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Some Identities Involving the High-Order Degenerate Type 2 Daehee Polynomials

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Abstract

Numerous polynomial variations and their extensions have been explored extensively and found applications in a variety of research fields. In 2019, T. Kim and D. Kim defined the type 2 Daehee polynomials by the generating function of the type 2 Bernoulli polynomials and express the central factorial numbers of the second kind in terms of type 2 Bernoulli and type 2 Daehee numbers of negative integral orders. In this paper, we define the generating function of the high-order degenerate type 2 Daehee polynomials, then we study the high-order degenerate type 2 Daehee numbers and polynomials by using the method of generating function and Riordon array. First, applying generating functions methods, we obtain some character involving the high-order degenerate type 2 Daehee polynomials. In addition, we establish some new equations and relations involving two classes of generalized Stirling numbers, generalized Lah numbers, high-order type 2 Bernoulli polynomials, the central factorial numbers of the second kind, generalized Harmonic numbers and so on.

Keywords

High-order degenerate type 2 Daehee numbers and polynomials, Generalized stirling numbers, Generalized Lah numbers, High-order type 2 Bernoulli polynomials, Generalized harmonic numbers, The bell numbers

1. Introduction

In recent years, many mathematicians have studied various degenerate versions of Daehee polynomials and numbers, obtaining many arithmetic and combinatorial results [1-4]. In reference [5], Taekyun Kim firstly defined type 2 Daehee polynomials and numbers and derived a series of identities about type 2 Daehee polynomials and numbers. The high-order type 2 Daehee polynomials which are defined by the generating function as follows [5]:

$$\left(\frac{\ln(1+t)}{(1+t)-(1+t)^{-1}}\right)^r(1+t)^x=\sum_{n=0}^{\infty}d_n^{(r)}(x)\frac{t^n}{n!}. \tag{1}$$

when $x=0, d_n^{(r)}(0)=d_n^{(r)}$ is called high-order type 2 Daehee numbers.

Motivated by the works of Kim [6], we define the high-order degenerate type 2 Daehee polynomials.

Definition 1.1 Let n be a non-negative integer, λ be a real number, $r\geq 0$ is an integer. The high-order degenerate type 2 Daehee polynomials are given by means of the following generating function:

$$\left(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}\right)^r(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x=\sum_{n=0}^{\infty}d_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2}$$

when $x=0, d_{n,\lambda}^{(r)}(0)=d_{n,\lambda}^{(r)}$ is called high-order degenerate type 2 Daehee numbers.

The generating functions of the relevant special combinatorial sequences involved in this paper are as follows [1-16]: The Stirling numbers of the first kind and the second kind are defined by

$$\sum_{n\geq k} s(n,k)\frac{t^n}{n!}=\frac{\ln^k(1+t)}{k!}, \tag{3}$$

$$\sum_{n\geq k} S(n,k)\frac{t^n}{n!}=\frac{(e^t-1)^k}{k!}. \tag{4}$$

The generalized Stirling numbers of the first kind and the second kind are defined by

$$\sum_{n\geq k} s(n,k;h)\frac{t^n}{n!}=(1+t)^{-h}\frac{(\ln(1+t))^k}{k!}, \tag{5}$$

$$\sum_{n\geq k} S(n,k;h)\frac{t^n}{n!}=e^{ht}\frac{(e^t-1)^k}{k!}. \tag{6}$$

The degenerate Stirling numbers of the first kind and the second kind are defined by

$$\sum_{n\geq k} s_{1,\lambda}(n,k)\frac{t^n}{n!}=\frac{(\frac{1}{\lambda}[(1+t)^{\lambda}-1])^k}{k!}, \tag{7}$$

$$\sum_{n\geq k} S_{2,\lambda}(n,k)\frac{t^n}{n!}=\frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^k}{k!}. \tag{8}$$

The generalized Lah numbers are defined by the generating function

$$\sum_{n=0}^{\infty} L(n,k;h)\frac{t^n}{n!}=(1+t)^h\frac{1}{k!}(\frac{-t}{1+t})^k. \tag{9}$$

For integer $n,h\geq 1$, the combinatorial numbers are defined by

$$\sum_{n=0}^{\infty}\binom{n+k}{k}P(h,n+k,k)t^n=\frac{(-\ln(1-t))^h}{(1-t)^{k+1}}. \tag{10}$$

The generalized Harmonic numbers are given by the generating function

$$\sum_{n=0}^{\infty} H_{n,h}t^n=\frac{(-\ln(1-t))^h}{h!(1-t)}. \tag{11}$$

The higher-order type 2 Bernoulli polynomials are given by the generating function

$$\sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}=(\frac{t}{e^t-e^{-t}})^r e^{xt}. \tag{12}$$

The second class of Bernoulli polynomials is given by the generating function

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x)\frac{t^n}{n!}=\frac{t}{\ln(1+t)}(1+t)^x. \tag{13}$$

The central factorial numbers of the second kind are defined by

$$\sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}=\frac{1}{k!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k. \tag{14}$$

The Bell numbers of the first kind and the second kind are defined by

$$\sum_{n=k}^{\infty} B(n,k)\frac{t^n}{n!}=\frac{1}{k!}(e^{e^t-1}-1)^k, \tag{15}$$

$$\sum_{n=k}^{\infty}\beta(n,k)\frac{t^n}{n!}=\frac{1}{k!}(\ln(1+ln(1+t)))^k . \tag{16}$$

Lemma 1 If $D=(g(t),f(t))=(d_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan matrix, let $h(t)=\sum_{k\geq 0}h_k t^k$ is the generating function of the sequence $(h_k)_{k\in\mathbb{N}}$. Then we have [15]

$$\sum_{k=0}^{n}d_{n,k}h_k=[t^n]g(t)h(f(t)) . \tag{17}$$

Lemma 2 Let f,g be functions defined on the set of positive integers, then we have the following inversion formula [16]

$$f_n=\sum_{k=0}^{n}s(n,k)g_k \Leftrightarrow g_n=\sum_{k=0}^{n}S(n,k)f_k , \tag{18}$$

$$f_n=\sum_{k=0}^{n}s(n,k;h)g_k \Leftrightarrow g_n=\sum_{k=0}^{n}S(n,k;h)f_k . \tag{19}$$

2. Properties about High-Order Degenerate Type 2 Daehee Numbers and Polynomials

This section derives some properties of high-order degenerate type 2 Daehee numbers and polynomials using the generating function and the method of taking coefficients.

Theorem 2.1 For non-negative integer n, integer $r_i\geq 0$, $m\geq 1$, the high-order degenerate type 2 Daehee polynomial has the following properties

$$d_{n,\lambda}^{(r_1+r_2+\cdots+r_m)}(x_1+x_2+\cdots+x_m)=\sum_{n_1+n_2+\cdots n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}d_{n_1,\lambda}^{(r_1)}(x_1)d_{n_2,\lambda}^{(r_2)}(x_2)\cdots d_{n_m,\lambda}^{(r_m)}(x_m) . \tag{20}$$

Proof By [2], we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}d_{n,\lambda}^{(r_1+r_2+\cdots+r_m)}(x_1+x_2+\cdots+x_m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}d_{n_1,\lambda}^{(r_1)}(x_1)\frac{t^n}{n!}\sum_{n=0}^{\infty}d_{n_2,\lambda}^{(r_2)}(x_2)\frac{t^n}{n!}\cdots\sum_{n=0}^{\infty}d_{n_m,\lambda}^{(r_m)}(x_m)\frac{t^n}{n!} .\\&=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}d_{n_1,\lambda}^{(r_1)}(x_1)d_{n_2,\lambda}^{(r_2)}(x_2)\cdots d_{n_m,\lambda}^{(r_m)}(x_m)\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ In both sides of the last equation, we get the identity.

Corollary 2.1 For $x_1=x_2=\cdots=x_m=0$ in [20], we obtain

$$d_{n,\lambda}^{(r_1+r_2+\cdots+r_m)}=\sum_{n_1+n_2+\cdots n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}d_{n_1,\lambda}^{(r_1)}d_{n_2,\lambda}^{(r_2)}\cdots d_{n_m,\lambda}^{(r_m)} . \tag{21}$$

Let $r_i=1,\ i\in[m]$ in [21], we obtain

$$d_{n,\lambda}^{(m)}=\sum_{n_1+n_2+\cdots n_m=n}\binom{n}{n_1,n_2,\cdots,n_m}d_{n_1,\lambda}d_{n_2,\lambda}\cdots d_{n_m,\lambda} . \tag{22}$$

Corollary 2.2 For$m=2$ in [20], we obtain

$$d_{n,\lambda}^{(r_1+r_2)}(x_1+x_2)=\sum_{l=0}^{n}\binom{n}{l}d_{l,\lambda}^{(r_1)}(x_1)d_{n-l,\lambda}^{(r_2)}(x_2) . \tag{23}$$

Let $x_2=0$ in [23], we obtain

$$d_{n,\lambda}^{(r_1+r_2)}(x_1)=\sum_{l=0}^{n}\binom{n}{l}d_{n-l,\lambda}^{(r_2)}d_{l,\lambda}^{(r_1)}(x_1) . \tag{24}$$

Let $x_1 = x_2 = 0$ in [23], we obtain

$$d_{n,\lambda}^{(r_1+r_2)} = \sum_{l=0}^{n} \binom{n}{l} d_{n-l,\lambda}^{(r_2)} d_{l,\lambda}^{(r_1)}. \tag{25}$$

Let $r_1 = 0,\ r_2 = r$ in [23], we obtain

$$d_{n,\lambda}^{(r)}(x_1 + x_2) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \lambda^{l-m} (x_1)_m s(l,m) d_{n-l,\lambda}^{(r)}(x_2). \tag{26}$$

Let $r = 1$ in [26], we obtain

$$d_{n,\lambda}(x_1 + x_2) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \lambda^{l-m} (x_1)_m s(l,m) d_{n-l,\lambda}(x_2). \tag{27}$$

Let $x_2 = 0$ in [27], we obtain

$$d_{n,\lambda}(x_1) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \lambda^{l-m} (x_1)_m s(l,m) d_{n-l,\lambda}. \tag{28}$$

Theorem 2.2 For integer $n \geq 1$, we obtain

$$d_{n,\lambda}^{(r)}(x+1) - d_{n,\lambda}^{(r)}(x) = \sum_{k=1}^{n} k\lambda^{n-k} s(n,k) d_{k-1}^{(r)}(x). \tag{29}$$

Proof By [2], we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} (d_{n,\lambda}^{(r)}(x+1) - d_{n,\lambda}^{(r)}(x)) \frac{t^n}{n!} \\
&= \left(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})\text{-}(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}} \right)^r (1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x \cdot \ln(1+\lambda t)^{\frac{1}{\lambda}} \\
&= \sum_{k=1}^{\infty} k\lambda^{-k} d_{k-1}^{(r)}(x) \frac{(\ln(1+\lambda t))^k}{k!} \\
&= \sum_{k=1}^{\infty} k\lambda^{-k} d_{k-1}^{(r)}(x) \sum_{n=k}^{\infty} \lambda^n s(n,k) \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} k\lambda^{n-k} s(n,k) d_{k-1}^{(r)}(x) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ In both sides of the last equation, we get the identity.

Corollary 2.3 For $r = 1$ in [29], we obtain

$$d_{n,\lambda}(x+1) - d_{n,\lambda}(x) = \sum_{k=1}^{n} k\lambda^{n-k} s(n,k) d_{k-1}(x). \tag{30}$$

Corollary 2.4 For $x = 0$ in [30], we obtain

$$d_{n,\lambda}(1) - d_{n,\lambda} = \sum_{k=1}^{n} k\lambda^{n-k} s(n,k) d_{k-1}. \tag{31}$$

Theorem 2.3 For integer $n \geq 1$, we obtain

$$\frac{\partial d_{n,\lambda}^{(r)}(x)}{\partial x} = \sum_{m=1}^{n} \sum_{k=1}^{m} k\lambda^{n-m} s(m,k) s(n,m) b_{k-1}^{(r)}(x). \tag{32}$$

Proof By [2], we obtain

$$\sum_{n=0}^{\infty}\frac{\partial d_{n,\lambda}^{(r)}(x)}{\partial x}\frac{t^n}{n!}=\left(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}\right)^r(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x\cdot ln(1+ln(1+\lambda t)^{\frac{1}{\lambda}})$$

$$=\left(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{e^{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}-e^{-\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}}\right)^r e^{x\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}\cdot\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})$$

$$=\sum_{k=0}^{\infty}b_k^{(r)}(x)\frac{[\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})]^k}{k!}\cdot\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})$$

$$=\sum_{k=1}^{\infty}kb_{k-1}^{(r)}(x)\frac{[\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})]^k}{k!}$$

$$=\sum_{k=1}^{\infty}kb_{k-1}^{(r)}(x)\sum_{m=k}^{\infty}\lambda^{-m}s(m,k)\frac{[\ln(1+\lambda t)]^m}{m!}$$

$$=\sum_{k=1}^{\infty}kb_{k-1}^{(r)}(x)\sum_{m=k}^{\infty}\lambda^{-m}s(m,k)\sum_{n=m}^{\infty}\lambda^n s(n,m)\frac{t^n}{n!}$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{k=1}^{m}k\lambda^{n-m}s(m,k)s(n,m)b_{k-1}^{(r)}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ In both sides of the last equation, we get the identity.

3. Identities about High-Order Degenerate Type 2 Daehee Numbers and Polynomials

In this section, by means of the Riordan matrix and generating functions, we derive some new equalities between High-order Degenerate Type 2 Daehee Polynomials and generalized Striling numbers, generalized Lah numbers, higher-order type 2 Bernoulli polynomials, Bell numbers and so on.

Theorem 3.1 For non-negative integer n, we obtain

$$\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l\langle h\rangle_l d_{n-l,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\lambda^{n-k}s(n,k;h)d_k^{(r)}(x). \tag{33}$$

Proof By [17] and [5], we obtain

$$\Re\left(\frac{k!}{n!}\lambda^{n-k}s(n,k;h)\right)=\left((1+\lambda t)^{-h},\frac{(\ln(1+\lambda t))}{\lambda}\right). \tag{34}$$

$$n!\sum_{k=0}^{n}\frac{k!}{n!}\lambda^{n-k}s(n,k;h)\frac{1}{k!}d_k^{(r)}(x)$$

$$=n![t^n](1+\lambda t)^{-h}\left[\left(\frac{\ln(1+y)}{(1+y)-(1+y)^{-1}}\right)^r(1+y)^x \;\Big|\; y=\frac{(\ln(1+\lambda t))}{\lambda}\right]$$

$$=n![t^n](1+\lambda t)^{-h}\left(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}\right)^r(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x$$

$$=n![t^n]\sum_{n=0}^{\infty}(-\lambda)^n\langle h\rangle_n\frac{t^n}{n!}\sum_{n=0}^{\infty}d_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}$$

$$=n![t^n]\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l\langle h\rangle_l d_{n-l,\lambda}^{(r)}(x)\frac{t^n}{n!}$$

$$=\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l\langle h\rangle_l d_{n-l,\lambda}^{(r)}(x),$$

which completes the proof.

Corollary 3.1 For x =0 in [33], we obtain

$$\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^{l}\langle h\rangle_{l}d_{n-l,\lambda}^{(r)}=\sum_{k=0}^{n}\lambda^{n-k}s(n,k;h)d_{k}^{(r)}. \tag{35}$$

Corollary 3.2 When h = 0 in [33], we obtain

$$d_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\lambda^{n-k}s(n,k)d_{k}^{(r)}(x). \tag{36}$$

Theorem 3.2 For non-negative integer n, we obtain

$$\sum_{l=0}^{n}\binom{n}{l}\lambda^{-l}h^{n-l}d_{l}^{(r)}(x)=\sum_{k=0}^{n}\lambda^{-k}S(n,k;h)d_{k,\lambda}^{(r)}(x). \tag{37}$$

Proof By [17] and [6], we obtain

$$\Re\left(\frac{k!}{n!}\lambda^{-k}S(n,k;h)\right)=\left(e^{ht},\frac{e^{t}-1}{\lambda}\right). \tag{38}$$

$$n!\sum_{k=0}^{n}\frac{k!}{n!}\lambda^{-k}S(n,k;h)\frac{1}{k!}d_{k,\lambda}^{(r)}(x)$$

$$=n![t^{n}]e^{ht}\left[\left(\frac{\ln(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{-1}}\right)^{r}(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{x}\,\Big|\,y=\frac{e^{t}-1}{\lambda}\right]$$

$$=n![t^{n}]e^{ht}\left(\frac{\ln(1+\frac{t}{\lambda})}{(1+\frac{t}{\lambda})-(1+\frac{t}{\lambda})^{-1}}\right)^{r}(1+\frac{t}{\lambda})^{x}$$

$$=n![t^{n}]\sum_{n=0}^{\infty}h^{n}\frac{t^{n}}{n!}\sum_{n=0}^{\infty}d_{n}^{(r)}(x)\lambda^{-n}\frac{t^{n}}{n!}$$

$$=n![t^{n}]\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\lambda^{-l}h^{n-l}d_{l}^{(r)}(x)\frac{t^{n}}{n!}$$

$$=\sum_{l=0}^{n}\binom{n}{l}\lambda^{-l}h^{n-l}d_{l}^{(r)}(x),$$

which completes the proof.

Corollary 3.3 By means of Lemma 2, the inverse relation [19], we obtain

$$d_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\lambda^{n-l}h^{k-l}s(n,k;h)d_{l}^{(r)}(x). \tag{39}$$

Corollary 3.4 For x = 0 in [37], we obtain

$$\sum_{l=0}^{n}\binom{n}{l}\lambda^{-l}h^{n-l}d_{l}^{(r)}=\sum_{k=0}^{n}\lambda^{-k}S(n,k;h)d_{k,\lambda}^{(r)}. \tag{40}$$

Corollary 3.5 When $h = 0$ in [37], we obtain

$$d_{n}^{(r)}(x)=\sum_{k=0}^{n}\lambda^{n-k}S(n,k)d_{k,\lambda}^{(r)}(x). \tag{41}$$

Theorem 3.3 For non-negative integer n, we obtain

$$d_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=0}^{k}b_{l}^{(r)}(x)\lambda^{n-k}s(k,l)s(n,k). \tag{42}$$

Proof By [2] and [12], we obtain

$$\sum_{n=0}^{\infty} d_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \Big(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}}\Big)^r (1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^x$$

$$= \Big(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{e^{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})} - e^{-\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}}\Big)^r e^{x\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}$$

$$= \sum_{l=0}^{\infty} b_l^{(r)}(x)\frac{[\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})]^l}{l!}$$

$$= \sum_{l=0}^{\infty} b_l^{(r)}(x)\sum_{k=l}^{\infty} s(k,l)\frac{[\ln(1+\lambda t)^{\frac{1}{\lambda}}]^k}{k!}$$

$$= \sum_{l=0}^{\infty} b_l^{(r)}(x)\sum_{k=l}^{\infty} s(k,l)\lambda^{-k}\sum_{n=k}^{\infty} s(n,k)\lambda^n \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k} b_l^{(r)}(x)\lambda^{n-k}s(k,l)s(n,k)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, we get the identity.
Corollary 3.6 For $x = 0$ in [42], we obtain

$$d_{n,\lambda}^{(r)} = \sum_{k=0}^{n}\sum_{l=0}^{k} b_l^{(r)}\lambda^{n-k}s(n,k)s(k,l). \tag{43}$$

Theorem 3.4 For non-negative integer n, we obtain

$$b_n^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\lambda^{k-l}S(k,l)S(n,k)d_{l,\lambda}^{(r)}(x). \tag{44}$$

Proof Replacing t by $\frac{e^{\lambda(e^t-1)}-1}{\lambda}$ in [2], we obtain

$$\sum_{n=0}^{\infty} d_{n,\lambda}^{(r)}(x)\frac{\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^n}{n!} = \left(\frac{t}{e^t-e^{-t}}\right)^r e^{xt} = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}.$$

On the other hand

$$\sum_{l=0}^{\infty} d_{l,\lambda}^{(r)}(x)\frac{\left(\frac{e^{\lambda(e^t-1)}-1}{\lambda}\right)^l}{l!} = \sum_{l=0}^{\infty} d_{l,\lambda}^{(r)}(x)\lambda^{-l}\sum_{k=l}^{\infty} S(k,l)\lambda^k\frac{(e^t-1)^k}{k!}$$

$$= \sum_{l=0}^{\infty} d_{l,\lambda}^{(r)}(x)\lambda^{-l}\sum_{k=l}^{\infty} S(k,l)\lambda^k\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\lambda^{k-l}S(k,l)S(n,k)d_{l,\lambda}^{(r)}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, we get the identity.
Corollary 3.7 For $x = 0$ in [44], we obtain

$$b_n^{(r)} = \sum_{k=0}^{n}\sum_{l=0}^{k}\lambda^{k-l}S(k,l)S(n,k)d_{l,\lambda}^{(r)}. \tag{45}$$

Corollary 3.8 By means of Lemma 2, the inverse relation [18], we obtain

$$\sum_{k=0}^{n} s(n,k)b_k^{(r)}(x)=\sum_{l=0}^{n}\lambda^{n-l}S(n,l)d_{l,\lambda}^{(r)}(x). \tag{46}$$

Theorem 3.5 For non-negative integer n, we obtain

$$\sum_{k=0}^{n} s_{1,\lambda}(n,k)d_{k,\lambda}^{(r)}(x)=\sum_{k=0}^{n} d_k^{(r)}(x)s(n,k). \tag{47}$$

$$\sum_{k=0}^{n} s_{1,\lambda}(n,k)d_{k,\lambda}^{(r)}(x)=\sum_{k=0}^{n} \beta(n,k)b_k^{(r)}(x). \tag{48}$$

Proof By [17] and [7], we obtain

$$\Re(\frac{k!}{n!}s_{1,\lambda}(n,k))=(1,\frac{1}{\lambda}[(1+t)^{\lambda}-1]). \tag{49}$$

$$\begin{aligned}
&n!\sum_{k=0}^{n}\frac{k!}{n!}s_{1,\lambda}(n,k)\frac{d_{k,\lambda}^{(r)}(x)}{k!}\\
&=n![t^n]\Big[\Big(\frac{\ln(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{-1}}\Big)^r(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^x \mid y=\frac{1}{\lambda}[(1+t)^{\lambda}-1]\Big]\\
&=n![t^n]\Big(\frac{\ln(1+\ln(1+t))}{(1+\ln(1+t))-(1+\ln(1+t))^{-1}}\Big)^r(1+\ln(1+t))^x\\
&=n![t^n]\sum_{k=0}^{\infty}d_k^{(r)}(x)\frac{[\ln(1+t)]^k}{k!}\\
&=n![t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}d_k^{(r)}(x)s(n,k)\frac{t^n}{n!}\\
&=\sum_{k=0}^{n}d_k^{(r)}(x)s(n,k).
\end{aligned}$$

Equation [47] complete the proof.

Observe the above proof process, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}s_{1,\lambda}(n,k)d_{k,\lambda}^{(r)}(x)\frac{t^n}{n!}=\Big(\frac{\ln(1+\ln(1+t))}{(1+\ln(1+t))-(1+\ln(1+t))^{-1}}\Big)^r(1+\ln(1+t))^x.$$

By [17] and [16], we obtain

$$\Re(\frac{k!}{n!}\beta(n,k))=(1,\ln(1+\ln(1+t))). \tag{50}$$

$$\begin{aligned}
&n!\sum_{k=0}^{n}\frac{k!}{n!}\beta(n,k)\frac{b_k^{(r)}(x)}{k!}\\
&=n![t^n]\Big[\Big(\frac{y}{e^y-e^{-y}}\Big)^r e^{xy}\mid y=\ln(1+\ln(1+t))\Big]\\
&=n![t^n]\Big(\frac{\ln(1+\ln(1+t))}{(1+\ln(1+t))-(1+\ln(1+t))^{-1}}\Big)^r(1+\ln(1+t))^x\\
&=n![t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}s_{1,\lambda}(n,k)d_{k,\lambda}^{(r)}(x)\frac{t^n}{n!}\\
&=\sum_{k=0}^{n}s_{1,\lambda}(n,k)d_{k,\lambda}^{(r)}(x).
\end{aligned}$$

Equation [48] complete the proof.

Corollary 3.9 By means of Lemma 2, the inverse relation [18], we obtain

$$d_n^{(r)}(x)=\sum_{l=0}^{n}\sum_{k=0}^{l}s_{1,\lambda}(l,k)S(n,l)d_{k,\lambda}^{(r)}(x). \tag{51}$$

Theorem 3.6 For non − negative integer n, we obtain

$$b_n^{(r)}(x)=\sum_{l=0}^{n}\sum_{k=0}^{l}B(n,l)s_{1,\lambda}(l,k)d_{k,\lambda}^{(r)}(x). \tag{52}$$

Proof The proof procedure from Theorem 3.5, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}s_{1,\lambda}(n,k)d_{k,\lambda}^{(r)}(x)\frac{t^n}{n!}=\Big(\frac{\ln(1+\ln(1+t))}{(1+\ln(1+t))\text{-}(1+\ln(1+t))^{-1}}\Big)^r(1+\ln(1+t))^x.$$

By [17] and [15], we obtain

$$\Re(\frac{k!}{n!}B(n,k)=(1,e^{e^t-1}-1). \tag{53}$$

$$\begin{aligned}
&n!\sum_{l=0}^{n}\frac{l!}{n!}B(n,l)\frac{1}{l!}\sum_{k=0}^{l}s_{1,\lambda}(l,k)d_{k,\lambda}^{(r)}(x)\\
&=n![t^n]\Big[\Big(\frac{\ln(1+\ln(1+y))}{(1+\ln(1+y))\text{-}(1+\ln(1+y))^{-1}}\Big)^r(1+\ln(1+y))^x \mid y=e^{e^t-1}-1\Big]\\
&=n![t^n]\Big(\frac{t}{e^t-e^{-t}}\Big)^r e^{xt}\\
&=n![t^n]\sum_{n=0}^{\infty}b_n^{(r)}(x)\frac{t^n}{n!}\\
&=b_n^{(r)}(x),
\end{aligned}$$

which completes the proof.

Theorem 3.7 For non −negative integer n, we obtain

$$\sum_{k=0}^{n}\lambda^{-k}S_{2,\lambda}(n,k)d_{k,\lambda}^{(r)}(x)=\sum_{k=0}^{n}d_k^{(r)}(x)\lambda^{n-2k}s(n,k). \tag{54}$$

Proof By [17] and [8], we obtain

$$\Re(\frac{k!}{n!}\lambda^{-k}S_{2,\lambda}(n,k))=(1,\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\lambda}). \tag{55}$$

$$\begin{aligned}
&n!\sum_{k=0}^{n}\frac{k!}{n!}\lambda^{-k}S_{2,\lambda}(n,k)\frac{d_{k,\lambda}^{(r)}(x)}{k!}=n![t^n]\Big[\Big(\frac{\ln(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})\text{-}(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{-1}}\Big)^r(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^x \mid y=\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\lambda}\Big]\\
&=n![t^n]\Big(\frac{\ln(1+\frac{1}{\lambda^2}\ln(1+\lambda t))}{(1+\frac{1}{\lambda^2}\ln(1+\lambda t))-(1+\frac{1}{\lambda^2}ln(1+\lambda t))^{-1}}\Big)^r(1+\frac{1}{\lambda^2}\ln(1+\lambda t))^x\\
&=n![t^n]\sum_{k=0}^{\infty}d_k^{(r)}(x)\lambda^{-2k}\frac{[\ln(1+\lambda t)]^k}{k!}\\
&=n![t^n]\sum_{k=0}^{\infty}d_k^{(r)}(x)\lambda^{-2k}\sum_{n=k}^{\infty}s(n,k)\lambda^n\frac{t^n}{n!}\\
&=n![t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}d_k^{(r)}(x)\lambda^{n-2k}s(n,k)\frac{t^n}{n!}=\sum_{k=0}^{n}d_k^{(r)}(x)\lambda^{n-2k}s(n,k),
\end{aligned}$$

which completes the proof.

Corollary 3.10 By means of Lemma 2, the inverse relation [18], we obtain

$$d_n^{(r)}(x)=\sum_{l=0}^{n}\sum_{k=0}^{l}\lambda^{2n-l-k}S_{2,\lambda}(l,k)S(n,l)d_{k,\lambda}^{(r)}(x). \tag{56}$$

Theorem 3.8 For non − negative integer n, we obtain

$$\mathcal{B}_n(x)+\mathcal{B}_n(x-1)=\sum_{k=0}^{n}\lambda^{n-k}d_{k,\lambda}^{(-1)}(x)S(n,k). \tag{57}$$

Proof Let us take $r=-1$ and t replaced by $\frac{e^{\lambda t}-1}{\lambda}$ in [2], we obtain

$$\sum_{n=0}^{\infty}d_{n,\lambda}^{(-1)}(x)\frac{(\frac{e^{\lambda t}-1}{\lambda})^n}{n!}=\frac{(1+t)-(1+t)^{-1}}{\ln(1+t)}(1+t)^x$$

$$=\frac{t}{\ln(1+t)}(1+t)^{x-1}[1+(1+t)]$$

$$=\frac{t}{\ln(1+t)}(1+t)^{x-1}+\frac{t}{\ln(1+t)}(1+t)^x$$

$$=\sum_{n=0}^{\infty}\mathcal{B}(x-1)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathcal{B}(x)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}(\mathcal{B}(x)+\mathcal{B}(x-1))\frac{t^n}{n!}.$$

On the other hand

$$\sum_{n=0}^{\infty}d_{n,\lambda}^{(-1)}(x)\frac{(\frac{e^{\lambda t}-1}{\lambda})^n}{n!}=\sum_{k=0}^{\infty}d_{k,\lambda}^{(-1)}(x)\lambda^{-k}\frac{(e^{\lambda t}-1)^k}{k!}$$

$$=\sum_{k=0}^{\infty}d_{k,\lambda}^{(-1)}(x)\lambda^{-k}\sum_{n=k}^{\infty}S(n,k)\lambda^n\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^{n-k}d_{k,\lambda}^{(-1)}(x)S(n,k)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, we get the identity.

Theorem 3.9 For non-negative integer n, we obtain

$$2^{n+k}T(n+k,k)=\binom{n+k}{k}\sum_{m=0}^{n}\sum_{l=0}^{m}d_{l,\lambda}^{(-k)}\lambda^{m-l}S(m,l)S(n,m). \tag{58}$$

Proof Let us take $r=-k$ in [2], by [2] with $x=0$ and t replaced by $\frac{e^{\lambda\left(e^{\frac{t}{2}}-1\right)}-1}{\lambda}$, we obtain

$$\sum_{n=0}^{\infty}d_{n,\lambda}^{(-k)}\frac{(\frac{e^{\lambda(e^{\frac{t}{2}}-1)}-1}{\lambda})^n}{n!}=(\frac{2}{t})^k(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k=\frac{2^k k!}{t^k}\frac{1}{k!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k$$

$$=\frac{2^k k!}{t^k}\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!}=2^k k!\sum_{n=0}^{\infty}T(n+k,k)\frac{t^n}{(n+k)!}$$

$$=2^k\sum_{n=0}^{\infty}T(n+k,k)\frac{1}{\binom{n+k}{k}}\frac{t^n}{n!}.$$

On the other hand

$$\sum_{l=0}^{\infty} d_{l,\lambda}^{(-k)} \frac{(\frac{e^{\lambda(e^{\frac{t}{2}}-1)}-1}{\lambda})^l}{l!} = \sum_{l=0}^{\infty} d_{l,\lambda}^{(-k)} \lambda^{-l} \sum_{m=l}^{\infty} S(m,l)\lambda^m \frac{(e^{\frac{t}{2}}-1)^m}{m!}$$
$$= \sum_{l=0}^{\infty} d_{l,\lambda}^{(-k)} \lambda^{-l} \sum_{m=l}^{\infty} S(m,l)\lambda^m \sum_{n=m}^{\infty} S(n,m)2^{-n} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{m=0}^{n} \sum_{l=0}^{m} d_{l,\lambda}^{(-k)} \lambda^{m-l} S(m,l) S(n,m) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, we get the identity.

Theorem 3.10 For non-negative integer n, we obtain

$$\sum_{k=0}^{n} \lambda^{-k} L(n,k;h) d_{k,\lambda}^{(r)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (-\lambda)^{-m} (h)_{n-l} s(l,m) d_m^{(r)}(x). \tag{59}$$

Proof By [17] and [9], we obtain

$$\Re(\frac{k!}{n!}\lambda^{-k} L(n,k;h)) = ((1+t)^h, \frac{-t}{\lambda(1+t)}). \tag{60}$$

$$n!\sum_{k=0}^{n} \frac{k!}{n!}\lambda^{-k} L(n,k;h) \frac{d_{k,\lambda}^{(r)}(x)}{k!}$$
$$= n![t^n](1+t)^h \Big[\Big(\frac{\ln(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})\text{-}(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{\text{-}1}} \Big)^r (1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^x \,\Big|\, y = \frac{-t}{\lambda(1+t)} \Big]$$
$$= n![t^n](1+t)^h \Big(\frac{\ln(1+\ln(1+t)^{\text{-}\frac{1}{\lambda}})}{(1+\ln(1+t)^{\text{-}\frac{1}{\lambda}})\text{-}(1+\ln(1+t)^{\text{-}\frac{1}{\lambda}})^{\text{-}1}} \Big)^r (1+\ln(1+t)^{-\frac{1}{\lambda}})^x$$
$$= n![t^n](1+t)^h \sum_{m=0}^{\infty} d_m^{(r)}(x)(-\lambda)^{-m} \frac{(\ln(1+t))^m}{m!}$$
$$= n![t^n]\sum_{n=0}^{\infty} (h)_n \frac{t^n}{n!} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-\lambda)^{-m} s(n,m) d_m^{(r)}(x) \frac{t^n}{n!}$$
$$= n![t^n]\sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (-\lambda)^{-m} (h)_{n-l} s(l,m) d_m^{(r)}(x) \frac{t^n}{n!}$$
$$= \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (-\lambda)^{-m} (h)_{n-l} s(l,m) d_m^{(r)}(x),$$

which completes the proof.

Corollary 3.11 For $h = 0$ in [59], we obtain

$$\sum_{k=0}^{n} \lambda^{-k} L(n,k) d_{k,\lambda}^{(r)}(x) = \sum_{m=0}^{n} (-\lambda)^{-m} s(n,m) d_m^{(r)}(x). \tag{61}$$

Theorem 3.11 For non-negative integer n, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \lambda^{-k} P(h,n,k) d_{k,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} \sum_{h=0}^{m} (-1)^{k-m} h! \lambda^{h-m} \binom{m}{h} s(k,m) d_{m-h}^{(r)}(x). \tag{62}$$

Proof By [17] and [10], we obtain

$$\Re(\binom{n}{k}\lambda^{-k}P(h,n,k))=(\frac{(-\ln(1-t))^{r}}{1-t},\frac{t}{\lambda(1-t)}). \tag{63}$$

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\lambda^{-k}P(h,n,k)\frac{d_{k,\lambda}^{(r)}(x)}{k!}\\
&=[t^n]\frac{(-\ln(1-t))^{h}}{1-t}[(\frac{\ln(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{-1}})^{r}\ (1+\ln(1+\lambda y)^{\frac{1}{\lambda}})^{x}\mid y=\frac{t}{\lambda(1-t)}]\\
&=[t^n]\frac{(-\ln(1-t))^{h}}{1-t}(\frac{\ln(1+\ln(1-t)^{-\frac{1}{\lambda}})}{(1+\ln(1-t)^{-\frac{1}{\lambda}})-(1+\ln(1-t)^{-\frac{1}{\lambda}})^{-1}})^{r}(1+\ln(1-t)^{-\frac{1}{\lambda}})^{x}\\
&=[t^n]\frac{(-\ln(1-t))^{h}}{1-t}\sum_{m=0}^{\infty}d_{m}^{(r)}(x)(-\lambda)^{-m}\frac{(\ln(1-t))^{m}}{m!}\\
&=[t^n]\frac{(-1)^{h}}{1-t}\sum_{m=h}^{\infty}d_{m-h}^{(r)}(x)(-\lambda)^{h-m}h!\binom{m}{h}\sum_{n=m}^{\infty}(-1)^{n}s(n,m)\frac{t^n}{n!}\\
&=[t^n]\sum_{n=0}^{\infty}t^n\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{h=0}^{m}(-1)^{n+h}h!(-\lambda)^{h-m}\binom{m}{h}s(n,m)d_{m-h}^{(r)}(x)\frac{t^n}{n!}\\
&=[t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{h=0}^{m}(-1)^{k-m}h!\lambda^{h-m}\binom{m}{h}s(k,m)d_{m-h}^{(r)}(x)\frac{1}{k!}t^n\\
&=\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{h=0}^{m}(-1)^{k-m}h!\lambda^{h-m}\binom{m}{h}s(k,m)d_{m-h}^{(r)}(x)\frac{1}{k!},
\end{aligned}$$

which completes the proof.

Theorem 3.12 For non-negative integer n, we obtain

$$\sum_{k=0}^{n}(-\lambda)^{n-2k}(n-k)!H_{n,k}d_{k}^{(r)}(x)=\sum_{k=0}^{n}d_{n-k,\lambda}^{(r)}(x). \tag{64}$$

Proof By [17] and [11], we obtain

$$\Re(h!(-\lambda)^{n-h}H_{n,h})=(\frac{1}{1+\lambda t},\frac{\ln(1+\lambda t)}{\lambda}). \tag{65}$$

$$\begin{aligned}
&\sum_{k=0}^{n}k!(-\lambda)^{n-k}H_{n,k}\frac{d_{k}^{(r)}(x)}{k!}=[t^n]\frac{1}{1+\lambda t}[(\frac{\ln(1+y)}{(1+y)-(1+y)^{-1}})^{r}(1+y)^{x}\mid y=\frac{\ln(1+\lambda t)}{\lambda}]\\
&=[t^n]\frac{1}{1+\lambda t}(\frac{\ln(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})-(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{-1}})^{r}(1+\ln(1+\lambda t)^{\frac{1}{\lambda}})^{x}\\
&=[t^n]\sum_{n=0}^{\infty}(-\lambda)^{n}t^{n}\sum_{n=0}^{\infty}d_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}=\sum_{k=0}^{n}(-\lambda)^{k}\frac{d_{n-k,\lambda}^{(r)}(x)}{(n-k)!},
\end{aligned}$$

which completes the proof.

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