat{S} = -M\hat{S} + b - \epsilon. \tag{11}$$

By solving equation (11), we obtain that

$$\hat{S} = \frac{b-\epsilon}{M} + \hat{C} e^{-Mt}$$

and

$$\hat{C} = e^{Mt_3}\left(\widehat{S_3} - \frac{b-\epsilon}{M}\right).$$

Therefore, we have

$$\frac{b-\epsilon}{\mu_1 + \beta} \le \hat{S}, \quad t \ge t_3$$

for small $\epsilon > 0$. Thus, we obtain

$$\frac{b-\epsilon}{\mu_1 + \beta} \le S$$

on $t \ge t_3,\ x \in \Omega$. By taking infimum, $t \to \infty$and later letting $\epsilon \to 0$ in the above inequality, we obtain

$$\frac{b}{\mu_1 + \beta} \le \liminf_{t\to\infty}\left[\inf_{x\in\overline{\Omega}} S(t,x)\right].$$

This completes the proof of Lemma 2.

Lemma 3. Under the assumptions(H_1) and (H_2), the solution $\left(S(t,x), I(t,x)\right)$ of equation (5) with (4) except for $R(0,x)$ satisfies the following inequality

$$\limsup_{t\to\infty}\left[\sup_{x\in\Omega} I(t,x)\right] \le M_2, \tag{12}$$

for some $M_2 > 0$.

Proof. For some $t_4 > t_0$, we can show that

$$I(t,x) \le \hat{I}(t) \quad t > t_4, x \in \overline{\Omega}. \tag{13}$$

Here, $\hat{I}(t)$ is the solution of ordinary differential equation

$$\frac{d}{dt}\hat{I}(t) = \beta\left(K - \hat{I}(t)\right) - (\mu_2 + \lambda)\hat{I}(t) + \epsilon, \quad t > t_4, \tag{14}$$

$$\hat{I}(t_4) = \widehat{I_4} \quad \text{and} \quad 0 < \widehat{I_4} \le sup_{x\in\overline{\Omega}} I(t_4, x) \le \frac{B}{A},$$

where $A = \beta + (\mu_2 + \lambda) > 0$ and $B = \beta K + \epsilon > 0$. To see this, we consider the function $w_2(t,x) := I(t,x) - \hat{I}(t)$ on $[t_4, \infty) \times \overline{\Omega}$. Then $w_2(0,x) = I(0,x) - \hat{I}(0) \le 0$ for $x \in \overline{\Omega}$, and moreover, since $\beta SI(t-h,x)/N \le \beta SI(t,x)/N \le \beta(N-I)I/N \le \beta(N-I)N/N \le \beta(K-I)$ by (H_2), we have

$$\partial w_2/\partial t = \partial I/\partial t - d\hat{I}(t)/dt$$

$$= d\Delta I + \beta SI(t-h,x)/N - (\mu_2+\lambda)I - \{\beta\left(K-\hat{I}(t)\right) - (\mu_2+\lambda)\hat{I}(t) + \epsilon\}$$
$$\le d\Delta I + \beta(K-I) - (\mu_2+\lambda)I - \{\beta\left(K-\hat{I}(t)\right) - (\mu_2+\lambda)\hat{I}(t) + \epsilon\}$$
$$\le d\Delta I - \beta I - (\mu_2+\lambda)I + \beta\hat{I} + (\mu_2+\lambda)\hat{I} - \epsilon.$$

Since $w_2(t,x) = I(t,x) - \hat{I}(t)$,

$$\partial w_2/\partial t \le d\Delta w_2 - \beta\left(w_2+\hat{I}\right) - (\mu_2+\lambda)\left(w_2+\hat{I}\right) + \beta\hat{I} + (\mu_2+\lambda)\hat{I} - \epsilon$$
$$= d\Delta w_2 - \beta w_2 - (\mu_2+\lambda)w_2 - \epsilon.$$

Hence,

$$d\Delta w_2 - \partial w_2/\partial t - \left(\beta + (\mu_2+\lambda)\right)w_2 \ge \epsilon > 0.$$

Therefore, by the same reasoning as the one for $w_0(t,x)$, of Lemma 1, one can see that $w_2(t,x) \le 0$ on $[t_4,\infty)\times\overline{\Omega}$. Thus, we must have (13). Moreover, from (14),

$$\frac{d}{dt}\hat{I}(t) = -\left(\beta + (\mu_2+\lambda)\right)\hat{I} + \beta K + \epsilon.$$

Here, for the simplicity, we use A and $B > 0$ in the above equation (14), then we obtain

$$\frac{d}{dt}\hat{I} = -A\hat{I} + B. \tag{15}$$

By solving equation (15), we have

$$\hat{I} = \frac{B}{A} + \hat{C}^* e^{-At} \tag{16}$$

and

$$\hat{C}^* = e^{At_4}\left(\hat{I}_4 - \frac{B}{A}\right).$$

Therefore, we obtain

$$\hat{I}(t) \le \hat{I}(t_4) \tag{17}$$

for $t \ge t_4$. By (13), (17), we have

$$I(t,x) \le \hat{I}(t_4)$$

on $[t_4,\infty)\times\overline{\Omega}$. By taking supremum, $t\to\infty$ and later letting $\epsilon\to 0$ in the above inequality, we obtain (12), i.e.

$$\limsup_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} I(t,x)\right] \le M_2,$$

where the positive number $M_2 = \frac{B'}{A}$, $B' = \beta K$. This completes the proof of Lemma 3.

Proof of Theorem 1. From Lemma 1, we have

$$N(t,x) \le K,$$

where $N(t,x) = S(t,x) + I(t,x)$. By Lemma 3,

$$\limsup_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} I(t,x)\right] \le M_2$$

for $M_2 > 0$, Thus, we hold that the solution $\left(S(t,x), I(t,x)\right)$ of equation (5) with initial condition (4) except for $R(0,x)$ satisfies

$$\limsup_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} S(t,x)\right] \le M_1 \tag{18}$$

for some $M_1 > 0$.

We can next show that the solution $\left(S(t,x), I(t,x)\right)$ of equation (5) with initial condition (4) except for $R(0,x)$ satisfies

$$\liminf_{t\to\infty}\left[\inf_{x\in\overline{\Omega}} I(t,x)\right] \ge \nu_2$$

for some $\nu_2 > 0$ which does not depend on the initial function in (4). To see this, it is sufficient to prove

$$I(t,x) \to \frac{\beta}{\beta+\sigma}K \quad as\, t \to \infty, \quad x \in \bar{\Omega}, \tag{19}$$

where $\sigma = \mu_2 + \lambda$. We define the function

$$f(t,x) = \frac{I(t,x)}{\sigma} - \frac{\beta}{(\beta+\sigma)\sigma}K.$$

Then

$$\frac{\partial f}{\partial t} \le \frac{1}{\sigma}(d\Delta I + \beta K - (\beta+\sigma)I)(\text{by } (H_2))$$
$$= \frac{d}{\sigma}\Delta I - \frac{\beta+\sigma}{\sigma}I + \frac{\beta}{\sigma}K.$$

We thus have the following differential inequality:

$$\frac{\partial f}{\partial t} \le d\Delta f - (\beta+\sigma)f. \tag{20}$$

Then, we can see that

$$f(t,x) \to 0 \quad as\, t \to \infty, x \in \bar{\Omega}.$$

If we set $W(t) := W(f)(t) = \int_\Omega f^2(t,x)dx, t \ge 0$, then, $W(t) \ge 0$ and we have

$$\frac{dW(t)}{dt} = 2\int_\Omega f\frac{\partial f}{\partial t}dx$$
$$\le 2\int_\Omega f(d\Delta f - (\beta+\sigma)f)dx$$
$$= -2d\int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 dx - 2(\beta+\sigma)\int_\Omega f^2\,dx. \tag{21}$$

H_1 and H_2 be defined by the following

$$H_1 = 2d, \quad H_2 = 2(\beta+\sigma).$$

Then $H_1 > 0$ and $H_2 > 0$. It follows from (21) that for $t > 0$,

$$W(t) + H_2\int_0^t\int_\Omega \left(\frac{\partial f(s,x)}{\partial x}\right)^2 dxds + H_2\int_0^t\int_\Omega f^2(s,x)\,dxds \le W(0). \tag{22}$$

Since $W(t) \ge 0$, we have from (22) that

$$\int_0^t\left[\int_\Omega\left(\frac{\partial f(s,x)}{\partial x}\right)^2 dx\right]ds < \frac{W(0)}{H_1},$$
$$\int_0^t\left[\int_\Omega f^2(s,x)dx\right]ds < \frac{W(0)}{H_2}. \tag{23}$$

Thus, we conclude from (21), (22) and (23) that $W(t) \in L^1[0,\infty)$ and $\frac{dW(t)}{dt} \in L^1[0,\infty)$. By Barbalate's lemma [19, Lemma 1.2.2.], we obtain $W(t) \to 0$ and thus, $f \to 0$ in L^2 as $t \to \infty$, that is

$$\|f(t,\cdot)\|_{L^2} \to 0 \qquad as \qquad t \to \infty, \tag{24}$$

where $\|\cdot\|_{L^2}$ denotes the L^2-norm of functions on Ω. We next prove that

$$\sup_{x\in\bar{\Omega}}|f(t,x)| \to 0 \quad as \quad t \to \infty. \tag{25}$$

To do this (cf.[14]), we take notice of the boundedness of $f(t,x)$ by (7) in Lemma 1. Thus, we see that the orbit for meaning of differential equation (except for inequality:<) in (20), that is, $\{f(t,\cdot)\,|t \ge 0\}$ has relatively compact. The assertion (25) follows from this fact. Indeed, if (25) is not true, then there exist sequences $\{t_n\}, t_n \to \infty$ as $n \to \infty$, and $\{x_n\} \subset \bar{\Omega}$ such that $|f(t_n,x_n)| \ge \epsilon > 0, n = 1,2,\cdots$for some $\epsilon > 0$. We can assume that $x_n \to x_0$ and $f(t_n,x_n) \to$

$\tilde{f}(x_0)$ uniformly on $\bar{\Omega}$ for some $x_0 \in \bar{\Omega}$ and $\tilde{f} \in C(\bar{\Omega})$ as $n \to \infty$, if necessary taking a subsequence of these. In particular, we get $|\tilde{f}(x_0)| \geq \varepsilon$. This is a contradiction, because $\int_\Omega \tilde{f}^2(x)\,dx = \lim_{n\to\infty} \|f(t_n,\cdot)\|_{L^2}^2 = 0$ by (24). Thus, we must have (25). From the definition of f, we have (19), that is $I \to \frac{\beta}{\beta+\sigma}K > 0$. Thus, (18) holds. Moreover, we easily have

$$0 < \nu_3 \leq \liminf_{t\to+\infty}\left[\inf_{x\in\bar{\Omega}} R(t,x)\right]$$

for some $\nu_3 > 0$. Thus, equation (1) has the property of permanence by Lemmas 1, 2 and 3. This proves Theorem 1.

3. Global attractor

Theorem 2. If $S_0^* < S^*$or $\phi_2 \equiv 0$, the disease-free equilibrium of (5) satisfies

$$lim_{t\to\infty}\left[\sup_{x\in\Omega} I(t,x)\right] = 0,$$

and

$$lim_{t\to\infty}\left[\sup_{x\in\Omega}\left|S(t,x) - \frac{b}{\mu_1}\right|\right] = 0$$

whenever the assumption (H_2) holds.

Proof. If $b/\mu_1 \leq N(0,x) \leq K$, then we can show that

$$N(t,x) \leq \hat{N}(t) \quad t > 0, x \in \bar{\Omega}, \tag{26}$$

where $\hat{N}(t)$ is the solution of ordinary differential equation

$$\frac{d}{dt}\hat{N}(t) = -\mu_1\hat{N}(t) + b, \qquad t > 0$$

and

$$\hat{N}(0) = K.$$

To see this, we consider the function $w_3(t,x) := N(t,x) - \hat{N}(t)$ on $[0,\infty)\times\bar{\Omega}$. Then $w_3(0,x) = N(0,x) - \hat{N}(0) \leq 0$ for $x \in \bar{\Omega}$, and moreover

$$\begin{aligned}\partial w_3/\partial t &= \partial N/\partial t - d\hat{N}(t)/dt \\ &= d\Delta N - \mu_1 N - pI + b - \lambda I + \mu_1\hat{N} - b \\ &= d\Delta w_3 - \mu_1(w_3 + \hat{N}) - (\lambda+p)I + \mu_1\hat{N}\end{aligned}$$

and hence

$$d\Delta w_3 - \partial w_3/\partial t - \mu_1 w_3 = (\lambda+p)I \geq 0.$$

Therefore, by the same reasoning as the one for $w(t,x)$, one can see that $w_3(t,x) \leq 0$. Thus, we must have (26). Since $\hat{N}(t) = \hat{C}e^{-\mu_1 t} + b/\mu_1, \hat{C} = K - b/\mu_1$, by letting $t \to \infty$ in the above inequality (26), we obtain

$$\limsup_{t\to\infty}\left[\sup_{x\in\bar{\Omega}} N(t,x)\right] \leq \frac{b}{\mu_1}.$$

Hence for the discussion of the asymptotic behavior of solutions as $t \to +\infty$ we can (without loss of generality) assume that

$$N(t,x) \leq b/\mu_1 \quad t > 0, x \in \bar{\Omega}. \tag{27}$$

We next define

$$f(t,x) = \frac{I(t,x)}{\sigma},$$

where $\sigma = \mu_2 + \lambda$. Since $S_0^* < S^*$, we have $\frac{b}{\mu_1} < \frac{b}{\beta+\mu_1-\sigma}$. Then $\frac{\beta}{\sigma} < 1$. Thus

$$\frac{\partial f}{\partial t} = \frac{1}{\sigma}\frac{\partial I}{\partial t} = \frac{1}{\sigma}\left(d\Delta I + \beta\frac{SI(t-h,x)}{N(t,x)} - \sigma I\right)$$

$$\leq \frac{d}{\sigma}\Delta I + \left(\frac{\beta}{\sigma}\frac{S}{N} - 1\right)I \qquad (by\ (H_2))$$
$$\leq \frac{d}{\sigma}\Delta I - \frac{1}{\sigma}\left(\beta\left(1 - \frac{S}{N}\right)\right)I$$
$$\leq \frac{d}{\sigma}\Delta I - \frac{1}{\sigma}Q^* I \qquad \left(by\ \frac{\sigma}{\beta} > 1\ \ and\ \ (18)\right),$$

where $Q^* = \beta\mu_1\nu_2/b > 0$. We thus have the following differential inequality of

$$\frac{\partial f}{\partial t} \leq d\Delta f - Q^* f. \tag{28}$$

Then, we can see that

$$f(t,x) \to 0 \quad as \quad t \to \infty, x \in \overline{\Omega},$$

by the same argument of the proof in Theorem 1. From the definition of f, we have

$$I(t,x) \to 0 \quad as \quad t \to \infty, x \in \overline{\Omega}. \tag{29}$$

Since S is bounded and I has the strong fading memory property (H_2) in theorem,

$$\frac{\beta_1 S I(t-h,x)}{N} \to 0 \quad as \quad t \to \infty, x \in \overline{\Omega}.$$

We next claim that

$$S \to \frac{b}{\mu_1} \quad as \quad t \to \infty, x \in \overline{\Omega}. \tag{30}$$

By (29), for any small $\epsilon > 0$, there exists a large time $t_5 > 0$ such that $I(t,x) \leq \epsilon$ for $t \geq t_5, x \in \overline{\Omega}$. Then, it is sufficient for (30) to prove

$$N(t,x) \geq \widetilde{N}(t) \quad t \geq t_5, x \in \overline{\Omega}, \tag{31}$$

where $\widetilde{N}(t)$ is the solution of ordinary differential equation

$$\frac{d}{dt}\widetilde{N}(t) = -\mu_1\widetilde{N}(t) + b - (\lambda + p)\epsilon, \quad t > t_5,$$
$$\widetilde{N}(t_5) = \widetilde{N_5} \quad and \quad 0 < \widetilde{N_5} \leq \sup_{x\in\overline{\Omega}} N(t_5,x) \leq \frac{b}{\mu_1}.$$

Then, we have

$$\widetilde{N}(t) \leq N(t,x) \leq \frac{b}{\mu_1} \quad t \geq t_5, x \in \overline{\Omega}.$$

Since $\widetilde{N}(t) = \tilde{C}e^{-\mu_1 t} + b/\mu_1 - (\lambda + p)\epsilon/\mu_1, \tilde{C} = e^{\mu_1 t_5}\left(\widetilde{N_5} - b/\mu_1 + (\lambda + p)\epsilon/\mu_1\right)$, by letting $t \to \infty$ and later letting $\epsilon \to 0$ in the above inequality, we obtain

$$\frac{b}{\mu_1} \leq \liminf_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} N(t,x)\right] \leq \lim_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} N(t,x)\right]$$
$$\leq \lim_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} S(t,x)\right] + \lim_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} I(t,x)\right] = \lim_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} S(t,x)\right]$$
$$\leq \limsup_{t\to\infty}\left[\sup_{x\in\overline{\Omega}} N(t,x)\right] \leq \frac{b}{\mu_1}.$$

To see (31), we consider the function $w_4(t,x) := \widetilde{N}(t) - N(t,x)$ on $[t_5,\infty) \times \overline{\Omega}$. Then $w_4(t_5,x) = \widetilde{N_5} - N(t_5,x) \leq 0$ for $x \in \overline{\Omega}$, and moreover

$$\partial w_4/\partial t = d\widetilde{N}(t)/dt - \partial N/\partial t$$
$$= -\mu_1\widetilde{N} + b - (\lambda + p)\epsilon - d\Delta N + \mu_1 N - b + (\lambda + p)I$$
$$= d\Delta w_4 + \mu_1\left(\widetilde{N} - w_4\right) - \mu_1\widetilde{N} + (\lambda + p)(I - \epsilon)$$

and hence, by $I \geq \nu_2$,

$$d\Delta w_4 - \partial w_4 / \partial t - \mu_1 w_4 \leq (\lambda + p)(\epsilon - \nu_2) < 0.$$

Therefore, by the same reasoning as the one for $w(t,x)$, one can see that $w_4(t,x) \leq 0$, that is (31) holds. Thus, we have (30). This completes the proof of Theorem 2.

We show also that the following theorem.

Theorem 3. If $S_0^* > S^*$ as the assumption (H_1) and $\phi_2 \not\equiv 0$, then, for each nonnegative continuous initial function, there is a unique positive equilibrium (S^*, I^*) of (5) satisfies

$$\lim_{t\to\infty}\left[\sup_{x\in\bar{\Omega}}|I(t,x) - I^*|\right] = 0$$

and

$$\lim_{t\to\infty}\left[\sup_{x\in\bar{\Omega}}|S(t,x) - S^*|\right] = 0$$

whenever the assumption (H_2) holds.

Proof of Theorem 3. In order to prove this theorem, we need the following Corollary in [8, pp.148-153]. As there are complete comments and references of this result for ordinary differential equations in [cf. 8, pp.159-160], we omit the proof of this corollary for simplicity.

Corollary. If $S_0^* > S^*$, and $I_0(x) \neq 0$, then there is a unique positive endemic equilibrium (S^*, I^*).

Now, from $N = S + I$, the system (5) drives to

$$\frac{\partial N}{\partial t} = d\Delta N - \mu_1 N - (\lambda + p)I + b, \tag{32}$$

$$\frac{\partial I}{\partial t} \leq d\Delta I + \beta I\left[\frac{(N-I)}{N}\right] - (\mu_2 + \lambda)I$$

$$= d\Delta I + \beta I\left[(N-I)/N - \frac{\sigma}{\beta}\right]. \tag{33}$$

The system (5) has the positive equilibrium (S^*, I^*) where $N^* = S^* + I^*$. We can rewrite (32) in the form

$$\frac{\partial N}{\partial t} = d\Delta N - \mu_1(N - N^*) - (\lambda + p)(I - I^*) \tag{34}$$

because $-\mu_1 N^* - (\lambda + p)I^* + b = 0$. Moreover, for the first equation of Sin (5), since

$$-\beta\frac{S^* I^*}{N^*} - \mu_1 S^* + b = 0,$$

we have

$$\beta\frac{S^*}{N^*} = (\lambda + \mu_2) = \sigma$$

by I^* in (5^*). Moreover, from (33),

$$\frac{\partial I}{\partial t} \leq d\Delta I + \beta I\{(N-I)/N + (-N^* + I^*)/N^*\}$$

$$= d\Delta I + \beta I\{G(N) - (I - I^*)/N\}. \tag{35}$$

Here,

$$G(N) = \frac{N - N^*}{N} - \frac{(N-N^*)(N^* - I^*)}{NN^*}$$

$$= I^*\frac{(N-N^*)}{NN^*}.$$

Then, $G(N) > 0$ for $N > N^*$ and $G(N) < 0$ for $N < N^*$. We now define a function $V(t)$ by

$$V(t) = V(N,I)(t)$$

$$= \int_\Omega \{\xi \int_{N^*}^{N} G(s)ds + (I - I^*) - I^* \log\frac{I}{I^*}\}dx,$$

where $\xi = \frac{\beta}{\lambda + p}$. Then $V(N^*, I^*)(t) = 0$ and $V(N,I)(t) > 0$ for other admissible (N, I). Furthermore, we calculate

dV/dt along the solution of (34) and (35).

$$\frac{dV(t)}{dt} = \int_\Omega \{\xi G(N)\frac{\partial N}{\partial t} + \frac{\partial I}{\partial t} - I^* \frac{\partial I/\partial t}{I}\} dx$$

$$\leq \int_\Omega [\xi G(N)\big(d\Delta N - \mu_1(N-N^*) - (\lambda+p)(I-I^*)\big)$$

$$+ d\Delta I + \beta I G(N) - \beta I(I-I^*)/N$$

$$-I^*\left\{d\frac{\Delta I}{I} + \beta G(N) - \beta(I-I^*)/N\}\right] dx$$

$$= d\xi \int_\Omega \Delta N G(N) dx - \xi\mu_1 \int_\Omega G(N)(N-N^*) dx$$

$$-\xi(\lambda+p)\int_\Omega G(N)(I-I^*)dx + d\int_\Omega \Delta I dx$$

$$+\beta \int_\Omega I G(N) dx - \beta \int_\Omega \frac{I(I-I^*)}{N} dx$$

$$-d\int_\Omega \frac{I^*\Delta I}{I} dx - \beta \int_\Omega I^* G(N) dx + I^*\beta \int_\Omega \frac{I-I^*}{N} dx$$

$$= d\xi \int_\Omega \Delta N G(N) dx - \xi\mu_1 \int_\Omega G(N)(N-N^*) dx$$

$$-\beta \int_\Omega G(N)(I-I^*) dx + d\int_\Omega \Delta I \frac{I-I^*}{I} dx$$

$$+\beta \int_\Omega I G(N) dx - \beta \int_\Omega I^* G(N) dx - \beta \int_\Omega \frac{(I-I^*)^2}{N} dx$$

$$< 0 \tag{36}$$

whenever $(N, I) \neq (N^*, I^*)$. To drive this, we continue to estimate for (36) in more detail.

$$d\xi \int_\Omega \Delta N G(N) dx = d\xi \left\{\left[\frac{\partial N}{\partial x} G(N)\right]_\Omega - \int_\Omega \frac{\partial N}{\partial x}\frac{\partial G(N)}{\partial x} dx\right\}$$

$$= -d\xi \int_\Omega \frac{\partial N}{\partial x}\frac{\partial G(N)}{\partial x} dx. \tag{37}$$

Here

$$\frac{\partial G(N)}{\partial x} = \frac{\partial}{\partial x}\left\{I^* \frac{(N-N^*)}{NN^*}\right\} = I^* \frac{\partial}{\partial x}\left(\frac{1}{N^*} - \frac{1}{N}\right) = I^* \frac{\partial N/\partial x}{N^2}.$$

Thus, expression (37) is

$$-d\xi I^* \int_\Omega \frac{(\partial N/\partial x)^2}{N^2} dx < 0.$$

Moreover, we have

$$-\xi\mu_1 \int_\Omega G(N)(N-N^*) dx \leq 0,$$

because $G(N) > 0$ for $N > N^*$ and $G(N) < 0$ for $N < N^*$.

And moreover,

$$-\beta \int_\Omega G(N)(I-I^*) dx + \beta \int_\Omega I G(N) dx - \beta \int_\Omega I^* G(N) dx = 0.$$

Similarly, we can check

$$d\int_\Omega \Delta I \frac{I-I^*}{I} dx = d\int_\Omega \Delta I \left(1 - \frac{I^*}{I}\right) dx = d\left[\frac{\partial I}{\partial x}\left(1 - \frac{I^*}{I}\right)\right]_\Omega - dI^* \int_\Omega \frac{(\partial I/\partial x)^2}{I^2} dx < 0.$$

Therefore, $V(t)$ is non-increasing in$t$that is there exists a constant $c_1 \geq 0$ such that $V(t) \to c_1$ as $t \to \infty$. From Lemma 1 and Theorem 1, $I(t,x)$ is uniformly bounded on $[0,\infty) \times \bar{\Omega}$. Thus, we see that for any $h > 0$, there exists $C(h) > 0$ such that $|I(t+h,\cdot) - I(t,\cdot)| \leq C(h)$ for $t \geq 0$. From (36), we have $\dot{V}(t) \leq -W(N,I)(t) \leq 0$ (included equilibrium point case), where $W(N,I)(t)$ is the function of right-hand side in (36). Suppose that $\dot{V}(t) \neq 0$. For any sequence$\{t_k\}, t_k \to \infty$ as $k \to \infty$ and some positive number γ, there exists $\delta > 0$ such that

$$\dot{V}(t) < -\gamma \tag{37}$$

if $|I(t+t_k,\cdot) - I(t,\cdot)| \leq \delta,\ 0 \leq t \leq \delta$ and k is sufficient large. For regions $[t_k, t_k + \delta]$, we can see that

$$V(t_k + \delta) \leq V(t_k) - \gamma\delta \tag{38}$$

to integral on $[t_k, t_k + \delta]$ for the both sides of (37). Since (38) is true for all large number k and $\lim_{t\to\infty} V(t) = c_1 \geq 0$, it contradicts by $\gamma\delta$ is positive. This shows that $\dot{V}(t) = 0$. Then, we have $W(N,I)(t) = 0$. We thus obtain $N \to N^*$and $I \to I^*$by continuity of V and W. The asymptotic behavior of Snow follows from the above result on the behavior of N and I. Thus, it is clear from $S = N - I$ that $S \to S^*$. This completes the proof.

4. Example

We consider the following concrete example of equation (5);

$$\begin{gathered}
\frac{\partial S}{\partial t}(t,x) = 0.1\Delta S(t,x) - 0.0425 S(t,x) I(t-12.0,x)/N(t,x) \\
-0.01 S(t,x) + 0.8 \quad t > 0, x \in \Omega, \\
\frac{\partial I}{\partial t}(t,x) = 0.1\Delta I(t,x) + 0.0425 S(t,x) I(t-12.0,x)/N(t,x) - 0.04 I(t,x) \\
t > 0, x \in \Omega,
\end{gathered} \tag{39}$$

where, in equation (5), $d = 0.1, h = 12, \beta = 0.0425, \mu_1 = 0.01, \mu_2 = 0.02, \lambda = 0.02, b = 0.8$ and $\sigma = \mu_2 + \lambda = 0.04$. Thus, we have

$$S_0^* = \frac{b}{\mu_1} = \frac{0.8}{0.01} = 80,$$

$$S^* = \frac{b}{\beta + \mu_1 - (\mu_2 + \lambda)} = \frac{0.8}{0.0125} = 64,\ \ therefore\ S_0^* > S^*.$$

$$E_{S_0^*} = (S_0^*, 0) = (80,0) \quad \text{and} \quad E^+ = (S^*, I^*) = (64,4).$$

Here

$$I^* = \frac{b - \mu_1 S^*}{\mu_2 + \lambda} = \frac{0.8 - 0.01 \times 64}{0.04} = 4 > 0.$$

The initial functions are

$$\begin{gathered}
S(\theta,x) = \phi_1(\theta,x) \equiv 10^4 > 0, \quad x \in \bar{\Omega}, \\
I(\theta,x) = \phi_2(\theta,x) \equiv 15 > 0, \quad x \in \bar{\Omega} \quad and \\
belong\ to\ the\ \ (\theta,x) \in [-2,0] \times C^1(\bar{\Omega}).
\end{gathered}$$

Conclusion. We obtain the results of Theorem 1, 2 and 3 that the asymptotic stability of the equilibrium point $E_{S_0^*}$,E^+[cf. 22] and the property of permanence for equation (5), by using the method of the strong maximum principle, the technique of Lyapunovfunctionals and others. Moreover, we have given the simple example for Theorem 3 that the equilibrium-point E^+ of equation (39), that is equation (5), is the asymptotically stable by assumptions(H_1), (H_2) and the strong maximum principle.

Figures. Figure 1 and Figure 2 denotes the asymptotic stability of the equilibrium point $E^+$of equation (5) satisfying our theorem 3, if assumptions (H_1) and (H_2) hold. In Figure 1, we denote measures of the susceptible individuals, $z = S(t,x), y$ =time t passes to the left side for the future and x =space variable x, and also, Figure 2, the infectious individual $z = I(t,x)$ for the partner $y = t$ passes to the left side for the future and x =space variable x. These are the graph of the trajectory of equation (39). In the case of Example, the solutions of (39) approach the equilibrium point(S^*, I^*).

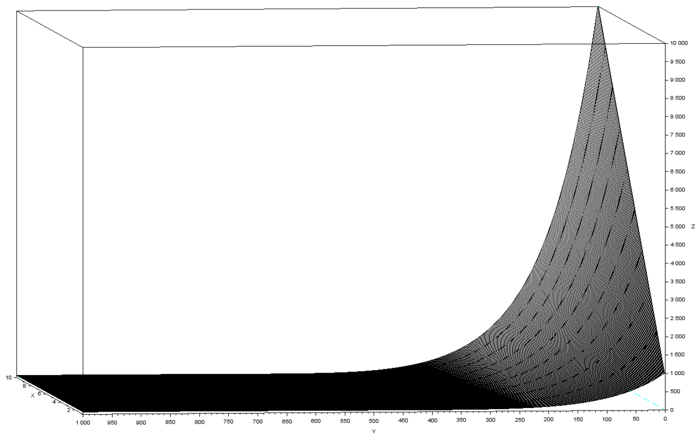

Figure 1. (t,x,z).

The Figure 2 is the phase space of (t,z)-plane in Figure 1.

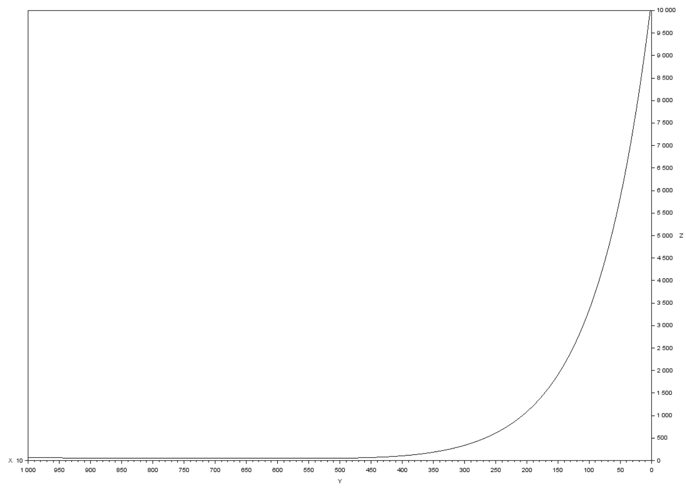

Figure 2. (t,z).

In the case of Example, the solutions of (39) approach the equilibrium point (S^*,I^*).

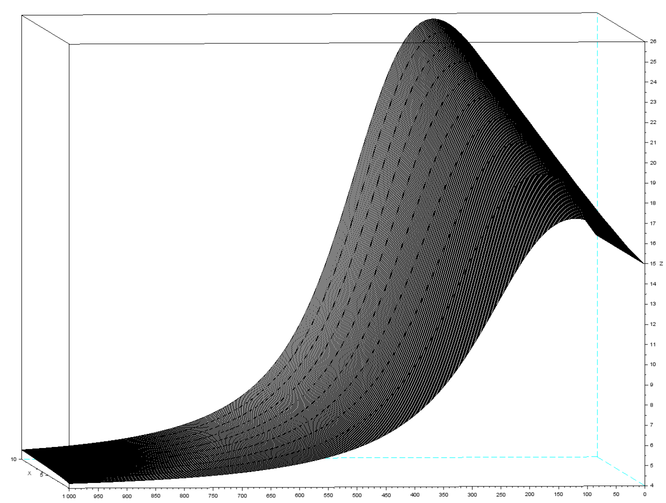

Figure 3. (t,x,z).

The Figure 4 is the phase space of (t, z)-plane in Figure 3.

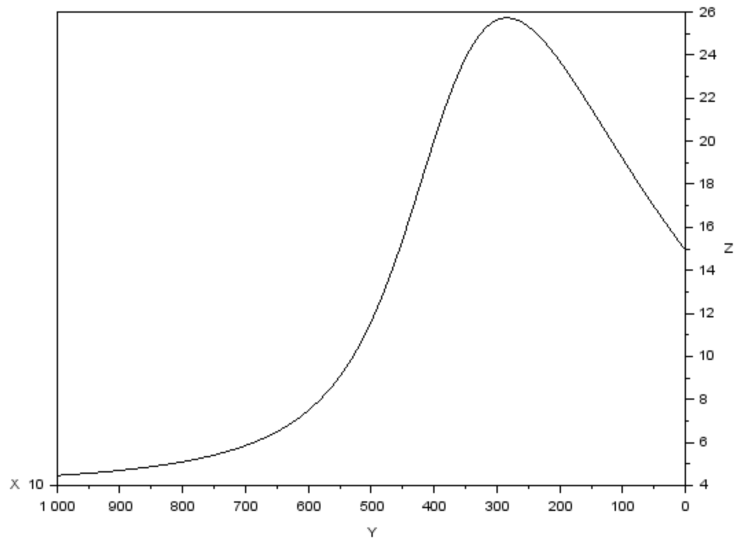

Figure 4. $(\boldsymbol{t}, \boldsymbol{z})$.

Acknowledgements

We wish to thank referees for their many useful comments concerning the content of this paper. The research of this article is supported by the JSPS KAKENHI Grant Number 21K03318.

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