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The Uniqueness of Sturm-Liouville Problems on a P-Star Graph

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Abstract

The inverse nodal problem is always an important research topic in mathematics, physics, biology, and many other fields. Such problems have many applications in mathematics and natural science. In this paper, we study the uniqueness of Sturm-Liouville equations on a p-star graph from paired-dense nodal data. Firstly, we establish some general uniqueness theorems on the component $q_l(x)$ for $l=\overline{1,p}$, and show that the component $q_l(x)$ up to a constant for the above problem can be uniquely determined by the paired-dense nodal subsets corresponding to a number of subsequences of eigenvalues in adjacent or, intersecting subintervals having the central vertex under some conditions. Then, without any nodal data on some component $q_{i_0}(x)$, $i_0\in\{1,2,\cdots,p\}$, adding some information on eigenvalues, we can also recover the other component $q_l(x)$ up to a constant from paired-dense nodal data. It is interesting that the length of each subinterval on each edge may be arbitrarily small.

Keywords

Inverse nodal problem, p-star graph, paired-dense, arbitrary subinterval

1. Introduction

Consider the partial inverse nodal problems for the Sturm-Liouville problem $B=\mathrm{B}(q,\alpha)$ on a p-star graph defined by

$$\begin{cases}-y_l''+q_l(x)y_l=\lambda y_l, \quad x\in(0,1), \quad l=\overline{1,p}, & (1.1)\\ y_l(0,\lambda)\cos\alpha_l+y_l'(0,\lambda)\sin\alpha_l=0, \quad l=\overline{1,p}, & (1.2)\\ y_1(1,\lambda)=y_l(1,\lambda)\ \text{(continuity condition)}, \quad l=\overline{2,p}, & (1.3)\\ \sum_{l=1}^{p}y_l'(1,\lambda)=0\ \text{(Kirchhoff's condition)}, & (1.4)\end{cases}$$

where λ is the spectral parameter, $q(x)=\{q_l(x)\}_{l=1}^{p}$, $\alpha=\{\alpha_l\}_{l=1}^{p}$, $\alpha_l\in[0,\pi)$, the potential $q_l(x)$ is an integrable and real-valued function on [0,1]. Recently the inverse problems for differential operators on graphs have been widely investigated [1, 3, 4, 7, 13-16, 18]. Inverse problems for Sturm-Liouville problem on quantum graphs have many applications in mathematics, chemistry, and engineering [7].

The greatest success in the inverse spectral and nodal theories has been obtained for the classical Sturm-Liouville operator (for details, see [2, 5, 6, 8, 9, 12, 17] and other works). The inverse nodal problems for the classical Sturm-Liouville operator were firstly studied in [8, 12], respectively. Then Yang [17] studied the partial inverse nodal problem for the classical Sturm-Liouville operator with the strong-dense nodal subset, which was generalized by the twin-dense nodal subset in [2]. In particular, Guo and Wei [8] first studied the uniqueness of the classical Sturm-Liouville operator with a paired-dense nodal subset in interior subintervals under some conditions instead of the twin-dense nodal subset. Note that the notion of paired-dense is a generalization of the notion of twin-dense in [6].

There were several recent investigations of inverse nodal problems on graphs with twin-dense nodal subsets [13-16]. Cheng [4] derived the asymptotic formulae of eigenvalues of B, and presented direct and explicit formulae on recovering the potential from dense nodal set on each edge. Wei, Cheng and Wang [15] studied the partial inverse nodal problems for B by paired-dense nodal subsets in interior subintervals.

In this paper, we study the uniqueness of Sturm-Liouville equations on a p-star graph from paired-dense nodal data and prove some uniqueness theorems on the potentials with paired-dense nodal subsets on arbitrary subintervals having the central vertex under some conditions. In particular, the length of the subinterval on each edge may be arbitrarily small.

We present preliminaries in Section 2. Our main results and their proofs are given in Section 3.

2. Preliminaries

Let $\varphi_l(x,\lambda)$ be solutions of (1.1) for each $l=\overline{1,p}$ associated with the initial conditions $\varphi_l(0,\lambda)=\sin\alpha_l$, $\varphi_l'(0,\lambda)=-\cos\alpha_l$. The following asymptotic formulae follow from the results in [3]

$$\begin{cases}\varphi_l(x,\lambda)=\dfrac{\sin\rho x}{\rho}-\dfrac{Q_l(x)\cos\rho x}{\rho^2}+o\left(\dfrac{e^{\tau x}}{\rho^2}\right), & 0<x<1,\\ \varphi_l'(x,\lambda)=\cos\rho x+\dfrac{Q_l(x)\sin\rho x}{\rho}+o\left(\dfrac{e^{\tau x}}{\rho}\right), & 0<x<1,\end{cases}\quad \text{if}\quad \alpha_l=0, \tag{2.1}$$

$$\begin{cases}\varphi_l(x,\lambda)=\sin\alpha_l\cos\rho x+\left(Q_l(x)\alpha_l-\cos\alpha_l\right)\dfrac{\sin\rho x}{\rho}+o\left(\dfrac{e^{\tau x}}{\rho}\right), & 0<x<1,\\ \varphi_l'(x,\lambda)=-\sin\alpha_l\sin\rho x+\left(Q_l(x)\alpha_l-\cos\alpha_l\right)\cos\rho x+o\left(e^{\tau x}\right), & 0<x<1,\end{cases}\quad \text{if}\quad 0<\alpha_l<\pi, \tag{2.2}$$

for $|\lambda|\to\infty$, where $\rho=\sqrt{\lambda}$, $\tau=|\operatorname{Im}\rho|$, $Q_l(x)=\int_0^1 q_l(x)dx$, $l=\overline{1,p}$. The characteristic function of B is

$$\Delta(\lambda)=\sum_{l=1}^{p}\varphi_l'(x,\lambda)\prod_{k=1,\ k\neq l}^{p}\varphi_l(x,\lambda)=B_1(\lambda)\varphi_{i_0}'(x,\lambda)+B_2(\lambda)\varphi_{i_0}(x,\lambda),\quad i_0\in\{1,p\}, \tag{2.3}$$

where

$$B_1(\lambda)=\prod_{l\neq i_0}^{p}\varphi_l(x,\lambda)\quad\text{and}\quad B_2(\lambda)=\sum_{l\neq i_0}^{p}\varphi_l'(x,\lambda)\prod_{k=i_0,\ k\neq l}^{p}\varphi_k(x,\lambda) \tag{2.4}$$

are analytic in λ. Denote

$$\sigma(B)=\bigcup_{m=1}^{p}M_m,\quad M_m=\left\{\lambda_{m,n}:\lambda_{m,n}=\rho^2_{m,n}\right\}_{n\in N},\quad m=\overline{1,p}.$$

Consider the following three cases:

$$\text{(I)}\ \alpha_l=0,\ l=\overline{1,p};\quad \text{(II)}\ \alpha_l\in(0,\pi),\ l=\overline{1,p};\quad \text{(III)}\ \alpha_l=0,\ l=\overline{1,T},\ \alpha_l\in(0,\pi),\ l=\overline{T+1,p},\ 1\le T\le p-1.$$

By Theorem 2.1 in [3], all eigenvalues are real, and there exist p sequences of eigenvalues $\lambda_{m,n}$ with the asymptotic formulae:

$$\rho_{1,n}=(n-1/2)\pi+\frac{\omega_0}{(n-1/2)\pi}+o\left(\frac{1}{n}\right),\quad \rho_{m,n}=n\pi+\frac{\Lambda_{m,n,1}}{n\pi}+o\left(\frac{1}{n}\right),\ m=\overline{2,p}\quad \text{for (I)}; \tag{2.5}$$

$$\rho_{1,n}=(n-1)\pi+\frac{1}{(n-1)\pi}\left(-\frac{A_1}{2p}+\omega_0\right)+o\left(\frac{1}{n}\right),\quad \rho_{m,n}=n\pi+\frac{\Lambda_{m,n,2}}{n\pi}+o\left(\frac{1}{n}\right),\ m=\overline{2,p}\quad \text{for (II)}; \tag{2.6}$$

$$\begin{cases}\rho_{m,n}=n\pi+(-1)^n a_1+\dfrac{\omega_1}{2n\pi}+o\left(\dfrac{1}{n}\right), & m=1,2,\\ \rho_{m,n}=n\pi+\dfrac{\Lambda_{m,n,3}}{n\pi}+o\left(\dfrac{1}{n}\right), & m=\overline{3,T+1},\\ \rho_{m,n}=\left(n-\dfrac{1}{2}\right)\pi+\dfrac{\Lambda_{m,n,4}}{\left(n-\dfrac{1}{2}\right)\pi}+o\left(\dfrac{1}{n}\right), & m=\overline{T+2,p},\end{cases}\quad \text{for (III)}. \tag{2.7}$$

for $\mathbf{n}\gg\mathbf{1}$, where $\Lambda_{m,n,\nu}$ is the m-th zero of the polynomial $p_\nu(\Lambda)$ of degree $p-1$, $\nu=1,2$, $\Lambda_{m,n,3}$ is the $(m-2)$-th zero of the polynomial $p_3(\Lambda)$ of degree $T-1$, $\Lambda_{m,n,4}$ is the $(m-T-1)$-th zero of the polynomial $p_4(\Lambda)$ of degree $p-T-1$,

$$p_1(\Lambda)=\sum_{l=1}^{p}\prod_{m\neq l}(\Lambda-Q_l(1)),\quad p_2(\Lambda)=\sum_{l=1}^{p}\prod_{m\neq l}(\Lambda-\cot\alpha_m+Q_m(1)),\quad p_3(\Lambda)=\sum_{l=1}^{T}\prod_{m\neq l}(\Lambda-Q_l(1))$$

$$p_4(\Lambda)=\sum_{l=T+1}^{p}\prod_{m\neq l,m=T+1}^{p}(\Lambda-\cot\alpha_m+Q_m(1)),\quad \omega_0=\frac{1}{p}\sum_{l=1}^{p}Q_l(1),\quad A_1=\sum_{l=1}^{p}\cot\alpha_l,\quad a_1=\arcsin\sqrt{\frac{T}{p}},$$

$$\omega_1=\frac{1}{p}\left((p-T)\sum_{l=1}^{T}Q_l(1)+T\sum_{l=1+T}^{p}Q_l(1)-TA_2\right),\quad A_2=\sum_{l=T+1}^{p}\cot\alpha_l.$$

In general, the multiplicity of eigenvalues of the Sturm-Liouville problem B is finite. We present an example for B with multiple eigenvalues.

Example 2.1 Let $q_l(x)\equiv q(x)$ and $\alpha_l=0$ for all $l=\overline{1,p}$. Then $\varphi_l(x,\lambda)=S(x,\lambda)$ [19]. By (2.3), we have the characteristic function of B

$$\Delta(\lambda)=pS^{p-1}(x,\lambda)S'(x,\lambda).$$

Therefore there only exist 2 sequences of various eigenvalues of B and 1 sequence of eigenvalues with their multiplicities $p-1$.

For each $l=\overline{1,p}$, the problem B_l is defined by(1.1), (1.2) and $\varphi_l(x,\lambda)=0$. The Weyl m-function $m_l(\lambda)$ of B_l is of the form [5]: $m_l(\lambda)=-\dfrac{\varphi_l'(1,\lambda)}{\varphi_l(1,\lambda)}$.

3. Main results and Proofs

In this section, we shall present our main results and their proofs. For convenience, we assume that another Sturm-Liouville problem $\tilde{B}=\tilde{B}(\tilde{q},\tilde{\alpha})$ on a p-star graph. Both the symbol γ of B and the corresponding symbol $\tilde{\gamma}$ of $\tilde{B}$ denote two analogous objects, and $\hat{\gamma}=\gamma-\tilde{\gamma}$. If $\varphi_l(x_0,\lambda_{m,n})=0$, then x_0 is called a nodal point of the component $\varphi_l(x,\lambda_{m,n})$. The component $\varphi_l(x,\lambda_{m,n})$ has exactly n-1 (simple) zeros inside the interval (0,1), namely:

$$0<x_{l,m,n}^1<x_{l,m,n}^2<\cdots<x_{l,m,n}^{n-1}<1.$$

Denote $X_{l,m}=\left\{x_{l,m,n}^{j}\right\}$ the set of nodal points of the l-th component $\varphi_l(x,\lambda_{m,n})$ corresponding to M_m. Then $X_{l,m}$ is dense on (0,1) (see below, Lemma 3.1). Let $I_m=\left\{n_{m,k}\right\}_{k=K_0+1}^{\infty}, m=\overline{1,k_0}$ and $I_m=\left\{n_{m,k}\right\}_{k=1}^{\infty}, m=\overline{k_0+1,p}$ be a strictly increasing subsequence in N (K_0 sufficiently large). Denote

$$M_{m,0}=\left\{\lambda_{m,n_{m,k}}:n_{m,k}\in I_m\right\},\ m=\overline{1,k_0},\ M_{m,1}=\left\{\lambda_{m,n_{m,k}}:n_{m,k}\in I_m\right\},\ m=\overline{k_0+1,p}.$$

We assume that each eigenvalue $\lambda_{m,n}\in M_{m,0}$ is always simple and all are disjoin. For the definition of the paired-dense nodal subset the readers refer to [6,15]. Denote $W_l[a,b]$ the paired-dense nodal subset of $X_{l,m}$.

For each $m=\overline{1,p}$, the counting function corresponding to $M_{m,\xi}$, $\xi=0,1,$ is defined by

$$N_{M_{m,\xi}}(t)=\sum_{\rho_{m,n}<t,\lambda_{m,n}\in M_{m,\xi}}1,\ t\in R^{+},\quad \xi=0,1.$$

For simplicity, we assume that all subintervals $[a_{l,m},b_{l,m}],m=\overline{1,k_0}$ on the l-th edge are defined in Theorems 3.2-3.5. Denote

$$c_1=2\hat{\omega}_0,\ \text{for (I)};\quad c_2=2\hat{\omega}_0-\frac{2A_1}{p},\ \text{for (II)};\quad c_3=\hat{\omega}_1,\ \text{for (III)}.$$

We get asymptotic behavior of nodal points from the first formula of (2.1), (2.2), and (2.5)-(2.7).

Lemma 3.1 The asymptotic formulae of $x_{l,m,n}^{j}$ of $\varphi_l(x,\lambda_{m,n})$ are as follows:

$$x_{l,m,n}^{j}=\begin{cases}\alpha_{1,n}^{j}+\dfrac{Q\left(\alpha_{1,n}^{j}\right)-\omega_0\alpha_{1,n}^{j}}{(n-1/2)^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ m=1,\ l=\overline{1,p},\\ \alpha_{m,n}^{j}+\dfrac{Q\left(\alpha_{m,n}^{j}\right)-\Lambda_{m,p,1}\alpha_{m,n}^{j}}{2n^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ m=\overline{2,k_0},\ l=\overline{1,p},\end{cases}\quad \text{for (I)}; \tag{3.1}$$

$$x_{l,m,n}^{j}=\begin{cases}\beta_{1,n}^{j}+\dfrac{-\cot\alpha_l+Q\left(\beta_{1,n}^{j}\right)+\left(A_1/p-\omega_0\right)\beta_{1,n}^{j}}{(n-1)^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ m=1,\ l=\overline{1,p},\\ \alpha_{m,n}^{j}+\dfrac{-\cot\alpha_l+Q\left(\alpha_{m,n}^{j}\right)-\Lambda_{m,p,2}\alpha_{m,n}^{j}}{2n^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ m=\overline{2,k_0},\ l=\overline{1,p},\end{cases}\quad \text{for (II)}; \tag{3.2}$$

$$x_{1,m,n}^{j}=\begin{cases}\gamma_{1,n}^{j}+\dfrac{a_1}{n\pi}+\dfrac{Q\left(\gamma_{1,n}^{j}\right)+\left(a_1^2-\omega_0\right)\gamma_{1,n}^{j}}{n^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ l=\overline{1,T},\\ \gamma_{1,n}^{j}+\dfrac{a_1}{n\pi}+\dfrac{-\cot\alpha_1+Q\left(\gamma_{1,n}^{j}\right)+\left(a_1^2-\omega_0\right)\gamma_{1,n}^{j}}{2n^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ l=\overline{T+1,p},\end{cases}\quad \text{for (III)}; \tag{3.3}$$

$$x_{2,m,n}^{j}=\begin{cases}\gamma_{2,n}^{j}-\dfrac{a_1}{n\pi}+\dfrac{Q\left(\gamma_{2,n}^{j}\right)+\left(a_1^2-\omega_0\right)\gamma_{2,n}^{j}}{n^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ l=\overline{1,T},\\ \gamma_{2,n}^{j}-\dfrac{a_1}{n\pi}+\dfrac{-\cot\alpha_1+Q\left(\gamma_{2,n}^{j}\right)+\left(a_1^2-\omega_0\right)\gamma_{2,n}^{j}}{2n^2\pi^2}+o\left(\dfrac{1}{n^2}\right),\ l=\overline{T+1,p},\end{cases}\quad \text{for (III)}; \tag{3.4}$$

$$x_{l,m,n}^{j}=\begin{cases}\gamma_{1,n}^{j}+\dfrac{Q\left(\gamma_{1,n}^{j}\right)-\Lambda_{m,p,3}\gamma_{1,n}^{j}}{n^2\pi^2}+o\left(\dfrac{1}{n^2}\right), & m=\overline{3,k_0},\ k_0\le T+1,\ l=\overline{1,T},\\ \gamma_{1,n}^{j}+\dfrac{-\cot\alpha_l+Q\left(\gamma_{1,n}^{j}\right)-\Lambda_{m,p,3}\gamma_{1,n}^{j}}{n^2\pi^2}+o\left(\dfrac{1}{n^2}\right), & m=\overline{3,k_0},\ k_0\le T+1,\ l=\overline{T+1,p},\\ \alpha_{m,n}^{j}+\dfrac{Q\left(\alpha_{m,n}^{j}\right)-\Lambda_{m,p,4}\alpha_{m,n}^{j}}{(n-1/2)^2\pi^2}+o\left(\dfrac{1}{n^2}\right), & m=\overline{T+2,k_0},\ l=\overline{1,T},\\ \beta_{m,n}^{j}+\dfrac{-\cot\alpha_l+Q\left(\beta_{m,n}^{j}\right)-\Lambda_{m,p,4}\beta_{m,n}^{j}}{(n-1/2)^2\pi^2}+o\left(\dfrac{1}{n^2}\right), & m=\overline{T+2,k_0},\ l=\overline{T+1,p},\end{cases}\quad \text{for (III)} \tag{3.5}$$

for $\mathbf{n}\gg\mathbf{1}$ uniformly in **j**, where

$$\alpha_{1,n}^{j}=\frac{j}{n-1/2},\ \alpha_{m,n}^{j}=\frac{j-1/2}{n},\ \beta_{1,n}^{j}=\frac{j-1/2}{n-1},\ \beta_{m,n}^{j}=\frac{j-1/2}{n-1},\gamma_{1,n}^{j}=\frac{j}{n},\ \gamma_{m,n}^{j}=\frac{j-1/2}{n}.$$

We still use the following conditions:

Condition 1: (1) For each $l=\overline{1,p}$, $a_{l,\mathrm{m}},b_{l,m}$, $m=\overline{1,k_0}$, $2\le k_0\le p$, satisfy

$0\le a_{l,1}<b_{l,1}=a_{l,2}<b_{l,2}<\cdots<a_{l,k_0}<b_{l,k_0}=1,\ 2\le k_0\le p$;

(2) The function $q_l(x)-\tilde{q}_l(x)$ is continuous at $x=b_{l,i},i=\overline{1,k_0-1}$.

Condition 2: For each $l=\overline{1,p}$, $a_{l,\mathrm{m}},b_{l,m}$, $m=\overline{1,k_0}$, $2\le k_0\le p$, satisfy

$0\le a_{l,1}<a_{l,2}<b_{l,1}<b_{l,2}<\cdots<a_{l,k_0}<b_{l,k_0-1}<b_{l,k_0}=1,\ 2\le k_0\le p$.

The main results are given as follows:

Theorem 3.2 Suppose that Condition 1 hold and the following conditions are satisfied:

For each $l=\overline{1,p}$ and $m=\overline{1,k_0}$, $\mathrm{W}_l[a_{l,m},b_{l,m}]=\tilde{\mathrm{W}}_l[a_{l,m},b_{l,m}]$;

There exist $t_0>0, 0\le\kappa_\xi\le 1$ and $\delta_\xi>0,\ \xi=0,1,$ such that

$$\sum_{m=1}^{k_0}N_{M_{m,0}}(t)\ge 2\beta_1\left\{\kappa_0\left[\frac{t}{\pi}+\frac{1}{2}\right]+(1-\kappa_0)\left(\left[\frac{t}{\pi}\right]+\frac{1}{2}\right)-\kappa_0+1+O\left(t^{-\delta_0}\right)\right\},\ \text{ for (I)} \tag{3.6}$$

$$\sum_{m=1}^{k_0}N_{M_{m,0}}(t)\ge 2\beta_1\left\{\kappa_1\left[\frac{t}{\pi}+\frac{1}{2}\right]+(1-\kappa_1)\left(\left[\frac{t}{\pi}\right]+\frac{1}{2}\right)-\kappa_1+O\left(t^{-\delta_1}\right)\right\},\ \text{ for (II), or (III)} \tag{3.7}$$

for sufficiently large $t>t_0$, and

$$\lim_{t\to\infty}\sum_{m=1}^{k_0}N_{M_{m,0}}(t)=\frac{2\beta_1}{\pi}\ \text{ and }\ \beta_1=\max_{1\le l\le k_0}\{a_{l,1}\}, \tag{3.8}$$

Then

$$q_l(x)-\tilde{q}_l(x)\overset{\text{a.e.}}{=}c_\mu\ \text{ on }[0,1],\ \mu=\overline{1,3},\ \alpha_l=\tilde{\alpha}_l\ \text{ for each } l=\overline{1,p}. \tag{3.9}$$

Proof. It follows from (3.1)-(3.5)

$$\begin{cases}q_l(x)-\tilde{q}_l(x)\overset{\text{a.e.}}{=}\hat{c}_\mu\ \text{ on }[a_{l,1},b_{l,1}],\ \mu=\overline{1,3},\ \text{ for each } l=\overline{1,p}\ \text{ and }\ T=\tilde{T}, & (3.10)\\ q_l(x)-\tilde{q}_l(x)\overset{\text{a.e.}}{=}\hat{d}_{m,\mu}\ \text{ on }[a_{l,m},b_{l,m}],\ \mu=\overline{1,3},\ \text{ for each } m=\overline{2,k_0},\ l=\overline{1,p}, & (3.11)\end{cases}$$

where

$\hat{d}_{m,\mu} = 2\hat{\Lambda}_{m,p,1}$, for (I); $\hat{d}_{m,\mu} = 2\hat{\Lambda}_{m,p,2}$, for (II); $\hat{d}_{1,\mu} = -\hat{\omega}_1/2$, for (III);
$\hat{d}_{2,\mu} = -\hat{\omega}_1$, for (III); $\hat{d}_{m,\mu} = 2\hat{\Lambda}_{m,p,3}, m = \overline{3,k_0}$, $k_0 \le T+1$, for (III); $\hat{d}_{m,\mu} = 2\hat{\Lambda}_{m,p,4}, m = \overline{T+2,k_0}$, for (III).

Applying the methods in [6,15], we firstly prove

$$\begin{cases} \lambda_{m,n_{1,k}} - \lambda_{m,n_{1,k}} \overset{\text{a.e.}}{=} \hat{c}_\mu, & n_{1,k} \in I_1, \quad \mu = \overline{1,3}, \quad (3.12) \\ \lambda_{m,n_{m,k}} - \lambda_{m,n_{m,k}} \overset{\text{a.e.}}{=} \hat{d}_{m,\mu}, & n_{m,k} \in I_1, \quad m = \overline{2,k_0}, \quad \mu = \overline{1,3}, \quad (3.13) \end{cases}$$

By (3.10) and (3.11) together with the function $q_l(x) - \tilde{q}_l(x)$ is continuous at $x = b_{l,m}$, $m = \overline{1,k_0-1}$, we have

$$\hat{c}_\mu = \hat{d}_{m,\mu}, \quad \mu = \overline{1,3}, \tag{3.14}$$

It follows from (3.10)-(3.14)

$$\begin{cases} q_l(x) - \tilde{q}_l(x) \overset{\text{a.e.}}{=} \hat{c}_\mu \text{ on } [a_{l,1},1], \quad \mu = \overline{1,3}, \text{ for each } l = \overline{1,p}, & (3.15) \\ \lambda_{m,n_{m,k}} - \lambda_{m,n_{m,k}} \overset{\text{a.e.}}{=} \hat{c}_\mu, \quad n_{m,k} \in I_m, \quad m = \overline{1,k_0} \quad . & (3.16) \end{cases}$$

By (3.6)-(3.8) together with the results on Levinson's estimate [10], the Phragmén-Lindelöf type result in [11] and the Weyl m-functions [5], we obtain

$$q_l(x) - \tilde{q}_l(x) \overset{\text{a.e.}}{=} \hat{c}_\mu \text{ on } [0,a_{l,1}], \quad \mu = \overline{1,3}, \text{ for each } l = \overline{1,p} \text{ and } \alpha_l = \tilde{\alpha}_l. \tag{3.17}$$

This completes proof of Theorem 3.2.

Remark 1: The length $1 - a_{l,1}$, $l = \overline{1,p}$, of all subintervals on the l-th edge may be arbitrarily small only if β_1 satisfies the condition (3.8). This implies that the nodal information on the potential $q_l(x)$ may be much less.

If there isn't any nodal data on the component $q_{i_0}(x)$, we had to add some information on eigenvalues, which satisfy the assumption (A) to guarantee the uniqueness. Without loss of generality, we assume $i_0 > k_0$.

Assumption (A): For each $\lambda_{m,n_{m,k}} \in M_{m,\xi}, \xi = 0,1, m = \overline{1,p}$, such that

$$B_1^2(\lambda_{m,n_{m,k}}) + B_2^2(\lambda_{m,n_{m,k}}) \ne 0, \ \forall \lambda_{m,n_{m,k}} \in M_{m,\xi}, \xi = 0,1, m = \overline{1,p}.$$

Furthermore we show that

Theorem 3.3 suppose that Condition 1 and the assumption (A) hold and conditions (1) and (2) in Theorem 3.2 hold, and satisfy the following conditions:

For each $m = \overline{k_0+1,p}$, $M_{m,1} = \tilde{M}_{m,1}$;

There exist $t_0 > 0, 0 \le \kappa_\xi \le 1$ and $\delta_\xi > 0, \xi = 2,3$, such that

$$\sum_{m=1}^{k_0} N_{M_{m,0}}(t) + \sum_{m=1+k_0}^{p} N_{M_{m,1}}(t) \ge 2\left\{\kappa_0\left[\frac{t}{\pi} + \frac{1}{2}\right] + (1-\kappa_2)\left(\left[\frac{t}{\pi}\right] + \frac{1}{2}\right) - \kappa_2 + 1 + O\left(t^{-\delta_2}\right)\right\}, \text{ for (I)} \tag{3.18}$$

$$\sum_{m=1}^{k_0} N_{M_{m,0}}(t) + \sum_{m=1+k_0}^{p} N_{M_{m,1}}(t) \ge 2\left\{\kappa_1\left[\frac{t}{\pi} + \frac{1}{2}\right] + (1-\kappa_3)\left(\left[\frac{t}{\pi}\right] + \frac{1}{2}\right) - \kappa_3 + O\left(t^{-\delta_3}\right)\right\}, \text{ for (II), or (II)} \tag{3.19}$$

for sufficiently large $t > t_0$, and

$$\lim_{t\to\infty}\left(\sum_{m=1}^{k_0} N_{M_{m,0}}(t) + \sum_{m=k_0+1}^{p} N_{M,1}(t)\right) = \frac{2}{\pi}, \tag{3.20}$$

Then (3.9) holds.

Proof. Applying the same arguments as the proof of Theorem 3.2, we have

$$\begin{cases} q_l(x)-\tilde{q}_l(x)\overset{\text{a.e.}}{=}\hat{c}_\mu \quad \text{on} \quad [0,1], \quad \mu=\overline{1,3}, \quad \text{for } l\neq i_0 \text{ and } \alpha_l=\tilde{\alpha}_l, & (3.21)\\ \lambda_{m,n_{m,k}}-\lambda_{m,n_{m,k}}\overset{\text{a.e.}}{=}\hat{c}_\mu, \quad n_{m,k}\in I_m, \quad m=\overline{1,k_0}, \quad T=\tilde{T}. & (3.22)\end{cases}$$

Let $\tilde{q}_{i_0,\mu}(x)=\tilde{q}_{i_0}(x)+c_\mu$, $\mu=\overline{1,3}$, $\hat{q}_{i_0,\mu}(x)=q_{i_0,\mu}(x)-\tilde{q}_{i_0}(x)$, and be $\tilde{\varphi}_{i_0,0}(x,\lambda)$ the solution of (3.23)

$$\begin{cases} -u''_{i_0}(x,\lambda)+\tilde{q}_{i_0,\mu}(x)u_{i_0}(x,\lambda)=\lambda u_{i_0}(x,\lambda), \quad x\in(0,1),\\ u_{i_0}(0,\lambda)=\sin\alpha_{i_0}, \quad u'_{i_0}(0,\lambda)=-\cos\alpha_{i_0}\end{cases} \tag{3.23}$$

The Wronskian of $\varphi_{i_0}(x,\lambda)$ and $\tilde{\varphi}_{i_0,0}(x,\lambda)$ is as follows:

$$<\varphi_{i_0},\tilde{\varphi}_{i_0,0}>(x,\lambda)=\varphi_{i_0}(x,\lambda)\tilde{\varphi}'_{i_0,0}(x,\lambda)-\varphi'_{i_0}(x,\lambda)\tilde{\varphi}_{i_0,0}(x,\lambda).$$

It follows from (1.1) and (3.23)

$$<\varphi_{i_0},\tilde{\varphi}_{i_0,0}>\left(1,\lambda_{m,n_{m,k}}\right)=\int_0^1\hat{q}_{i_0,0}(x)\varphi_{i_0}\left(x,\lambda_{m,n_{m,k}}\right)\tilde{\varphi}_{i_0,0}\left(x,\lambda_{m,n_{m,k}}\right)\mathrm{d}x \tag{3.24}$$

for $\lambda_{m,n}\in M_{m,0}$, or $\lambda_{m,n}\in M_{m,1}$. By (3.21) and (3.22), we get

$$\varphi_l(x,\lambda)\equiv\tilde{\varphi}_{l,0}(x,\lambda), \quad 0\le x\le 1, \quad \text{for each } l\neq i_0,$$

which together with (2.4) implies that

$$B_1(\lambda)\equiv\tilde{B}_{1,0}(\lambda), \qquad \text{and} \qquad B_2(\lambda)\equiv\tilde{B}_{2,0}(\lambda). \tag{3.25}$$

It follows from (2.3) and (3.25) together with the assumption (A)

$$<\varphi_{i_0},\tilde{\varphi}_{i_0,0}>(1,\lambda_{m,n_{m,k}})=0 \quad \text{for each} \quad n_{m,k}\in I_m, \quad m=\overline{1,p}. \tag{3.26}$$

Define the function $\mathrm{K}_{i_0}(\lambda)$ by

$$\mathrm{K}_{i_0}(\lambda)=\frac{<\varphi_{i_0},\tilde{\varphi}_{i_0,0}>(1,\lambda)}{\prod_{m=1}^{k_0}\prod_{\lambda_{m,n_{m,k}}\in M_{m,0}}\left(1-\frac{\lambda}{\lambda_{m,n_{m,k}}}\right)\cdot\prod_{m=k_0+1}^{p}\prod_{\lambda_{m,n_{m,k}}\in M_{m,1}}\left(1-\frac{\lambda}{\lambda_{m,n_{m,k}}}\right)}. \tag{3.27}$$

If $\lambda_{m,n_{m,k}}=0$, we substitute $1-\dfrac{\lambda}{\lambda_{m,n_{m,k}}}$ by λ in (3.27). Note that the subset $M_{m,0}\cup M_{m,1}$ may contain multiple eigenvalues. By the asymptotic formulae (2.5)-(2.7), we see that there can be only a finite number of them. For instance, let $\lambda_0\in M_{m,0}\cup M_{m,1}$ be a double eigenvalue of B. Then

$$\Delta(\lambda_0)=\left.\frac{\mathrm{d}(\Delta(\lambda))}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_0}=0. \tag{3.28}$$

By virtue of (2.3) and (3.28), this yields

$$\begin{cases} B_1(\lambda_0)\varphi'_{i_0}(1,\lambda_0)+B_2(\lambda_0)\varphi_{i_0}(1,\lambda_0)=0,\\ B_1(\lambda_0)\left.\dfrac{\mathrm{d}\varphi'_{i_0}(1,\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0}+B_2(\lambda_0)\left.\dfrac{\mathrm{d}\varphi_{i_0}(1,\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0}+B'_1(\lambda_0)\varphi'_{i_0}(1,\lambda_0)+B_2'(\lambda_0)\varphi_{i_0}(1,\lambda_0)=0,\end{cases} \tag{3.29}$$

Similar to the proof of (3.29), we obtain

$$\begin{cases} B_1(\lambda_0)\tilde{\varphi}'_{i_0,0}(1,\lambda_0)+B_2(\lambda_0)\tilde{\varphi}_{i_0,0}(1,\lambda_0)=0, \\ B_1(\lambda_0)\dfrac{d\tilde{\varphi}'_{i_0,0}(1,\lambda)}{d\lambda}\Big|_{\lambda=0}+B_2(\lambda_0)\dfrac{d\tilde{\varphi}_{i_0,0}(1,\lambda)}{d\lambda}\Big|_{\lambda=0}+B_1'(\lambda_0)\tilde{\varphi}'_{i_0,0}(1,\lambda_0)+B_2'(\lambda_0)\tilde{\varphi}_{i_0,0}(1,\lambda_0)=0. \end{cases} \tag{3.30}$$

By virtue of (3.29) and (3.30), we show that the function $<\varphi_{i_0},\tilde{\varphi}_{i_0,0}>(1,\lambda)$ is a double zero at $\lambda=\lambda_0$. Since the above method can be applied for multiple eigenvalues with minor modifications, the function $K_{i_0}(\lambda)$ is an entire function in λ. Applying the same arguments as the proof of (3.17) in Theorem 3.2, we have

$$m_{i_0}(\lambda)=-\frac{\varphi'_{i_0}(1,\lambda)}{\varphi_{i_0}(1,\lambda)}=-\frac{\tilde{\varphi}'_{i_0,0}(1,\lambda)}{\tilde{\varphi}_{i_0,0}(1,\lambda)}=\tilde{m}_{i_0,0}(\lambda),$$

this implies that (3.9) is valid for $l=i_0$. Consequently the proof of Theorem 3.3 is completed.

Using Condition 2 instead of Condition 1, we obtain Theorems 3.4 and 3.5, which proofs are omitted here.

Theorem 3.4 Let us Condition 2 hold. If the conditions (1) and (2) in Theorem 3.2 are satisfied, then (3.9) holds.

Theorem 3.5 Let us Condition 2 hold. If the conditions (1) and (2) in Theorems 3.2 and 3.3 are satisfied, then (3.9) holds.

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