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Obtaining Approximate Solutions of Differential Equations Using the Method of Moving Nodes

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Abstract

The article discusses obtaining an approximate analytical solution to the Cauchy problem for an ordinary differential equation and an initial-boundary value problem for a linear parabolic equation. Obtaining an approximate analytical solution is based on the method of moving nodes. Various numerical methods are known for such problems. Using explicit and implicit Euler methods for solving the Cauchy problem in combination with ideas from the method of moving nodes, the possibility of obtaining an approximate analytical form of solving the problem is indicated. To refine the solution, a multi-point moving node was used. The method of using a multi-point movable unit taking into account linearization allows obtaining an approximate analytical solution to the Cauchy problem. Obtaining an approximate analytical expression for the initial boundary value problem for a one-dimensional parabolic equation is considered in various aspects. For a numerical solution, there are various numerical methods (finite difference method, control volume method, etc.). The finite-difference method, taking into account the movability of the node, allowed us to obtain a simple approximate solution. Approximation of derivatives in a differential equation is carried out in various ways. In the first version, both partial derivatives were approximated by a finite difference with the moving node. Using the resulting equation, an approximate solution was determined. In the second and third options, the approximation was carried out using only one of the variables, and by solving this, we obtained an approximate solution. Various examples are considered and the resulting solutions are compared.

Keywords

Cauchy problem, moving node. differential equation, approximate, boundary value problem

1. Introduction

Most problems in mathematical physics come down to solving the Cauchy problem or nonstationary differential equations with initial and boundary conditions. To solve such problems, there are various analytical and numerical methods [1-3].

Analytical methods have a relatively low degree of universality for solving such problems. More universal are approximate analytical methods (projection, variational methods, small parameter methods, operational methods, and various iterative methods) [4-7].

Some aspects of solving such problems using movable nodes are given in [8-12]. The idea of the method of moving nodes is that discrete equations (in particular, a finite-difference equation) are transformed into a continuum form based on which an approximately analytical solution of the equation is obtained. Some possibilities for developing this approach are suggested here.

Various versions of the Euler method are widely used to numerically solve an ordinary differential equation with an initial condition. Applying the method of moving nodes to the Euler method, we obtain an approximately analytical expression for the solution of the Cauchy problem.

A method for obtaining an approximately analytical solution to a parabolic equation under various boundary conditions is considered. In this case, the construction of an approximate solution is constructed using the method of straight lines.

2. Obtaining an approximately analytical solution for the Cauchy problem

Consider the Cauchy problem

$$y' = \frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0, \tag{1.1}$$

into segments t∈[0, T]. Let t be an arbitrary point on the segments [0, T].

2.1 Explicit Euler method

To numerically solve the problem (1.1), the explicit Euler method is often used (put)

$$y_{k+1} = y_k + hf(t_k, y_k), \quad k = 0, 1, ..., N-1. \tag{1.2}$$

Here $t_k = kh, \ h = \frac{T}{N}.$

To obtain an approximately analytical solution, we proceed as follows. Let *t* be an arbitrary point on the segments [0, T]. Considering t to be a moving node, and assuming $h = t - t_0$, following Euler's method, we have:

$$U_1^1 = y_0 + tf(t_0, y_0). \tag{1.3}$$

(1.3) is an approximately analytical solution to the problem (1.1), i.e. represents a rough approximate solution (linear function). To clarify the approximate solution (1.3), we divide the segment [0, t] into two parts: [0, t/2], [t/2,t].

Let's write an expression like (1.3) for each segment:

From here we can get

$$U_2^2 = y_0 + \frac{t}{2}\left[f(0, y_0) + f\left(\frac{t}{2}, y_0 + \frac{t}{2} f\left(0, y_0\right)\right)\right]. \tag{1.4}$$

(1.4) is a more accurate approximation compared to (1.3) for the entire interval [0, t].

Now we divide the segment [0, t] into n parts: [0, t/n], [t/n,2t/n],…,[(n-1)t/n,t]. For each segment, we apply Euler's method, and as a result, we obtain

$$U_n^n = y_0 + \frac{t}{n}\left[f(0, y_0) + f\left(\frac{t}{n}, U_n^1\right) + ... + f\left(\frac{(n-1)t}{n}, U_n^{n-1}\right)\right]. \tag{1.5}$$

Here

$$U_n^i = y_0 + \frac{t}{n} f\left(\frac{it}{n}, U_n^{i-1}\right), \quad i = 1, 2, ..., n-1.$$

Calculating using formula (1.5), we obtain an approximately analytical solution to problem (1.1).

2.2 Implicit Euler method for linear equation

Let the right-hand side of the problem (1.1) be a linear function with respect to y:

$$y' = P(t)y + Q(t). \tag{1.6}$$

Here $P(t), Q(t)$ are the given continuous functions.

We will proceed from the implicit Euler method. If we approximate (1.6) with one moving node, taking into account the initial condition, we have

$$U_1^1 = y_0 + tP(t)U_1^1 + tQ(t). \tag{1.7}$$

From here

$$U_1^1 = \frac{y_0 + tQ(t)}{1 - tP(t)}. \tag{1.8}$$

Expression (1.8) is an approximately analytical solution to the Cauchy problem for equation (1.6).
Let us now consider the segments [0, t/2], [t/2,t] and apply (1.8) to each segment

$$U_1^2 = \frac{y_0 + \frac{t}{2}P\left(\frac{t}{2}\right)}{1 - \frac{t}{2}Q\left(\frac{t}{2}\right)}, \quad U_2^2 = \frac{U_1^2 + tP(t)}{1 - tQ(t)},$$

From here excluding U_1^2 we get

$$U_2^2 = \frac{y_0 + \frac{t}{2}Q(t)}{\left(1 - \frac{t}{2}P\left(\frac{t}{2}\right)\right)\left(1 - \frac{t}{2}P(t)\right)} + \frac{\frac{t}{2}Q(t)}{1 - \frac{t}{2}P(t)}, \tag{1.9}$$

If we divide the segment [0, t] into three parts: [0, t/3], [t/3,2t/3], [2t/3,t] and for each segment we write a formula like (1.8), and in the end, we get

$$U_3^3 = \frac{1}{\left(1 - \frac{t}{3}P\left(\frac{t}{3}\right)\right)\left(1 - \frac{t}{3}P\left(\frac{2t}{3}\right)\right)\left(1 - \frac{t}{3}P(t)\right)} \left[y_0 + \frac{t}{3}Q\left(\frac{t}{3}\right) + \frac{t}{3}Q\left(\frac{2t}{3}\right)\left(1 - \frac{t}{3}P\left(\frac{t}{3}\right)\right) + \frac{t}{3}Q(x)\left(1 - \frac{t}{3}P\left(\frac{t}{3}\right)\right)\left(1 - \frac{t}{3}P\left(\frac{2t}{3}\right)\right)\right] \tag{1.10}$$

If we divide the segment with moving nodes into n parts, we obtain an analytical expression in the form

$$U_n^n = \frac{1}{\prod_{i=1}^{n}\left(1 - \frac{t}{n}P\left(\frac{it}{n}\right)\right)}\left[y_0 + \frac{t}{n}Q\left(\frac{t}{n}\right) + \frac{t}{n}\sum_{i=2}^{n} Q\left(\frac{it}{n}\right)\prod_{k=1}^{i-1}\left(1 - \frac{t}{n}P\left(\frac{kt}{n}\right)\right)\right] \tag{1.11}$$

Formula (1.11) is an approximately analytical solution to the Cauchy problem for equation (1.6).

2.3 Implicit Euler method for nonlinear equation

Let us now consider the implicit Euler method for the problem (1.1) using a moving node. To do this, we linearize the right side of equation (1.1) with respect to the variable y:

$$f(t, y) \approx f(t, y_0) + f_y'(t, y_0)(y - y_0) \tag{1.12}$$

Taking into account linearization (1.12) and introduce the notation as follows

$$P(t) = f_y'(t, y_0), \qquad Q(t) = f(t, y_0) - y_0 f_y'(t, y_0), \tag{1.13}$$

we obtain a linear equation of type (1.6). It should be noted that in expressions in the form (1.13) it is used in the case of one moving node; when using a multipoint floatable node is used instead.

Examples. Let us present some comparison results with the exact solution.

2.3.1 Consider the problem

$$y' = -ty + \sin t, \quad y(0) = 1. \tag{1.14}$$

Figure 1 shows the results of solving problem (1.14) on the interval [0, 5] using formula (1.5) for different partition numbers. Black lines correspond to the exact solution, and red lines correspond to the approximate one.

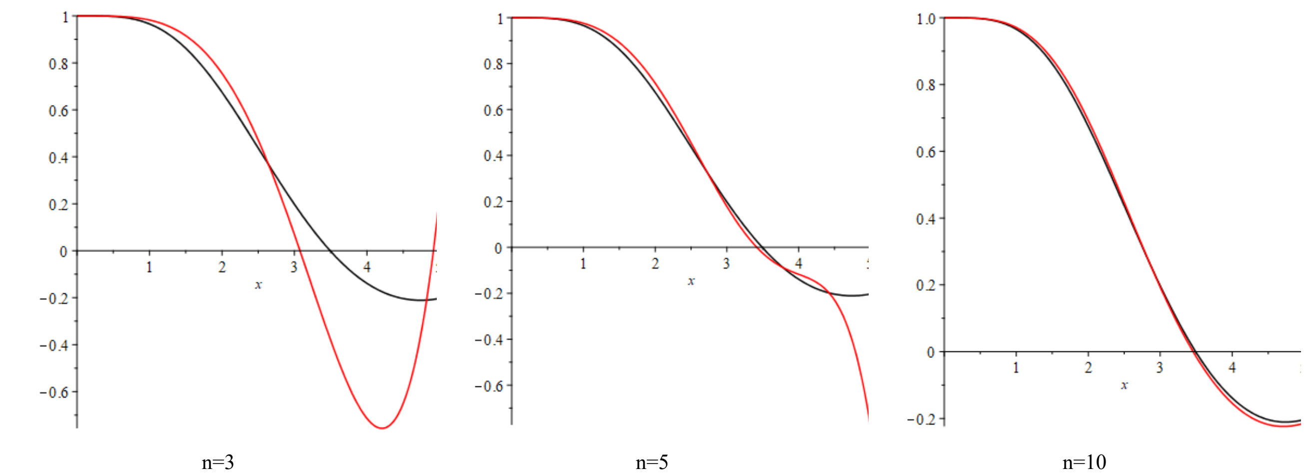


Figure 1. Comparison of approximate solutions with exact solutions of problem (1.14).

Figure 2 shows the results of solving problem (1.14) on the interval [0,10] using formula (1.11) for different partition numbers.

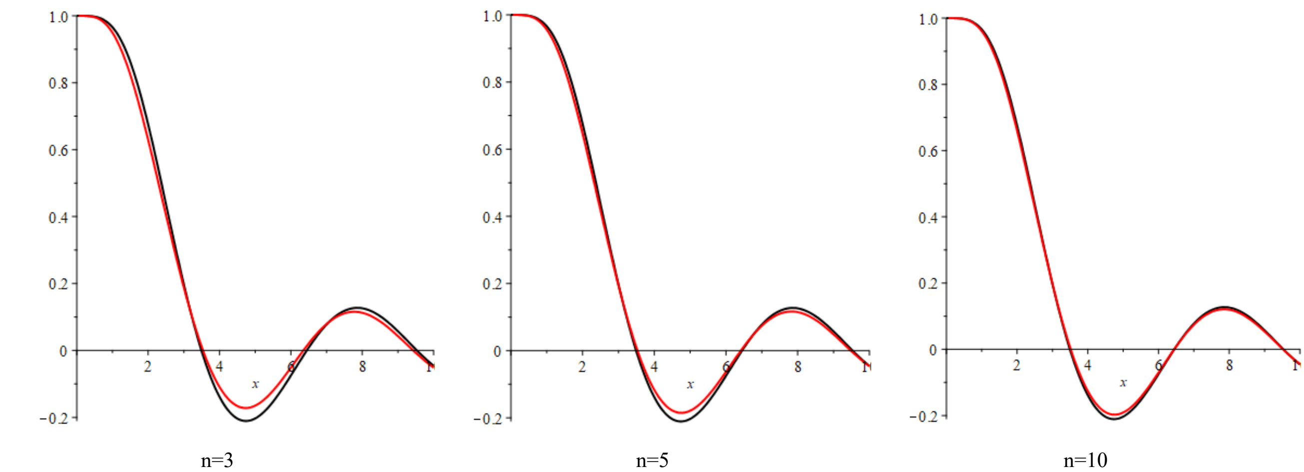


Figure 2. Comparison of approximate solutions with exact ones of problem (1.14) according to formula (1.11).

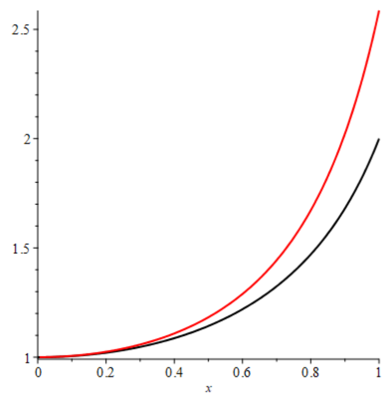
2.3.2 Nonlinear problem

$$y' = ty^2, \quad y(0) = 1. \tag{1.15}$$

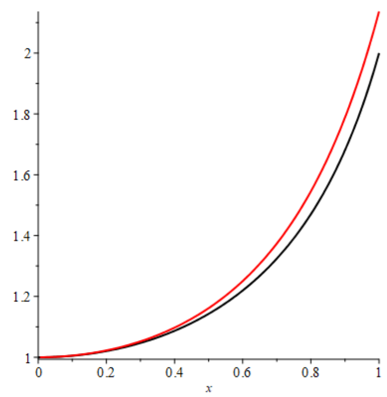
Figure 3 compares the solution to problem (1.15) using linearization (1.13) and formula (1.11) taking into account the number of partitions.

3. Obtaining an approximate solution for the initial boundary value problem of a parabolic equation

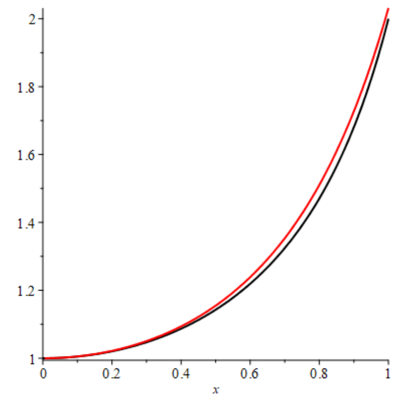
In [9], an approximately analytical solution was obtained for a linear parabolic equation using the method of moving nodes. Various aspects of the application of the moving nodes technique for initial boundary value problems were considered. Using one moving node, approximating a differential equation (approximation of differential operators for all changes, for one variable), an approximately analytical expression for solving the problem is obtained. Calculation examples show that partial approximation prevails in accuracy compared to the standard scheme. Here we will look at some schemes obtained based on the moving nodes method.



n=5



n=10



n=15

Figure 3. Comparison of approximate solutions with exact ones of problem (1.15).

Consider the boundary value problem

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{2.1}$$

with initial

$$u(x,0) = \varphi(x) \tag{2.2}$$

and boundary conditions

$$u(0,t) = u_0(t), \ u(b,t) = u_b(t). \tag{2.3}$$

In (2.1) – (2.3) $f(x,t), \varphi(x)$ the known function, $u_0(t)$ and $u_b(t)$ the given functions, Equation (2.1) is considered in the domain $0 \le x \le b, 0 \le t \le T, \sigma > 0$.

We use the method of moving nodes to obtain an approximate analytical solution to the problem with three methods in (2.1) – (2.3) a known function, and given functions, Equation (2.1) is considered in the domain.

We use the method of moving nodes to obtain an approximate analytical solution to the problem with three approaches.

3.1 Approximations in both variables

To simply solve problem (2.1) – (2.3), we select an arbitrary point (x, t) inside the region (Fig. 1). We approximate equation (2.1) as follows (using the finite-difference approximation of the derivatives appearing in the differential equation):

$$\frac{U(x,t)-U(x,0)}{t} = \sigma \frac{2}{b}\left[\frac{U(b,t)-U(x,t)}{b-x} - \frac{U(x,t)-U(0,t)}{x}\right] + f(x,t), \tag{2.4}$$

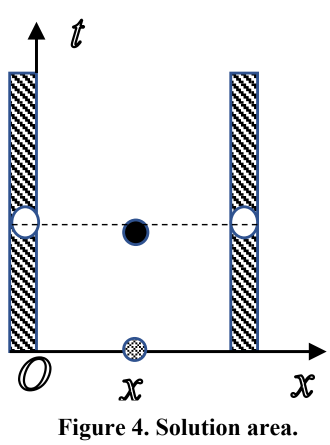


Figure 4. Solution area.

Here U(x, t) is an approximate solution to problem (2.1)-(2.3) at point (x, t). (2.4) represents one algebraic equation, the solution of which gives us an approximate analytical solution. Using the boundary condition, we obtain

$$U(0,t). = u(0,t) = u_0(t),$$
$$U(b,t). = u(b,t) = u_b(t),$$

From equation (2.4) taking into account (2.2) and (2.3), we obtain

$$U(x,t). = \frac{x(b-x)}{2\sigma t + x(b-x)}\varphi(x) + \frac{2\sigma t[u_b(t)x + u_0(t)(b-x)]}{b(2\sigma t + x(b-x))} + \frac{x(b-x)t}{x(b-x)+2\sigma t}\cdot f(x,t). \tag{2.5}$$

3.2 Approximation with respect to the variable t

In the previous paragraph, the derivatives were approximated with respect to both variables. Now let's approximate the derivative with respect to the variable t. Then we get (U(x, t) approximate solution)

$$\frac{U(x,t)-U(x,0)}{t} = \sigma\frac{d^2U}{dx^2} + f(x,t) \tag{2.6}$$

From here,

$$\sigma t\frac{d^2U}{dx^2} - U(x,t) = -t\cdot f(x,t) - u(x,0) \tag{2.7}$$

In numerical analysis [13-15], the straight line method is widely used. Here equation (2.7) is considered as an ordinary differential equation with respect to the variable x and the variable t is used as a parameter.

We solve the non-homogeneous differential equation (2.7). If the exact solution of the homogeneous equation (2.7) does not depend on x, it has the form

$$U^* = C_1\exp\left(-\sqrt{p}x\right) + C_2\exp\left(\sqrt{p}x\right).$$

Constant coefficients C1 and C2 are determined based on the boundary condition (2.2):

$$C_1 = \frac{\exp\left(-\sqrt{p}b\right)\left(\varphi(0)-Q(0,t)\right) - \left(\varphi(b)-Q(b,t)\right)}{\exp\left(-\sqrt{p}b\right) - \exp\left(\sqrt{p}b\right)},$$
$$C_2 = \frac{\left(\varphi(b)-Q(b,t)\right) - \exp\left(\sqrt{p}b\right)\left(\varphi(0)-Q(0,t)\right)}{\exp\left(-\sqrt{p}b\right) - \exp\left(\sqrt{p}b\right)} \tag{2.8}$$

3.3 Approximations with respect to the variable x

$$\frac{dU(x,t)}{dt} = \sigma\frac{b}{2}\left[\frac{U(b,t)-U(x,t)}{b-x} - \frac{U(x,t)-U(0,t)}{x}\right] + f(x,t) \tag{2.9}$$

From here

$$\frac{dU(x,t)}{dt} + p(x)\cdot U = q(x,t) \tag{2.10}$$

It's supposed to be here

$$p(x) = \frac{2\sigma}{(b-x)x},\ q(x,t) = \frac{2\sigma}{b}\cdot\frac{u_b(t)x + u_0(t)(b-x)}{x(b-x)}. + f(x,t)$$

We have obtained a first order linear differential equation. We solve equation (2.10). Considering x as a parameter and t as an independent variable, we obtain the solution

$$U(x,t) = \exp\left(-\int p(x)dt\right)\left(\int_0^t q(x,t)\exp\left(\int p(x)dt\right)dt + \varphi(x)\right) \tag{2.11}$$

If σ does not depend on t

$$\exp\left(-\int p(x)dt\right) = \exp\left(-\frac{2\sigma t}{x(b-x)}\right).$$

In this case, the solution has the form

$$U(x,t) = \exp\left(-\frac{2\sigma t}{x(b-x)}\right)\left[\varphi(x) + \int_0^t q(x,t)\exp\left(\frac{2\sigma t}{x(b-x)}\right)dt\right] \tag{2.12}$$

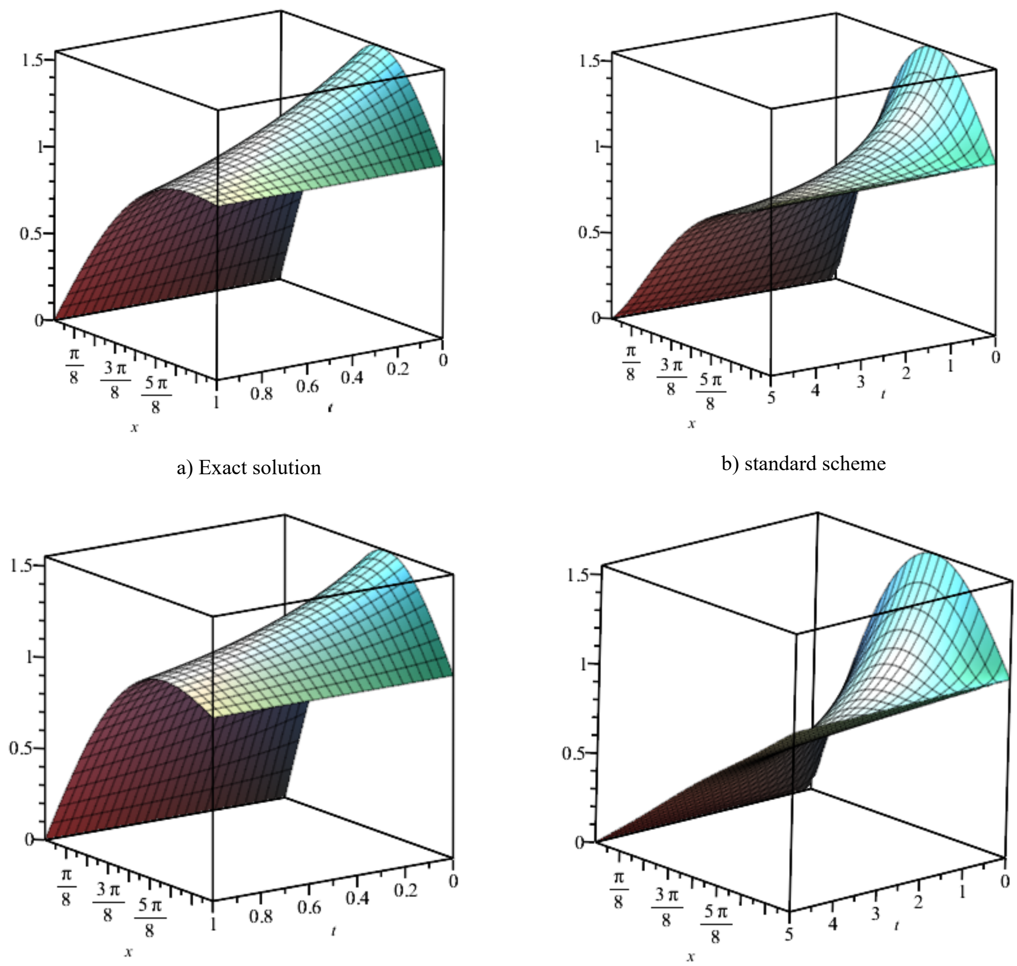
If the boundary conditions do not depend on t, we obtain a solution in simple form

$$U(x,t) = \varphi(x)\exp(-p\cdot t) + \frac{2\sigma}{b}\left(xu_b + (b-x)u_0\right)\left(1-\exp(-p\cdot t)\right) \tag{2.13}$$

Examples. Let's look at some comparison results with the exact solution**.**

3.3.1 Consider the problem of solving equation (2.1) with the conditions

$$\varphi(x) = \sin(x) + \frac{x}{\pi},\ b = \pi,\ u_0(t) = 0,\ u_b(t) = 1. \tag{2.14}$$



a) Exact solution

b) standard scheme

C) discrete by variable t

d) discrete by variable x

Figure 5. Comparison of solutions to problem (2.1), (2.14).

Figure 5 shows the solution to problem (2.1), (2.14) with three approaches to approximating derivatives.

Comparisons of the results of approximate solutions with exact ones are carried out using the mean square error according to the formula

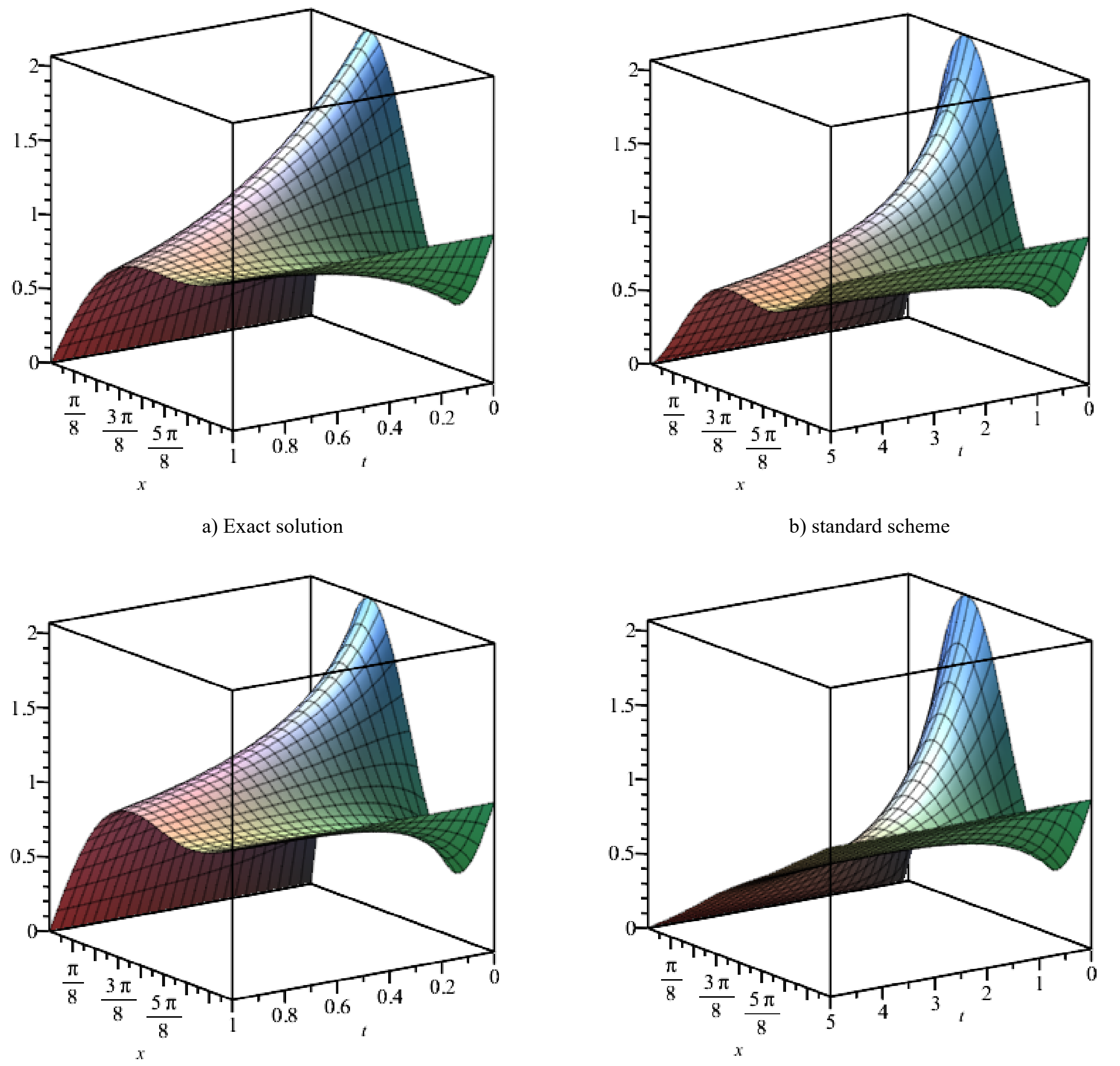
$$S = \frac{\sum_{i,j=1}^{MN} \left(u_{ij} - U_{ij}\right)^2}{\sqrt{NM}} .$$

It's supposed to be here $u_{ij} = u(t_i, x_j),\ U_{ij} = U(t_i, x_j),\ N = 100,\ M = 100.$

The calculated values of the standard deviation using three approximation methods are respectively equal to b) S=0.067477; c) S=0.051968; d) S=0.048134.

3.3.2 The solution of equation (2.1) is considered under the conditions

$$\varphi(x) = \sin(x) + \sin(2x),\ b = \pi,\ u_0(t) = 0,\ u_b(t) = 1. \tag{2.15}$$



a) Exact solution

b) standard scheme

C) discrete by variable t

d) discrete by variable x

Figure 6. Comparison of solutions to problem (2.1), (2.15).

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