HiLL PUBLISHING

Journal of Applied Mathematics and Computation, 2023, 7(4), 473-477
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

Generating a Series of Euler Sums with Harmonic Numbers via (arctan x)^p Expansions

Xiaoyu Liu

Department of Mathematics and Three Gorges Mathematics Research Center, China Three Gorges University, Yichang, Hubei, China.

How to cite this paper: Xiaoyu Liu. (2023) Generating a Series of Euler Sums with Harmonic Numbers via (arctan x)^p Expansions. *Journal of Applied Mathematics and Computation*, 7(4), 473-477.
DOI: 10.26855/jamc.2023.12.006

Received: November 28, 2023
Accepted: December 26, 2023
Published: January 24, 2024

***Corresponding author:** Xiaoyu Liu, Department of Mathematics and Three Gorges Mathematics Research Center, China Three Gorges University, Yichang, Hubei, China.

Abstract

This note primarily focuses on the expansion of the powers of arctan(x) series, exploring variable transformations, differentiation, integration, and subsequent evaluations for such series. Firstly, by substituting specific values for the variable x in the series with powers of arctan(x) and its derivative, some Euler sum identities involving harmonic numbers can be derived. Secondly, an alternative series representation for the powers of arctan(x) was identified, revealing a connection between these two forms of representation. Additionally, ongoing calculations were performed for the integration of powers of arctan(x), and Taylor series expansions of such integrals were carried out. This process led to the derivation of additional Euler sum identities containing harmonic numbers. Moreover, we observed that these derived Euler sum identities are closely linked to certain special constants, such as being expressible in terms of π, Catalan constant, and others. Some theorems and corollarys are provided, along with examples where specific values are assigned to the variables.

Keywords

Euler-like sum, Harmonic numbers, Integral representation, Arctangent function

1. Introduction

Euler sums [1] have been the subject of scholarly investigation since their discovery, as researchers have consistently sought methods to compute these infinite series or represent them using special constants. In recent studies [2-4], various authors have derived numerous identities for a class of Euler-like sums, which consist of partial sums involving harmonic numbers. These identities were obtained through the computation of integrals involving inverse trigonometric and hyperbolic functions. Furthermore, in another study [5], the authors have obtained a multitude of Euler-like sums by employing the expansion of $(\arcsin(x))^p$ and computing integrals involving $(\arcsin(x))^p$. In contrast, in studies such as [6-9], the authors have calculated integrals utilizing known Euler sums.

On this note, our objective is to obtain new Euler sums. By substituting values into the expansions of $(\arctan x)^p$ and calculating the derivatives of these expansions, then taking special values to the parameters, a set of Euler sum identities will be derived. Subsequently, we will employ different methods to expand the powers of $(\arctan(x)/x)^p$ and compare these two classes of expansions to reveal their interrelation. Finally, through the evaluation of integrals involving $\arctan(x)$, new Euler sum identities will be obtained.

The following notation and results will be useful in the subsequent sections of this paper. Harmonic and odd harmonic series are defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k}, h_n := H_{2n} - \frac{1}{2} H_n = \sum_{k=1}^{n} \frac{1}{2k-1}.$$

The Riemann zeta function is

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The constant G is Catalan constant [10], defined by

$$G:=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$

2. Main results

2.1 Euler-like sums via $\arctan(x)$

By the expansion in [2], then

$$(\arctan(x))^2=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n x^{2n}}{n}, \tag{1}$$

$$(\arctan(x))^3=3\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}x^{2j+1}. \tag{2}$$

Setting $x=1,\sqrt{3},\sqrt{3}/3$ in (1) leads to

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n}{n}=\frac{\pi^2}{16},\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n 3^n}{n}=\frac{\pi^2}{9},\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n}{3^n n}=\frac{\pi^2}{36},$$

whereas

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n}{n}=\frac{\pi^2}{16},$$

is already a well-known result.

Differentiating (1) at $x=1,\sqrt{3}$ shows that

$$\sum_{n=1}^{\infty}(-1)^{n+1}h_n=\frac{\pi}{8},\sum_{n=1}^{\infty}(-1)^{n+1}3^n h_n=\frac{\sqrt{3}\pi}{12}.$$

Setting $x=1,\sqrt{3},\sqrt{3}/3$ in (2) gives

$$\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}=\frac{\pi^3}{192},\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}3^j=\frac{\sqrt{3}\pi^3}{243},\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}\cdot\frac{1}{3^j}=\frac{\sqrt{3}\pi^3}{648}.$$

Differentiating (2) at $x=1,\sqrt{3}$, then

$$\sum_{j=1}^{\infty}(-1)^{j+1}\sum_{i=1}^{j}\frac{h_i}{i}=\frac{\pi^2}{32},\sum_{j=1}^{\infty}(-1)^{j+1}\sum_{i=1}^{j}\frac{h_i}{i}\cdot 3^j=\frac{\pi^2}{36}.$$

2.2 Sum involving Milgram's constants via $\arctan(x)$

From [11], since

$$\left(\frac{\arctan x}{x}\right)^p=\frac{\Gamma(p+1)}{2^p}\sum_{n=0}^{\infty}\frac{(-x^2)^n t_n(p-1)}{n+p/2},p\in\mathbb{N},$$

where the Milgram constant $t_k(n)$ is defined in [11] as

$$t_n(p-1)=\sum_{k_{p-1}=0}^{n}\frac{1}{k_{p-1}+\frac{p-1}{2}}\sum_{k_{p-2}=0}^{k_{p-1}}\frac{1}{k_{p-2}+\frac{p-2}{2}}\cdots\sum_{k_2=0}^{k_3}\frac{1}{k_2+1}\sum_{k_1=0}^{k_2}\frac{1}{k_1+\frac{1}{2}},$$

for $\left(\frac{\arctan(x)}{x}\right)^p$, consider some special cases of the parameter p, then

$$\left(\frac{\arctan x}{x}\right)^2=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n+1}t_n(1), \tag{3}$$

$$\left(\frac{\arctan x}{x}\right)^3=\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n+3/2}t_n(2), \tag{4}$$

2.3 The relationship between Euler sums and sums involving Milgram's Constants

Comparing the expansions of the previous two types of $(\arctan(x))^p\,(p\in\mathbb{N})$ leads to the following corollary.

Corollary 1. For $n\in(1,\infty), j\in(1,\infty)$, then

$$h_n=\frac{1}{2}t_{n-1}(1),$$

$$\sum_{k=1}^{j}\frac{h_k}{k}=\frac{1}{2}t_{n-1}(2).$$

Proof. By the Taylor expansion in [2]

$$\left(\frac{\arctan(x)}{x}\right)^2=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n x^{2n-2}}{n},\left(\frac{\arctan(x)}{x}\right)^3=3\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}x^{2j-2},$$

on the other hand, changing the counter n in both equations (3) and (4) leads to

$$\left(\frac{\arctan(x)}{x}\right)^2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-x^2)^{n-1}t_{n-1}(1)}{n},\left(\frac{\arctan(x)}{x}\right)^3=\frac{3}{4}\sum_{n=1}^{\infty}\frac{(-x^2)^{n-1}t_{n-1}(2)}{n+1/2}.$$

Comparing the two different expansions of $\left(\frac{\arctan(x)}{x}\right)^2$ yields $h_n=\frac{1}{2}t_{n-1}(1)$, using the same method yields $\sum_{k=1}^{j}\frac{h_k}{k}=\frac{1}{2}t_{n-1}(2)$, and Corollary 1 is proved.

2.4 Evaluating Euler Sums through Integration

In this section, we first provide the results of a certain type of integration, then proceed to calculate the integral using a series expansion, and finally obtain an identity involving Euler sums.

Theorem 2. For $k\in\mathbb{N}$, then

$$\begin{aligned}&\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+2k+2)}=\int_0^1 x^{2k}(\arctan(x))^3dx=\frac{\pi^3}{64(2k+1)}\\&-\frac{3}{2k+1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{4(k+1-j)}\times\left[\frac{\pi^2}{8}-(-1)^{k+1-j}\frac{\pi^2}{8}-\sum_{j'=1}^{k+1-j}(-1)^{j'+1}\times\frac{\pi+H_{\frac{k-j-j'}{2}}-H_{\frac{k-j-j'+1}{2}}}{2k-2j-2j'+3}\right]\\&-\frac{(-1)^k 3}{64(2k+1)}(16G\pi-\pi^2\ln 4-21\zeta(3)),\end{aligned} \tag{5}$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+2k+3)}=\int_0^1 x^{2k+1}(\arctan(x))^3\,dx=\frac{\pi^3}{128(k+1)}-\frac{3}{2k+2}$$
$$\times\sum_{j=1}^{k+1}\frac{(-1)^{j+1}}{2(2k+3-2j)}\times\left[\frac{\pi^2}{8}-(-1)^{k+1-j}2G+\frac{(-1)^{k+1-j}\pi\ln 2}{2}\right. \tag{6}$$
$$\left.-\sum_{j'=1}^{k+1-j}(-1)^{j'+1}\frac{\pi+H_{\frac{2k-2j-2j'+1}{4}}-H_{\frac{2k-2j-2j'+3}{4}}}{2k-2j-2j'+4}\right]-\frac{(-1)^{k+1}\pi^3}{128(k+1)}.$$

Proof. On the one hand,

$$\int_0^1 x^a(\arctan(x))^3\,dx=\frac{\pi^3}{64(a+1)}-\frac{3}{a+1}\int_0^1\frac{x^{a+1}}{1+x^2}(\arctan(x))^2\,dx,$$

By partial fractions, $\int_0^1\frac{x^{a+1}}{1+x^2}(\arctan(x))^2\,dx$ becomes

$$\int_0^1\frac{x^{2k+1}(\arctan(x))^2}{1+x^2}dx=\int_0^1\sum_{j=1}^{k}(-1)^{j+1}x^{2k+1-2j}(\arctan(x))^2\,dx+\int_0^1\frac{(-1)^k x(\arctan(x))^2}{1+x^2}dx,$$
$$\int_0^1\frac{x^{2k+2}(\arctan(x))^2}{1+x^2}dx=\int_0^1\sum_{j=1}^{k+1}(-1)^{j+1}x^{2k+2-2j}(\arctan(x))^2\,dx+\int_0^1\frac{(-1)^{k+1}(\arctan(x))^2}{1+x^2}dx.$$

Where $\int_0^1 x^a(\arctan(x))^2\,dx$ can be evaluated

$$\int_0^1 x^{2k}(\arctan(x))^2\,dx=\frac{1}{2(2k+1)}\left[\frac{\pi^2}{8}-\sum_{j=1}^{k}(-1)^{j+1}\frac{\pi+H_{\frac{2k-2j-1}{4}}-H_{\frac{2k-2j+1}{4}}}{2k-2j+2}-(-1)^k 2G+\frac{(-1)^k\pi\ln 2}{2}\right],$$
$$\int_0^1 x^{2k+1}(\arctan(x))^2\,dx=\frac{1}{4(k+1)}\left[\frac{\pi^2}{8}-\sum_{j=1}^{k+1}(-1)^{j+1}\frac{\pi+H_{\frac{k-j}{2}}-H_{\frac{k-j+1}{2}}}{2k-2j+3}-\frac{\pi^2(-1)^{k+1}}{8}\right].$$

One has

$$\int_0^1\frac{x(\arctan(x))^2}{1+x^2}dx=\frac{1}{64}(16G\pi-\pi^2\ln 4-21\zeta(3)),\int_0^1\frac{(\arctan(x))^2}{1+x^2}dx=\frac{\pi^3}{192},$$

From these, the second equation in (5) and the second equation in (6) can be obtained.
On the other hand, using the Taylor expansion

$$(\arctan(x))^3=3\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}x^{2j+1},$$

one has

$$\int_0^1 x^a(\arctan(x))^3\,dx=\int_0^1 3\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j+1}\sum_{i=1}^{j}\frac{h_i}{i}x^{2j+1+a}dx=\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+a+2)},$$

and Theorem 2 is proved.

Example 3. Let $k=2$ in $\int_0^1 x^{2k}(\arctan(x))^2dx$ and $\int_0^1 x^{2k+1}(\arctan(x))^2\,dx$ then

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n}{n(2n+5)}=\frac{1}{240}(-64-48G+3\pi(8+\pi+\ln 16)),$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}h_n}{n(2n+6)}=\frac{1}{720}(-52-52\pi+15\pi^2+184\ln 2).$$

Example 4. Let $k=1,2$ in (5) and (6), then

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+4)}=\frac{1}{192}(48(1+G)\pi+\pi^3-96\ln 2-6\pi^2(2+\ln 2)-63\zeta(3)),$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+5)}=\frac{1}{32}(-8-32G+\pi(4+\pi+\ln(256))),$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+6)}=\frac{1}{320}(-16+160\ln 2+\pi(-64-48G+\pi(12+\pi+\ln(64)))+63\zeta(3)),$$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}\frac{(-1)^{j+1}3h_i}{i(2j+1)(2j+7)}=\frac{1}{960}(288+736G-26\pi^2+5\pi^3-8\pi(16+23\ln 2)).$$

3. Conclusion

This note is based on the expansion of the $(\arctan(x))^p\,(p\in\mathbb{N})$ series, a set of series identities involving harmonic numbers can be obtained. It is noteworthy that these identities can be expressed using constants such as $\pi,\ln 2,\zeta(2n+1),n\in\mathbb{N}$, Catalan's constant G and others.

Acknowledgments

The author thanks the referee for her/his interest in our work and for helpful comments that greatly improved the manuscript.

References

[1] Euler L. (2015). Correspondence of Leonhard Euler with Christian Goldbach. Birkhäuser.

[2] Sofo A, Nimbran A S. (2019). Euler sums and integral connections. Mathematics, 7(9): 833.

[3] SMezo I, CENKCİ M. (2017). Evaluation of Euler-like sums via Hurwitz zeta values. Turkish Journal of Mathematics, 41(6): 1640-1655.

[4] Kunzhen Zhang, Xinhua Xiong. (2023). Harmonic Number Identities from Log-integral Transformation. Journal of Applied Mathematics and Computation, 7(1), 83-89.

[5] Jung M H, Cho Y J, Choi J S. (2004). Euler sums evaluatable from integrals. Communications of the Korean Mathematical Society, 19(3): 545-555.

[6] Xu C, Yan Y, Shi Z. (2016). Euler sums and integrals of polylogarithm functions. Journal of Number Theory, 165: 84-108.

[7] Sofo A. (2022). Evaluating log-tangent integrals via Euler sums. Mathematical Modelling and Analysis, 27(1): 1-18.

[8] Sofo A, Batir N. (2022). Moments of log-tanh integrals. Integral Transforms and Special Functions, 33(6): 434-448.

[9] Freitas P. (2005). Integrals of polylogarithmic functions, recurrence relations, and associated Euler sums. Mathematics of Computation, 74(251): 1425-1440.

[10] Bradley D M. (1999). A class of series acceleration formulae for Catalan's constant. The Ramanujan Journal, 3: 159-173.

[11] Milgram M. (2006). A new series expansion for integral powers of arctangent. Integral Transforms and Special Functions, 17(7): 531-538.