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A Simple Calculation of the First Order Melnikov Function for a Non-smooth Hamilton System

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Abstract

One of the important problems in differential systems is the study of the number of limit cycles, which is closely related to the famous Hilbert's 16th problem. The first order Melnikov function plays a key role in determining the lower and upper bounds of the number of limit cycles for smooth and non-smooth Hamilton systems. But the calculation of some Melnikov functions is very complicated, especially for non-smooth differential systems. In the present article, we use a simple approach to obtain the first order Melnikov function of a non-smooth differential system with a generalized heteroclinic loop through a cusp. We first prove that the first order Melnikov function can be expressed as combinations of some generator functions with polynomial coefficients, which can save a large amount of calculation. Then the upper bound of the number of limit cycles of the perturbed differential system can be obtained by using the existing methods. This method can be applied to other discontinuous differential systems.

Keywords

Non-smooth differential system, Melnikov function, generalized heteroclinic loop

1. Introduction

Due to the influence of natural laws and various factors, the bifurcation theory of limit cycles of smooth and non-smooth integrable differential systems is widely used in many practical problems, such as electronic engineering, neural network, automatic control, biological mathematics, etc. Moreover, this theory is related to the Hilbert's 16th problem [1, 2]. Therefore, the study on bifurcation of limit cycles for smooth and nonsmooth integrable differential systems has become one of the hot topics in recent years. The number of their limit cycles (the so-called limit cycle refers to an isolated periodic orbit of the system) has been widely concerned by scholars. At present, there are many research papers on the bifurcation of limit cycles of two-dimensional nonsmooth differential systems. As far as we know, there are mainly two related research methods: the Melnikov function method established by Han Maoan and the average method established by Llibre. Yang and Zhao extended the Picard-Fuchs equation method to the study of bifurcation of limit cycles for two-dimensional nonsmooth differential systems. The Melnikov function method plays an important part in dealing with this problem, see [3, 4] and references therein. However, calculating the detailed expression of Melnikov functions of differential equations is a difficult problem.

In [5], Yang, Zhang and Liu investigated a non-smooth Hamilton system with piecewise perturbation as follows

$$\begin{cases} \dot{x} = y + \varepsilon f^{+}(x,y), & x \geq 0\,, \\ \dot{y} = -\frac{3}{2}x^2 + 3x - \frac{3}{2} + \varepsilon g^{+}(x,y), & x \geq 0, \\ \dot{x} = y + \varepsilon f^{-}(x,y), & x < 0, \\ \dot{y} = 1 + x + \varepsilon g^{-}(x,y), & x < 0, \end{cases} \tag{1}$$

where

$$f^{\pm}(x,y)=\sum_{i+j=0}^{n} a_{i,j}^{\pm}\, x^i y^j,\ \ g^{\pm}(x,y)=\sum_{i+j=0}^{n} b_{i,j}^{\pm}\, x^i y^j \tag{2}$$

where $a_{i,j}^{\pm}, b_{i,j}^{\pm} \in R$ in (2). When $\varepsilon=0$, the corresponding Hamilton functions of (1) are the following (3) and (4)

$$H^{+}(x,y)=\frac{1}{2}y^2+\frac{1}{2}x^3-\frac{3}{2}x^2+\frac{3}{2}x, x\geq 0, \tag{3}$$

and

$$H^{-}(x,y)=\frac{1}{2}y^2-\frac{1}{2}x^2-x+\frac{1}{2}, x<0. \tag{4}$$

When $\varepsilon=0$, system (1) has a family of periodic orbits as follows

$$\Gamma_h=\left\{(x,y)\middle|H^{+}(x,y)=\frac{h}{2}, x\geq 0\right\}\cup\left\{(x,y)\middle|H^{-}(x,y)=\frac{h-1}{2}, x<0\right\}:=\Gamma_h{}^{+}\cup\Gamma_h{}^{-}, \tag{5}$$

with $h\in(0,1)$ in (5). If h approaches to 1, L_h tends to origin that is an elementary center of parabolic-focus type (see [6, 7]). And if $h\to 0$, $L_h\to L_0$, where L_0 is a generalized heteroclinic loop with a cusp (1,0); see Fig. 1.

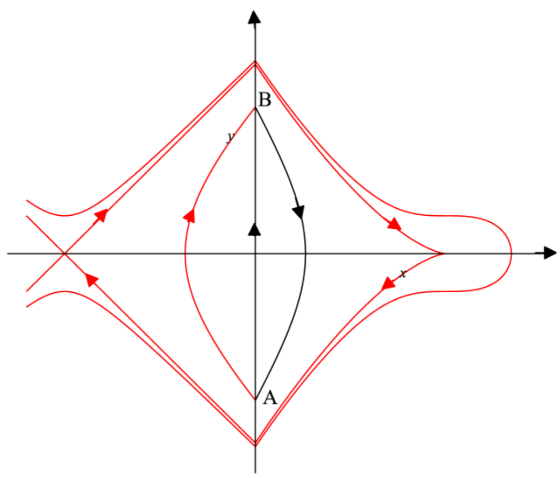

Figure 1. Phase diagram of system $(\mathbf{1})|_{\varepsilon=\mathbf{0}}$.

From the Th 1.1 in [8], one gets the generalized first order Melnikov function of system (1) as follows (6)

$$M(h)=\frac{H_y^{+}(B)}{H_y^{-}(B)}\left[\frac{H_y^{-}(A)}{H_y^{+}(A)}\int_{\Gamma_h{}^{+}} g^{+}(x,y)dx-f^{+}(x,y)dy+\int_{\Gamma_h{}^{-}} g^{-}(x,y)dx-f^{-}(x,y)dy\right], h\epsilon(0,1). \tag{6}$$

One also knows that the number of zeros of formula (6) controls the number of limit cycles of the perturbed system (1) for $|\boldsymbol{\varepsilon}|$ small enough. Since $H_y^{+}(A)=H_y^{-}(A)$ and $H_y^{+}(B)=H_y^{-}(B)$, (4) can be written as

$$M(h)=\int_{\Gamma_h{}^{+}} g^{+}(x,y)dx-f^{+}(x,y)dy+\int_{\Gamma_h{}^{-}} g^{-}(x,y)dx-f^{-}(x,y)dy:=\Phi(h)+\Psi(h) \tag{7}$$

In [5], the authors obtained $M(h)$ of system (1) as shown in Theorem A (see Th. 1.4 in [5]):

Theorem A. The first order Melnikov function of system (1) is

$$M(h)=\sqrt{h}f_n(\sqrt{h})+g_{\left[\frac{n-1}{2}\right]}(h)I_1(h)+\bar{g}_{\left[\frac{3n-5}{6}\right]}(h)I_2(h)+\tilde{g}_{\left[\frac{n-1}{2}\right]}(h)J_1(h) \tag{8}$$

where $0<h<1$, $f_i(t), g_i(t), \bar{g}_i(t)$ and $\tilde{g}_i(t)$ are polynomials of t with degree *i* in (8), and

$$I_i(h)=\int_0^{\sqrt{h}}(1+y^2-h)^{\frac{i}{3}}dy, J_1(h)=\int_0^{\sqrt{h}}(1+y^2-h)^{\frac{1}{2}}dy. \tag{9}$$

here *i*=1,2 in (9).

In the present article, we will use a simple approach to find $M(h)$ of (1) that can greatly simplify computations. Denote

$$I_{i,j}(h)=\int_{\Gamma_h{}^{+}} x^i y^j dy, J_{i,j}(h)=\int_{\Gamma_h{}^{-}} x^i y^j dy, h\in(0,1), \tag{10}$$

where $i,j \in \mathbb{N}$ in (10).

2. Algebraic structure of the first order Melnikov function M (h)

It is easy to find that, the orbits $\Gamma_h^{\pm}$ are symmetric with respect to the x-axis when $0 < h < 1$. Thus, $I_{i,2j+1}(h) = J_{i,2j+1}(h) \equiv 0$. Hence one only considers $I_{i,2j}(h)$ and $J_{i,2j}(h)$ when $0 < h < 1$.

Lemma 1.1. When $0 < h < 1$, the $\Phi(h)$ and $\Psi(h)$ in (7) can be written as

$$\Phi(h) = \alpha(h)I_{0,0}(h) + \beta(h)I_{1,0}(h) + \gamma(h)I_{2,0}(h), \Psi(h) = \xi(h)J_{0,0}(h) + \eta(h)J_{1,0}(h), \tag{11}$$

where (11) $\alpha(h)$, $\beta(h)$, $\gamma(h)$, $\xi(h)$ and $\eta(h)$ are polynomials in h with $deg\alpha(h)$, $deg\xi(h) \le \left[\frac{n}{2}\right]$, $deg\beta(h)$, $deg\eta(h) \le \left[\frac{n-1}{2}\right]$ and $deg\gamma(h) \le \left[\frac{n-2}{2}\right]$. Hence,

$$M(h) = \alpha(h)I_{0,0}(h) + \beta(h)I_{1,0}(h) + \gamma(h)I_{2,0}(h) + \xi(h)J_{0,0}(h) + \eta(h)J_{1,0}(h). \tag{12}$$

Proof. Denote $\boldsymbol{\Omega}$ by the interior of $\Gamma_h^{+} \cup \overrightarrow{AB}$, see Fig. 1. By Green Formula, one has

$$\int_{\Gamma_h^+} x^i y^j\,dx = \oint_{\Gamma_h^+\cup\overrightarrow{AB}} x^i y^j\,dx = j\iint_{\Omega} x^i y^{j-1} dx\,dy, \tag{13}$$

$$\int_{\Gamma_h^+} x^i y^j\,dy = \oint_{\Gamma_h^+\cup\overrightarrow{AB}} x^i y^j\,dy = -i\iint_{\Omega} x^{i-1} y^{j}\,dx\,dy. \tag{14}$$

Hence, by (13) and (14), one has

$$\int_{\Gamma_h^+} x^i y^j\,dx = -\frac{j}{i+1} I_{i+1,j-1}(h) \tag{15}$$

Similarly, one gets

$$\int_{\Gamma_h^-} x^i y^j\,dx = -\frac{j}{i+1} I_{i+1,j-1}(h) \tag{16}$$

A simple straightforward calculation gives

$$M(h) = -\sum_{i+j=1,j\ge1}^{n}\frac{j}{i+1}b_{i,j}^{+}\int_{\Gamma_h^+} x^{i+1}y^{j-1}dy - \sum_{i+j=0}^{n}a_{i,j}^{+}\int_{\Gamma_h^+} x^i y^j\,dy - \sum_{i+j=1,j\ge1}^{n}\frac{j}{i+1}b_{i,j}^{-}\int_{\Gamma_h^+} x^{i+1}y^{j-1}dy - \sum_{i+j=0}^{n} a_{i,j}^{-}\int_{\Gamma_h^+} x^i y^j\,dy := \sum_{i+j=0}^{n}\sigma_{i,j}\,I_{i,j}(h) + \sum_{i+j=0}^{n}\tau_{i,j}\,J_{i,j}(h), \tag{17}$$

in view of (15) and (16).

For simplicity, one only verifies the first equality of (11), and the other one can be proved in a similar way. Differentiating both sides of (3) with respect to y yields that

$$2y + 3x^2\frac{\partial x}{\partial y} - 6x\frac{\partial x}{\partial y} + 3\frac{\partial x}{\partial y} = 0 \tag{18}$$

Multiplying (18) by $x^i y^{j-1}dy$ and by (15), one has

$$I_{i,j} = \frac{3(j-1)}{2}\left[\frac{1}{i+3}I_{i+3,j-2} - \frac{2}{i+2}I_{i+2,j-2} + \frac{1}{i+1}I_{i+1,j-2}\right] \tag{19}$$

Similarly, multiplying (3) by $x^{i-2}y^j$ dy and integrating over Γ_h^{-} gives

$$I_{i,j} = hI_{i-3,j} + 3I_{i-1,j} - I_{i-3,j+2} - 3I_{i-2,j} \tag{20}$$

Elementary manipulations reduces Eqs. (19) and (20) to

$$I_{i,j} = \frac{3(j-1)}{2i+3j+3}\left[hI_{i,j-2} - \frac{2i}{i+1}I_{i+1,j-2} + \frac{i}{i+2}I_{i+2,j-2}\right] \tag{21}$$

and

$$I_{i,j} = \frac{2i}{2i+3j+3}\left[hI_{i-3,j} - \frac{3(i+j)}{i-1}I_{i-1,j} + \frac{3(3i+j-3)}{2(i-2)}I_{i-2,j}\right] \tag{22}$$

We will prove the (11) by mathematical induction. For $n = 2,3$, (21) and (22) imply (23)

$$\begin{cases} I_{0,2} = \frac{1}{3}hI_{0,0}, \\ I_{1,2} = \frac{3}{11}hI_{1,0} - \frac{3}{11}I_{2,0} + \frac{1}{11}I_{3,0}, \\ I_{3,0} = \frac{2}{3}hI_{0,0} - 6I_{1,0} + 3I_{2,0}, \end{cases} \tag{23}$$

(23) gives the equality when $n = 2,3$. Now suppose the conclusion holds for $i + j \le k - 1 (k \ge 3)$. If k is odd, then

for $i+j=k$, letting $(i,j)=(1,k-1),(3,k-3),(5,k-5),\cdots,(k-2,2)$ in (21) and $(i,j)=(k,0)$ in (22), respectively, one gets (24)

$$A\begin{pmatrix} I_{1,k-1} \\ I_{3,k-3} \\ I_{5,k-5} \\ \vdots \\ I_{k-2,2} \\ I_{k,0} \end{pmatrix}=\begin{pmatrix} \frac{3(k-2)}{3k+2}(hI_{1,k-3}-I_{2,k-3}) \\ \frac{k-4}{k}(hI_{3,k-5}-\frac{3}{2}I_{4,k-5}) \\ \frac{3(k-6)}{3k-2}(hI_{5,k-7}-\frac{5}{3}I_{6,k-7}) \\ \vdots \\ \frac{3}{2k+5}(hI_{k-2,0}-\frac{2(k-2)}{k-1}I_{k-1,0}) \\ \frac{2k}{2k+3}(hI_{k-3,0}+\frac{3k}{k-1}I_{k-1,0}-\frac{9(k-1)}{2k-4}I_{k-2,0}) \end{pmatrix}, \tag{24}$$

where A is a constant matrix as shown in (25)

$$A=\begin{pmatrix} 1 & \frac{k-2}{3k+2} & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{3(k-4)}{5k} & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{3(k-2)}{k(2k+5)} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{25}$$

Hence, for $i+2j=k$ and $j=1,2,\cdots,\frac{k-1}{2}$,

$$\begin{aligned} I_{i,2j}(h)&=\alpha_{k-1}(h)I_{0,0}(h)+\beta_{k-1}(h)I_{1,0}(h)+\gamma_{k-1}(h)I_{2,0}(h)+h[\alpha_{k-2}(h)I_{0,0}(h)+\beta_{k-2}(h)I_{1,0}(h)+\gamma_{k-2}(h)I_{2,0}(h)] \\ &:=\alpha_k(h)I_{0,0}(h)+\beta_k(h)I_{1,0}(h)+\gamma_k(h)I_{2,0}(h). \end{aligned} \tag{26}$$

In (26) $\alpha_{k-s}(h)$, $\beta_{k-s}(h)$ and $\gamma_{k-s}(h)(s=1,2)$ are the polynomials in h satisfying

$$deg\alpha_{k-s}(h)\le\left[\frac{k-s}{2}\right],\ deg\beta_{k-s}(h)\le\left[\frac{k-1-s}{2}\right], deg\gamma_{k-s}(h)\le\left[\frac{k-2-s}{2}\right], \tag{27}$$

deg in (27) represents the degree of a polynomial. It is easy to verify (28)

$$deg\alpha_k(h)\le\left[\frac{k}{2}\right],\ deg\beta_k(h)\le\left[\frac{k-1}{2}\right],\ deg\gamma_k(h)\le\left[\frac{k-2}{2}\right] \tag{28}$$

If $j=0$, one has (29)

$$\begin{aligned} I_{i,2j}(h)&=\alpha_{k-1}(h)I_{0,0}(h)+\beta_{k-1}(h)I_{1,0}(h)+\gamma_{k-1}(h)I_{2,0}(h)+\alpha_{k-2}(h)I_{0,0}(h)+\beta_{k-2}(h)I_{1,0}(h)+\gamma_{k-2}(h)I_{2,0}(h)+ \\ &h[\alpha_{k-3}(h)I_{0,0}(h)+\beta_{k-3}(h)I_{1,0}(h)+\gamma_{k-3}(h)I_{2,0}(h)]:=\alpha_k(h)I_{0,0}(h)+\beta_k(h)I_{1,0}(h)+\gamma_k(h)I_{2,0}(h) \end{aligned} \tag{29}$$

where $\alpha_{k-3}(h),\beta_{k-3}(h)$ and $\gamma_{k-3}(h)$ represent the polynomials in h satisfying that (30)

$$deg\alpha_{k-3}(h)\le\left[\frac{k-3}{2}\right],\ deg\beta_{k-3}(h)\le\left[\frac{k-4}{2}\right],\ deg\gamma_{k-3}(h)\le\left[\frac{k-5}{2}\right] \tag{30}$$

we can also get (31)

$$deg\alpha_k(h)\le\left[\frac{k}{2}\right],\ deg\beta_k(h)\le\left[\frac{k-1}{2}\right],\ deg\gamma_k(h)\le\left[\frac{k-2}{2}\right] \tag{31}$$

If k is even, one can shown analogously. This proof is completed. ◊

3. Proof of the Theorem 1.1

Theorem 1.1. $M(h)$ in (7) is

$$M(h)=P(h)\sqrt{h}+Q(h)I_1(h)+R(h)I_2(h)+S(h)J_1(h), h\in(0,1), \tag{32}$$

where $I_1(h)$, $I_2(h)$ and $J_1(h)$ are defined by (8) and $P(h)$, $Q(h)$, $R(h)$, and $S(h)$ are polynomials in h satisfying that (33)

$$\deg P(h)\le\left[\frac{n}{2}\right],\ degQ(h),degR(h)\le\left[\frac{n-1}{2}\right],\ degS(h)\le\left[\frac{n-2}{2}\right] \tag{33}$$

Proof. By some straightforward calculations, we obtain

$$\begin{cases} I_{0,0}(h) = 2\sqrt{h}, I_{1,0}(h) = 2\sqrt{h} - 2I_1(h), \\ I_{2,0}(h) = 2\sqrt{h} - 4I_1(h) + 2I_2(h), \\ J_{0,0}(h) = 2\sqrt{h}, J_{1,0}(h) = 2J_1(h) - 2\sqrt{h}. \end{cases} \tag{34}$$

Substituting (34) into (12) we obtain (32). This proof is completed. ◊

Remark 1.1.

(i) Comparing (7) in the Theorem A with (32) in the Theorem1.1 and noting that $\left[\frac{n-2}{2}\right] = \left[\frac{3n-5}{6}\right]$, we know that (7) coincides with (32), and the computational process in this paper is more simple. Hence, by Theorem 1.5 in [5], we know that a upper bound of the number of zeros of $M(h)$ is $12n + 7$.

(ii) We can also use this method to get the first order Melnikov functions in [9].

(iii) It is worth noting that the outline of the proof Lemma 1.1 in this paper comes from [10].

4. Conclusion and discussion

The purpose of this article is to get an improved method to compute the first order Melnikov function of a nonsmooth Hamilton system. Because this is a necessary step to estimate the number of limit cycles of some Hamilton systems. Using mathematical induction, we prove that the first-order Melnikov function can be written as a combination of generators with polynomial coefficients. This can save a lot of computation.

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