Engineering Advances, 2023, 3(4), 363-368
https://www.hillpublisher.com/journals/ea/
ISSN Online: 2768-7961

Analytic Theory of the Evolution of Circle Pearcey Beam

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How to cite this paper: Minghan Liu, Gouhua Fu. (2023) Analytic Theory of the Evolution of Circle Pearcey Beam. *Engineering Advances*, 3(4), 363-368.
DOI: 10.26855/ea.2023.08.017

Received: July 18, 2023
Accepted: August 15, 2023
Published: September 11, 2023

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Abstract

With the research of abrupt autofocusing beams deeper than ever before, the application of that turns wider, such as the particle manipulation, biomedicine, optical communication, vortex beams and so on, arousing scientists' interest further. However, the analytical solution of circle Pearcey beams (CPBs) in paraxial propagation has not yet been proposed so far. In this work, we first propose the analytical solution of circle Pearcey beams (CPBs) transmitting on axi, and obtain semi-analytical solution of which transmitting off axis under paraxial condition. Particularly, the complex amplitude and analytical expressions of the circle Pearcey beam (CPB) at any point in the free space under the paraxial approximation are obtained by means of the stationed phase method, the asymptotic theory and decouple theory–separation variable method, which contains separation variable method. It's fantastic that the analytical solution we acquire is well consistent with the numerical solution gained by three-step Fourier algorithm. Furthermore, compared with the three-step method, we theoretically get the self-focusing distance of circle Pearcey beams (CPBs), expounding the phenomena of autofocusing and the law of spatial evolution under paraxial approximation.

Keywords

Asymptotic theory, Stationed phase method, Self-focusing, Differential equation

1. Introduction

In recent years, the circle Airy beams (CABs) and the circle Pearcey beam (CPBs) have been proposed. During the propagation, they all self-focus and oscillate before and after the focus [1]. Different from those in nonlinear media, their self-focusing effects are caused by the structure of the beam itself. This unique feature makes them essential for frontier application in optical manipulation, particle capture, optical communication and other fields [2]. For example, spatiotemporal self-focusing effect can be produced by the combination of circle Pearcey pulse and Gaussian pulse. The auto-focusing phenomenon of circle Airy beam has been analyzed and explained by an analytical solution, while the self-focusing behavior of CPB has not been analyzed and explained by an analytical solution. For the analysis of spatial evolution behavior of CPB, there is only numerical calculation by "three-step method" at present.

Recently, Riccardo Borghi has proposed some methods to solve the integration of phase functions with higher degree polynomials [3]. Compared with BH method, the sequence obtained by WT has better convergence, smaller numerical error and better approximation effect. The Fresnel diffraction integral of CPB also faces the same problem that the phase function is a higher degree polynomial. However, it is worth noting that in calculating the aperture integral, the amplitude is not constant, and it cannot even be regarded as a moderated term in general, for which super-asymptotic method is ineffective, and so we can only improve the original method to solve the problem.

Consequently, here, we obtain the analytic solution of the Fresnel diffraction integration on and off the axis of a CPB under paraxial conditions using the method of simplified series. Under paraxial approximation, for the light intensity on the axis, we use the stationed phase method to approximate the integral, the series and integral transformation to get the analytic expression of the complex amplitude and light intensity on the axis [4]. For the light intensity outside the axis, the multiple integrals are transformed by modeling. Then, the coupled multiple integrals of the integrand become the product of two simple independent line integrals that is a semi-analytic expression. Ultimately, we will make a detailed analysis on the solution of Fresnel diffraction integral under paraxial condition and on and off the axis under paraxial condition [5].

2. Theory

2.1 Transmission under paraxial condition of CPB

The definition of CPB which is a definite integral is similar to the two-dimensional Pearcey beam. The integration is defined as [6]:

$$Pe(-\frac{\rho}{p_s},0)=\int_{-\infty}^{+\infty}e^{i(s^4-\frac{\rho}{p_s}s^2)}ds. \tag{1}$$

According to the definition of the CPB, the solution of the integral is circular symmetric. The transmission of CPB in free space, especially under paraxial conditions, is very interesting and will generate self-focusing effect. Pearcey beam will generate oscillations during propagation. Compared with Airy beam, the oscillations after the focus are significantly weakened, the autofocusing energy is more concentrated, and the peak contrast is more obvious [7]. Supposing that the light field at the initial plane $Z = 0$ is a given CPB, then under paraxial conditions, the complex amplitude when the beam propagated an arbitrary distance Z can be calculated by the Fresnel diffraction integral formula:

$$\begin{aligned}U(r,Z)&=\frac{1}{2iZ\pi}\int_0^{2\pi}\int_0^{+\infty}Pe(-\frac{\rho}{p_s},0)e^{i\frac{r^2+\rho^2+2r\rho cos(\theta-\varphi)}{2Z}}\rho d\rho d\varphi\\&=\frac{1}{Z}e^{i\frac{r^2}{2Z}}e^{-i\frac{\pi}{2}}\int_0^{+\infty}Pe(-\frac{\rho}{p_s},0)e^{i\frac{\rho^2}{2Z}}J_0\left(\frac{r\rho}{Z}\right)\rho d\rho.\end{aligned} \tag{2}$$

We take $Z=\frac{z}{kw_0^2}$ as the normalized transmission distance, and k denotes the wave vector, r shows the normalized radius of the Z plane, ρ is the normalized radius of the incident plane, all of which are normalized by $x_0=w_0=1$. Substituting equation (1) into equation (2), we can obtain:

$$U(r,Z)=\frac{1}{Z}e^{i\frac{r^2}{2Z}}e^{-i\frac{\pi}{2}}\int_0^{+\infty}\int_{-\infty}^{+\infty}e^{i(s^4-\frac{\rho}{p_s}s^2)}e^{i\frac{\rho^2}{2Z}}J_0\left(\frac{r\rho}{Z}\right)\rho dsd\rho. \tag{3}$$

After that, we introduce the function T:

$$\begin{aligned}T&=\int_0^{+\infty}\int_{-\infty}^{+\infty}e^{i(s^4-\frac{\rho}{p_s}s^2)}e^{i\frac{\rho^2}{2Z}}J_0\left(\frac{r\rho}{Z}\right)\rho dsd\rho\\&=\int_{-\infty}^{+\infty}e^{is^4}\left[\int_0^{+\infty}e^{i\frac{\rho^2}{2Z}}e^{-i\frac{s^2}{p_s}\rho}J_0\left(\frac{r\rho}{Z}\right)\rho d\rho\right]ds.\end{aligned} \tag{4}$$

Besides, we create the function Q defined by:

$$\begin{aligned}Q&=\int_0^{+\infty}e^{-i\frac{s^2}{p_s}\rho}e^{i\frac{\rho^2}{2Z}}J_0\left(\frac{r\rho}{Z}\right)\rho\, d\rho\\&=\int_0^{r_0}e^{i\left(\frac{\rho^2}{2Z}-\frac{s^2}{p_s}\rho\right)}J_0\left(\frac{r\rho}{Z}\right)\rho\, d\rho\,,\end{aligned} \tag{5}$$

where, r_0 is the maximum hole radius of the initial incident plane after normalization.

2.2 Theoretical analysis of transmission on axis

According to Watson lemma, [8] Q can be expanded into series:

$$\begin{aligned}Q&=\int_0^{r_0}e^{i(\frac{\rho^2}{2Z}-\frac{s^2}{p_s}\rho)}\sum_{m=0}^{+\infty}\left[\frac{(-1)^m}{2^{2m}(m!)^2}\left(\frac{r\rho}{Z}\right)^{2m}\right]\rho d\rho\\&=\sum_{m=0}^{+\infty}\left[\frac{(-1)^m r^{2m}}{(2Z)^{2m}(m!)^2}\int_0^{r_0}e^{i(\frac{\rho^2}{2Z}-s^2\frac{\rho}{p_s})}\rho^{2m+1}d\rho\right]\end{aligned} \tag{6}$$

The sum function of the series is obtained by expanding the series of the first class of zero-order Bessel functions and summing up by integrating term by term. Each term contains a definite integral, we can write:

$$L=\int_0^{r_0} e^{i\left(\frac{\rho^2}{2Z}-s^2\frac{\rho}{p_s}\right)}\rho^{2m+1}d\rho. \tag{7}$$

The integral interval is [0, r_0]. When $r = 0$, the series only remains the term which is the function of ρ. We get:

$$L=\int_0^{r_0} e^{i\left(\frac{\rho^2}{2Z}-s^2\frac{\rho}{p_s}\right)}\rho d\rho. \tag{8}$$

When $\ll r_0^2$, the amplitude function $U(\rho)=\rho$ can be regarded as a slow varying function within the integral interval, and the integral can be approximated by the stationed phase theory. We can get the saddle point of the phase function" $\psi(\rho)=\frac{\rho^2}{2Z}-\frac{s^2}{p_s}\rho$ " by taking the derivative of it:

$$\psi'(\rho)=\frac{\rho}{Z}-\frac{s^2}{p_s}. \tag{9}$$

$$\psi''(\rho)=\frac{1}{Z}. \tag{10}$$

$$\psi'''(\rho)=0. \tag{11}$$

set $\psi'(\rho)=0$, we have the stationed phase point $\rho_0=\frac{Z}{p_s}s^2$. Because $s\in(-\infty,+\infty)$, we can always find that s enable $\rho_0\in[0,r_0]$. If the stationed phase point is not within the integral interval, the integral is 0. The integral can be approximated as:

$$\begin{aligned}L&\approx\rho_0 e^{i\left(\frac{\rho_0^2}{2Z}-s^2\frac{\rho_0}{p_s}\right)}(2Z\pi)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\\&=\frac{Z}{p_s}s^2e^{-i\frac{Z}{2p_s^2}s^4}(2Z\pi)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\end{aligned} \tag{12}$$

where, the integral interval is taken $s\in[-a_0,+a_0]$, and we set $a_0=\left(\frac{r_0p_s}{Z}\right)^{\frac{1}{2}}, \alpha=1-\frac{Z}{2p_s^2}$. We deduce that:

$$\begin{aligned}T&=\frac{Z}{p_s}(2Z\pi)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\int_{-a_0}^{a_0}s^2e^{i\left(1-\frac{Z}{2p_s^2}\right)s^4}ds\\&=2\frac{Z}{p_s}(2Z\pi)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\sum_{n=0}^{+\infty}\frac{(i\alpha)^n}{n!}\frac{a_0^{4n+3}}{4n+3}.\end{aligned} \tag{13}$$

Considering that the sum function of this series is not the corresponding basic elementary function, we need to make an identity transformation:

$$\begin{aligned}T&=\frac{Z}{p_s}(2Z\pi)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\frac{1}{2}a_0^3\sum_{n=0}^{+\infty}\frac{(i\alpha a_0^4)^n}{n!(n+\frac{3}{4})}\\&=\frac{Z}{p_s}(2Z\pi)^{\frac{1}{2}}e^{i\frac{\pi}{4}}\frac{1}{2}a_0^3\int_0^1 t^{-\frac{1}{4}}e^{i\alpha a_0^4 t}dt,\end{aligned} \tag{14}$$

where, $a_0=\left(\frac{r_0p_s}{Z}\right)^{\frac{1}{2}}, \alpha=1-\frac{Z}{2p_s^2}$. In the light of the amplitude obtained above, the corresponding axial light intensity function after the distance Z can be deduced:

$$\begin{aligned}I(0,Z)&=|T|^2\left|\frac{e^{-i\frac{\pi}{2}}}{Z}\right|^2\\&=\frac{Z\pi a_0^6}{2p_s^2}\left|\int_0^1 t^{-\frac{1}{4}}e^{i\alpha a_0^4 t}dt\right|^2\end{aligned} \tag{15}$$

where, $r_0 = 1.5$, $p_s = 0.1$. As we can see, the final form of I contains a simple exponential integral, which can be obtained by subtracting the gamma function from the exponential integral [9].

Fig.1 indicates the results of the intensity varies with Z, where green curve shows the analytical solution in case of r=0, red curve and blue curve represents the numerical solution when r=0 and when the intensity value gets maximum under r is taken as arbitrary respectively. It is evident that both the analytical and numerical light intensity reach the peak when $Z = 0.02$, the analytical peak intensity is 2356, and the numerical peak intensity is 2340. Therefore, $Z = 0.02$ can be called the focus on the axis of CPBs [10]. It is interesting that the three curve corresponds each other deeply well, especially in focal zone. Besides, the red curve overlaps with the blue completely near focus zone, indicating that the autofocusing occurs in transmission axis and the focus spot locates in z-axis. However, there is distinct diversity between them out of focus zone, for instance, the amplitude of the small oscillation varied as z. For the green curve, we have the conclusion that the analytical solution represented by the green and numerical solution indicated by the red are in excellent agreement before the focus, particularly the small peaks of the oscillation are basically the same; while oscillations behave slightly differently after the self-focus only with a softly different zero-intensity position. It is noticeable that in the interval where the transmission distance is less than the self-focus, the intensity of the light oscillates dramatically. However, it is still not enough to explain the self-focusing behavior of CPB [11].

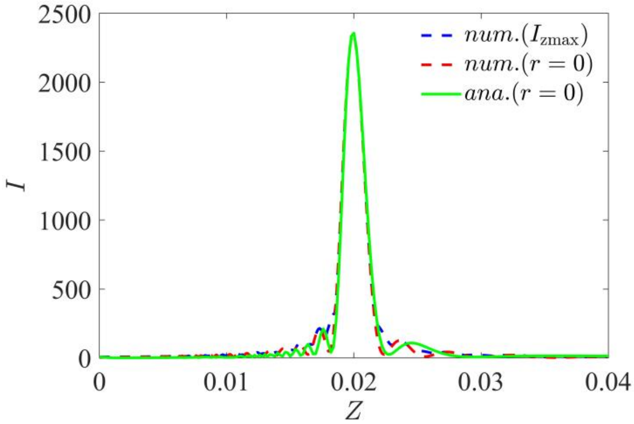

Figure 1. Light intensity's transformation transmitted with Z.

Definitely, the integral form like eq. (15) is a function of the propagation distance *Z*. This integral is an oscillation integral, regarded as the superposition of plane waves whose phase functions indicate the polynomial with different frequency. When the phase function is 0, the amplitude superposed reaches the maximum. When the wave vector k' is large, the wave vibrates very quickly, and the phase changes much faster than the amplitude. The modulus of k' first decreases and then increases with the increase of *Z*. The minimum value is $k' = 0$ when $L = L_p$. The curve's change rates behind and after L_p are different, leading it is not symmetric about L_p. In the case of $Z = 0$, the integral function will appear "singular", oscillating with extremely sharp amplitude attenuation. For CPBs, the near field condition is no longer correct, thus its violent oscillations and self-focusing occur under the "near field condition" defined traditionally [12]. The near field conditions for higher order phase functions should be more strictly defined.

2.3 Theoretical analysis of off-axis transmission

The Fresnel integral of CPB transmitted under paraxial approximation has been given above, which has been developed into series by Watson lemma. Ad for non-axial points, the method of calculation of integration becomes difficult, especially for that of aperture. However, we can still utilize some interesting methods to transform the integral equation into a differential equation, dividing the coupled multiple integral into an independent line integral to solve the amplitude distribution at any point (*r, Z*) [13]. The integration is a function of *s*. Thus, we have:

$$L(s) = \int_0^{r_0} \rho^{2m+1} e^{i(\frac{\rho^2}{2Z} - \frac{\rho}{p_s} s^2)} d\rho, \tag{16}$$

where, *m* denotes the number of terms of the series developed by the Bessel function, *Z* represents the normalized transmission distance, p_s displays the initial given parameter, and *s* is the integral variable of Pearcey function [14]. They can be all viewed as constants in this integral. Obviously, this integral is a function of *s*. The first derivative of this function is obtained by:

$$L'(s) = -i\frac{2s}{p_s} \int_0^{r_0} \rho^{2m+2} e^{i(\frac{\rho^2}{2Z} - \frac{\rho}{p_s} s^2)} d\rho. \tag{17}$$

And we set $L_2(s) = \int_0^{r_0} \rho^{2m+2} e^{i\left(\frac{\rho^2}{2Z} - \frac{s^2}{p_s}\rho\right)} d\rho$. The form of this integral is considerably resemble to $L(s)$, and we find a very intriguing and special relationship between them. We express the relation between the two integrals as $L_2(s) = L(s) + \triangle L(s)$. For a definite *Z* and *s*, the ratio modulus $t = \frac{\Delta L(s)}{L(s)}$ is almost between [0, 1] as shown in fig.2. *m* is the number of terms of the series. Fig.2 shows the variation of ratio function modulus under different parameters. In fig.2(a), *Z* is fixed but *s* is different, and the ratio function modulus .

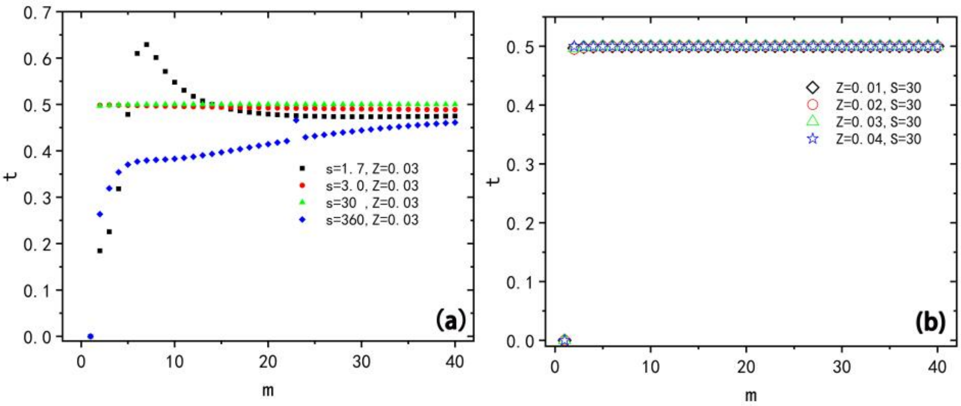

Figure 2. The integral ratio function modulus of ρ varies with the number of series terms m (a) Z is fixed; (b) s is fixed.

is converge on 0.5 when $10 \le m \le 40$ while the ratio function's phase is going to 0. Obviously, the similar conclusion can be obtained by Fig.2 (b). Thus, we substitute the value 0.5 for the ratio function modulus to solve the differential equation. In practice, t is a function of Z and s. We get the form:

$$L'(s) = -i\frac{2s}{p_s}(1+t)L(s). \quad (18)$$

The analytic solution of this differential equation can be evaluated directly with the help of the method of separating variables: [13]

$$L(s) = L(0)e^{-i\frac{3}{2p_s}s^2}, \quad (19)$$

where, $L(0) = \int_0^{r_0} \rho^{2m+2} e^{i\frac{\rho^2}{2Z}} d\rho$. In this way, the analytical form of $L(s)$ can be given. In view of the properties of the series, it is not difficult to find that the integrals of s and ρ are independent of each other. But the integral over ρ also depends on the series terms m, causing the calculation is still difficult. So the expanded series can be restored to a zero-order Bessel function to make numerical analysis [3, 16]. What's interesting is that the integral over s, we can get some useful results from some analysis [15]. This integral can be written as:

$$P_r = \int_{-\infty}^{+\infty} e^{i(s^4 - \frac{3}{2p_s}s^2)} ds. \quad (20)$$

Now we can write Eq. (3) as:

$$U(r,Z) = \frac{1}{Z} e^{i\frac{r^2}{2Z}} e^{-i\frac{\pi}{2}} P_r \int_0^{r_0} e^{i\frac{\rho^2}{2Z}} J_0\left(\frac{r\rho}{Z}\right) \rho d\rho. \quad (21)$$

From this formula, we can see that the diffraction integral result given in this paper is directly proportional to the product of the two integral parts. The light intensity is directly proportional to the square of the mode length of the two integrals. In fact, the aperture integral which is difficult to calculate in series can be represented by gamma function and incomplete gamma function [16]. In the light of the above method, we can turn it into a differential equation to find its approximate analytical solution. However, it will lead to a greater error in subsequent summation, which cannot be adopted. When $p_s = 0.1$, the calculating precise value of P_r is $0.623305+0.248356i$. The value calculated on off-axis that is far less than which on axis can be neglected although the results exist error [17].

3. Summary and Conclusions

In this paper, we have solved the Fresnel diffraction integral of circle Pearcey beam (CPB) on and off the axis under paraxial condition, obtaining the analytical and semi-analytical expressions. The theory of stationed phase is used to approximate the integral of the aperture, getting the final analytical solution on axis. This solution is compared with the numerical solution of the axial light intensity obtained by the "three-step" method, both of which the consistency is relatively good [18]. The most useful conclusion that we can get from the analytical formula is that the "axial" self-focusing distance of the CPB is L at the focal point (focal plane) of the whole CPB. To solve the analytical solution of the off-axis light intensity, we divide the coupled multiple integrals into the product of two independent line integrals by modeling and mathematical transformation [19]. The sum of the light intensity outside and on the axis can clearly show that the self-focusing distance of CPB is L_p, which is consistent with the result of the numerical "three-step" method. While there is a large error in off-axis situation, it can be reduced by other special approximation. Despite this, we are still able to get the above satisfactory results [20].

Acknowledgments

This research was supported by the Natural Science Foundation of Guangdong Province (Grant No. 2018A 030313062).

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