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Investigation of Chaotic Features of the Family of Generalized Tent Maps

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Abstract

The goal of this paper is to investigate some interesting dynamical and chaotic features of a family of tent maps. Such dynamical properties include fixed points and their stability, period orbits, visualizing the iterations using a kind of plot called a cobweb plot, and demonstrating a bifurcation diagram for T_m. Furthermore, detecting the presence of chaos in T_m is investigated. The Tent map family has chaotic regions for $m > 1$. Furthermore, the Cantor sets that occur as non-wandering sets for $m > 2$ are studied. Finally, we develop MATLAB computer programs that reflect the results interpreting such dynamical behavior. The bifurcation analysis is considered and the presence of chaotic behavior of the discrete dynamical system T_m is solved by investigating the sensitive dependence on the initial condition. The most complex dynamics-chaos-occur only for large values of the parameter *m*; so the prediction of a chaotic time series could be demonstrated. In the case of the tent map, when *m=2*, it has become chaotic.

Keywords

Fixed points, periodic points, periodic orbits, bifurcation

1. Introduction

The mathematician Edward Lorenz first discovered chaotic behaviour in the 1950s. He discovered how a small difference in initial values can result in a large difference in long-term results. He summarized it as: "Chaos: When the present determines the future, but the approximate present does not approximately determine the future." He came across chaos by accident when he was working on weather predictions. The mathematician Robert L. Devaney was the first to formally define chaos in 1986 [1].

The Tent map family is a family of functions. Its iterations form a discrete dynamical system. This is the Tent map family for some value of m:

$$x_{n+1} = T_m(x_n) = \begin{cases} mx_n & \text{for } x_n < \frac{1}{2} \\ m(1-x_n) & \text{for } \frac{1}{2} \leq x_n \end{cases}$$

where m is a positive real constant, and $x_n \in [0,1]$. Although the form of the tent map is simple and the equations involved are linear, for certain parameter values, this system can display highly complex behavior and even chaotic phenomena. For certain parameter values, the mapping undergoes stretching and folding transformations and displays sensitivity to initial conditions and periodicity. Its dynamics exhibit various features that are commonly used to identify chaotic systems. Chaos theory describes complex motion and the dynamics of sensitive systems. Chaotic systems are mathematically deterministic [2] but nearly impossible to predict Chaos is more evident in long-term systems than in short-term systems. Behavior in chaotic systems is periodic, in other words, no variable describes the state of the system that undergoes a regular repetition

of values. A chaotic system can evolve in a way that appears to be smooth and ordered; however, chaos refers to the issue of whether or not it is possible to make accurate long-term predictions of any system if the initial conditions are known to an accurate degree. In addition to the sensitivity condition, and a regularity condition called density of periodic points [3].

The first section is the introduction and the second section describes the mathematical preliminaries which are requirements for the subsequent section. In Section 3, we discuss the Chaotic behavior of the family of the tent map using the Lyapunov exponent, the dynamical behavior for this discrete dynamical system T_m such as fixed points and their stability and period orbits which present cobweb plots. Section four demonstrates period doubling and obtains the bifurcation diagram of T_m, the parameter $0 < m \leq 2$.

2. Mathematical Preliminaries

In this section, we present some definitions and theorems which are essential for finding the chaotic dynamical behavior of a one-dimensional map in the next section.

Definition 2.1 (**Fixed points and periodic points**)

A point x is a fixed point for f if $f(x) = x$. A point x is called a periodic point of period n if $f^n(x) = x$. The set of periodic points of period n is denoted by $\mathrm{Per}_n(f)$.

Definition 2.2 (**Eventually periodic points**)

A point x is eventually periodic of period n there exists $m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i \geq m$.

The map $Q(x) = x^2$ has fixed points $x = 1$ and $x = 0$ since $Q(1) = 1$ and $Q(0) = 0$. There is one eventually fixed point at $x = -1$ since $Q(-1) = 1$ and $x = 1$ is a fixed point.

Definition 2.3 (**Critical points**)

A point x is called a critical point if $f'(x) = 0$. A critical point is non-degenerate if $f''(x) \neq 0$ and degenerate if $f''(x) = 0$.

Definition 2.4 (Dense)

Suppose X is a set and Y is a subset of X. Then we say that Y is dense [4] in X if, for any point $x \in X$, there is a point y in the subset Y arbitrarily close to x.

Definition 2.5 (Transitive)

Let (X, U) be a dynamical system. Then f is a transitive [4] if, for any two non-empty open subsets, $U, of\ X$ there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$.

Definition 2.6 (Sensitive)

Let (X, f) be dynamical systems where X is a metric d. Then f is sensitive if there exists some $\delta > 0$ with the property that $\forall x \in X$ and $\forall \varepsilon > 0, \exists\ y \in (x, \varepsilon)$and $\exists\ n \in N$ such that $d\left(f^n(x), f^n(y)\right) > \delta$.

These three properties of dense, transitivity, and sensitivity are the basic ingredients of the chaotic system.

Definition (Devaney's Definition of Chaos) 2.7: Let X be a compact metric space. A continuous map $f : X \to X$ is said to be chaotic on X if it satisfies the following properties:

(i) For any $x \in X$ and $\varepsilon > 0$, we have $U(x, \varepsilon) \cap per(f) \neq \phi.$

(ii) For any $x \in X$, $y \in Y$ and, $\varepsilon > 0$ there exists $z \in X$ and $k \in N$ such that $d(x, z) < \varepsilon$ and $d(f^k(z), y) < \varepsilon$.

(iii) There exists $\delta > 0$ which satisfies that for any $x \in X$ and any neighborhood N_x of x there exists $y \in N_x$ and $k \in N$ such that $d(f^k(x), f^k(y)) \geq \delta$.

Definition (Chaos in the Li-Yorke Sense) 2.8: A dynamical system $F : I \to I$ is continuous and I is a bounded interval. Then F is chaotic in I in the Li-Yorke sense [5] if F has a periodic point in I of period 3.

Definition 2.9 (Lyapunov number): Let f be a smooth map of the real line. The Lyapunov number $L(x)$ of the orbit $\{x_1, x_2, x_3, x_4, \dots\}$ is defined as

$L(x_1) = \lim_{n\to\infty} (|f'(x_1)| \dots |f'(x_n)|)^{\frac{1}{n}}$ if this limit exists.

The Lyapunov exponent $h(x_1)$ is defined as

$$h(x_1) = \lim_{n\to\infty} \frac{1}{n}[ln|f'(x_1)| + \cdots + ln|f'(x_n)|] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} ln|f'(x_i)| = \lim_{n\to\infty} \frac{1}{n}\left[ln\prod_{i=1}^{n}|f'(x_i)|\right]$$

if this limit exists.

Theorem 2.1: Suppose $F: R \to R$ is a continuous function and has a periodic point of period three. Then *F* has a periodic point of all other periods.

Definition 2.10 (Bifurcation): Let f_c be a parameterized family of functions. Then there is a **bifurcation** at c_0 if there exists $\epsilon > 0$ such that whenever *a* and *b* satisfy $c_0 - \epsilon < a < c_0$ and $c_0 < b < c_0 + \epsilon$, then the dynamics of f_a are different from the dynamics of f_b. In other words, the dynamics of the function change when the parameter value crosses through the point c_0.

3. Chaotic Features of the tent map

3.1 Chaotic Features of the tent map using Lyapunov exponents

Theorem 3.1.1 If at least one of the average Lyapunov exponents is positive, then the system is chaotic; if the average Lyapunov exponent is negative, then the orbit is periodic; and when the average Lyapunov exponent is zero, a bifurcation occurs [6].

Problem 3.1.1: The Tent map $T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$ is chaotic on [0, 1] using the Lyapunov exponent.

Solution: In this case, $|T'(x)| = 2 > 1$. So, the tent map is not attracted to a sink and then is not asymptotically periodic. It is easy to compute the Lyapunov exponent as

$$h(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|T'(x_i)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln 2 \quad = \ln 2 > 0.$$

Since each orbit *T(x)* is asymptotically periodic and Lyapunov exponent $h(x_1) > 0$, the tent map is chaotic on [0, 1].

3.2 Chaotic Features of the Tent Map using Devaney's Definitions:

Definition (Devaney's Definition of Chaos) 2.7:

Let X be a compact metric space. A continuous map $f : X \to X$ is said to be chaotic [1] on X if it satisfies the following properties:

(i) For any $x \in X$ and $\varepsilon > 0$, we have $U(x, \varepsilon) \cap per(f) \neq \phi.$

(ii) For any $x \in X$, $y \in Y$ and, $\varepsilon > 0$ there exists $z \in X$ and $k \in N$ such that $d(x, z) < \varepsilon$ and $d(f^k(z), y) < \varepsilon$.

(iii) There exists $\delta > 0$ which satisfies that for any $x \in X$ and any neighborhood N_x of x there exists $y \in N_x$ and $k \in N$ such that $d(f^k(x), f^k(y)) \geq \delta$.

Proposition 3.2.1: The set of periodic points under *T* is dense in [0, 1].

Proof: Let (a, b) be any open interval in (0, 1). Let *p* be an odd positive integer large enough so that $\frac{1}{2.p} < (b - a)$. There is a least positive integer *k* such that $\frac{k}{p} \in (a, \frac{a+b}{2})$. If k is even, then we are done. If *k* is an odd then $\frac{k+1}{p} \in (a, b)$ will do.

Proposition 3.2.2: The tent map *T* is topologically transitive.

Proof: The periodic point of period n are

$$\begin{cases} p_n(T) = \{x \in X : T^n(x) = x\} \\ per(T) = \bigcup_{i=1}^{\infty} p_n(T) \end{cases}$$

For any $x \in X$ and $n \in \mathbb{N}$, $p_n(T) \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}) \neq \emptyset$

Let $x \in X$ = [0,1] and an open set *U* be given such that $x \in U$ then there exists $\varepsilon > 0$ such that $U(x,\varepsilon) \subset U$. Assume that $\varepsilon < 1$ then $\exists\ n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$ then

$$\left(x - \frac{1}{2^n}, x + \frac{1}{2^n}\right) \subset U(x, \varepsilon) \subset U \ldots\ldots\ldots (1)$$

For any $x \in X$ and $n \in \mathbb{N}$, $(x-\frac{1}{2^n}, x+\frac{1}{2^n}) \supset (\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}})$, k= {0, 1, 2,…..}

Moreover $\exists\ x \in U\ \varepsilon > 0$ such that $(x-\varepsilon,\ x+\varepsilon) \subset U$ and $\frac{1}{2^n} < \varepsilon$ so we have $(x-\frac{1}{2^n}, x+\frac{1}{2^n}) \subset (x-\varepsilon,\ x+\varepsilon) \subset U$

Moreover, there exists k such that $(\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}) \subset (x-\frac{1}{2^n}, x+\frac{1}{2^n}) \subset U$ *So* $T^n((\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}))$ = [0, 1]. Thus, $T^n \supset T^n((\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}))$ = [0, 1].

Therefore $T^n(U) \cap V = [0,1] \cap V \neq \emptyset$. That is $T^n(U) \cap V \neq \emptyset$.

Hence T is topologically transitive.

Proposition 3.2.3: The tent map *T* has a sensitive dependence on initial conditions on [0, 1].

Proof: Let $x \in$ [0,1]. First, we will show that if *v* is any dyadic rational number (of the form $j/2^m$ in lowest terms) in [0, 1] and *w* is any irrational number [8] in [0, 1] then there is a positive integer *n* such that

$$|T^n(v) - T^n(w)| > \frac{1}{2} \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Toward that goal, if $v = j/2^m$, then $T^m(v) = 1$ and $T^{m+k}(v) = 0$ for all $k > 0$.

By contrast, if *w* is any irrational number in [0, 1], then since *T* doubles each number in (0, 1/2), there exists an $n > m$ such that $T^n(w) > 1/2$. Since $n > m$ it follows that $T^n(v) = 0$.

So that (1) is valid. Next, let, $\delta > 0$. Then there exists a dyadic rational *v* and an irrational number *w* in [0, 1] such that $|x - v| < \delta$ and $|x - w| < \delta$. Therefore equation (1) implies that either $|T^n(x) - T^n(v)| > \frac{1}{4}$ or $|T^n(x) - T^n(w)| > \frac{1}{4}$.

Thus, if we let $\varepsilon = 1/4$ then sensitive dependence on initial conditions at the arbitrary number *x* and hence on [0, 1], is proved.

$d(T^n(x), T^n(y) > \frac{\varepsilon}{2}$. Moreover $\exists\ x_n$ such that $x_n \epsilon\ p_n(T) \cap (x-\frac{1}{2^n}, x+\frac{1}{2^n})$. Since equation (1) holds it follows that $x_n \epsilon$ per $(T) \cap U$. Hence per(T) is dense in *X*. Let $x \in X$ and $\varepsilon > 0$ so $\exists$ $n \in N$ such that $\frac{1}{2^n} < \varepsilon$. Assume that $x \in (\frac{k}{2^n}, \frac{k+1}{2^n})$ for some $k \epsilon \{0,1,2, \dots\dots\}$. Then $T^n((\frac{k}{2^n}, \frac{k+1}{2^n}))$ = [0, 1]. So, if $(\frac{k}{2^n}, \frac{k+1}{2^n}) \subset U\ (x, \varepsilon)$ and we have, $d(T^n(x), T^n(y) > \frac{\varepsilon}{2} = \delta$.

Thus, *T* exhibits sensitive dependence on initial conditions.

Since the tent map satisfies the above three conditions of Devaney's definition.

Therefore, *T* is chaotic.

4. Cobweb Plots of Tent Map

Here we will introduce a very helpful graphical view of the iteration process called cobweb plot. The cobweb plot makes use of a plot of the mapping and a plot of the reference line $y = x$. From these two curves, one may construct graphically an iteration sequence. We construct the plot first and then explain it. We do this for the case of three iterations. Here is the starting value of x is 0.23, the number of iterations is 3, the repeat count for mapping (the level of map composition), and at the parameter $m = 1$.

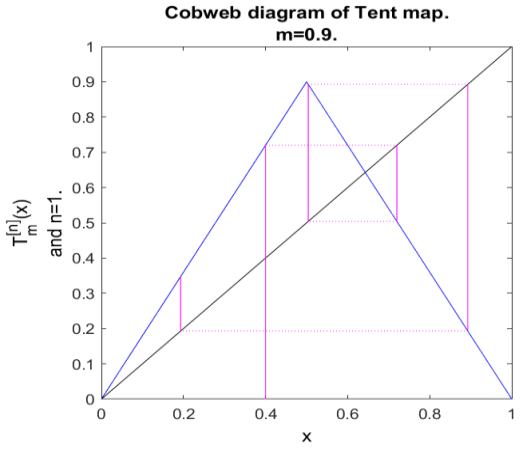

Figure 1. Cobweb diagram for Tent map $m = 0.9$.

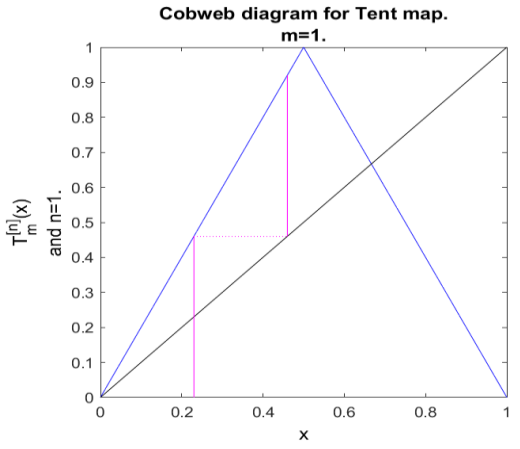

Figure 2. Cobweb diagram for Tent map $m = 1$.

Figure 2 shows a cobweb plot of three iterations for the tent map. The starting value is $x = 0.23$. Starting at that value on the x-axis, we move vertically upward until we hit the graph of cobwebtent(x). The y-value at that point is the value of the first iterate. Now we move horizontally until we hit the reference line $y = x$. The x-coordinate of that point is the value of the iterate. Next, we move vertically until we reach the graph of cobwebtent(x). The y-coordinate of that point is the second iterate. We repeat the steps to get the third iterate.

We begin by reducing m to a value of $1/3$, for which we have a single stable equilibrium. We then construct the cobweb plot of the approach to this equilibrium. Figure 3 shows a cobweb plot of 10 iterations for the tent map. The starting value is $x = 0.789$. This shows nicely the approach to the origin.

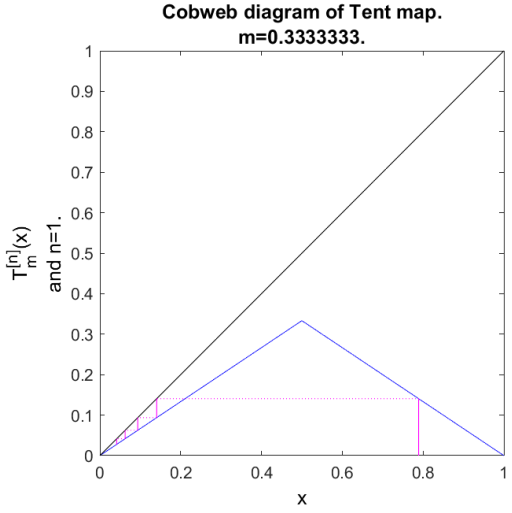

Figure 3. Cobweb diagram for Tent map $m = 0.33$.

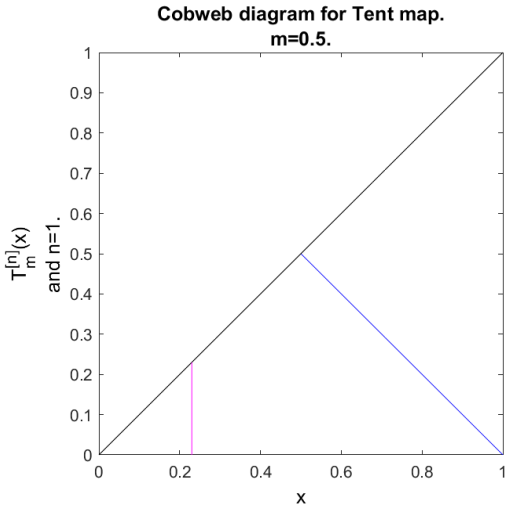

Figure 4. Cobweb diagram for Tent map $m = 0.5$.

Now let's increase m to 1 (here the starting value of $x = 0.4$ and iterations is 10) and look at a periodic orbit. In a way, we are cheated by staring at the orbit (see Figure 4). Let's do this again, but this time we start off the orbit (here $m = 1$, the starting value of $x = 0.4$ and the iterations are 10). After that, we choose a rational number for the initial x so that all calculations are done exactly, for example, $x = 230457/1000000$, iterations is 200 and $m = 1$ (see Figure 5). This is a chaotic orbit for the tent map.

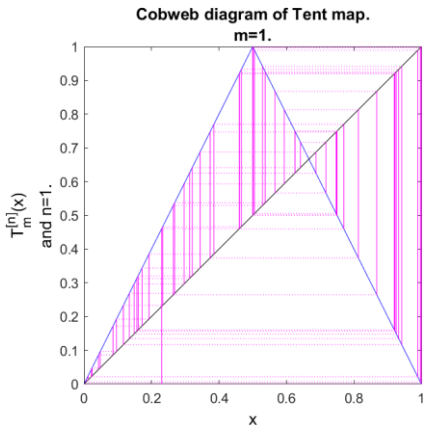

Figure 5. Cobweb diagram for Tent map $m = 1$.

5. Bifurcation diagram of tent map

In this section, we are going to explore the concept of bifurcations and how MATHEMATICA can help us visualize these changes. We will begin with some basic definitions, then examine some plots, and finally create a bifurcation diagram. A bifurcation is a fundamental change like a solution. When studying dynamical systems, we often want to know when the system undergoes a bifurcation. A bifurcation diagram is a kind of plot that shows the value of the changing parameter, m in our case, on one axis, and the solution to the system on the other axis. Our next goal is to produce the bifurcation diagram

of the 'tent' maps T_m.

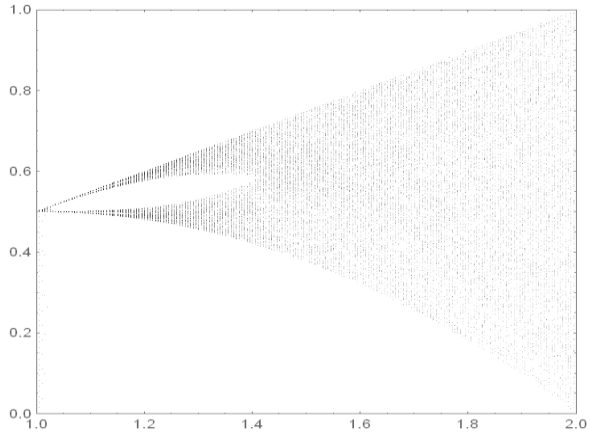

Figure 6(a). Bifurcation diagram of Tent map.

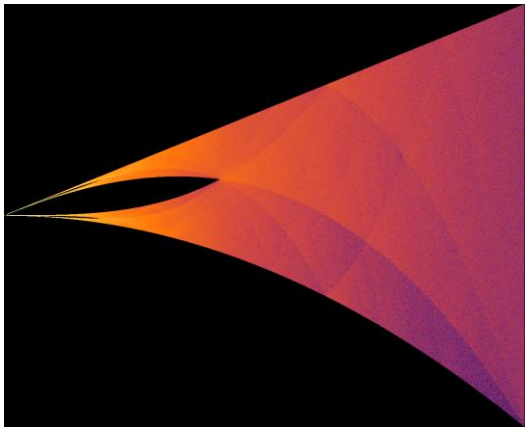

Figure 6(b). Bifurcation diagram of the Tent map.

Often, we want to know how system behavior depends on parameters. In the case of the tent map T_m, we have a single parameter m , and we have already known that there is a stable equilibrium for $m < \frac{1}{2}$. We also know that $m > \frac{1}{2}$, there are two unstable equilibria, some unstable periodic solutions, and, as we have just seen, chaotic solutions. We can get an overview of how all of this depends on the parameter *m* by a bifurcation diagram. That is a plot in which the abscissa is the value, and on the ordinate, we plot all of the *x*-values from an orbit for that value m. Fixed points will show up as a single point, a periodic orbit as several points, and a chaotic orbit as a band or several bands of points.

6. The Cantor Set a non-wandering set of the Tent map family

After studying the definition and an example of a Cantor set, we now have the tools to look specifically at the Tent map family.

Definition 6.1 (Cantor Middle third set $C_{\frac{1}{3}}$):

Beginning with the unit interval $I = [0,1]$, define closed subsets $A_1 \supset A_2 \supset \cdots$ in I as follows: we obtain A_1 by removing the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from I, A_2 is then obtained by removing from A_1 the open intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$. In general, having A_{n-1}, A_n is obtained by removing the open middle thirds from each of the 2^{n-1} closed intervals that make up A_{n-1}. The Cantor middle third set is the subspace $C_{1/3} = \bigcap A_n$.

Example 6.1 (Cantor Set on Tent map family)**.** for some value of *c*. For $m > 2$, the top of the Tent map family is higher than 1. This is shown in Figure 7. It can be observed that a point x_0 in this top leaves the interval $\boldsymbol{I}$ after the first iteration. Furthermore, we see that the iteration of such a point will go to $-\infty$.

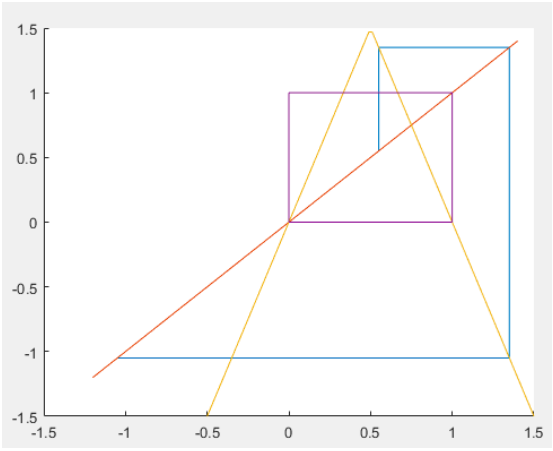

Figure 7. Dynamics of the Tent map family for $m > 2$. The purple square indicates the interval [0, 1] and the horizontal $T(x) = 0$ and $T(x) = 1$.

Denote the points that leave I after the first iteration by C_0. Inductively, we denote the points that leave I after the n^{th} iteration by C_{n-1}. The Cantor set related to the Tent map family is

$\Lambda_T = I - (\bigcup_{n=0}^{\infty} C_n)$. This Cantor set is formed for all $m > 2$.

Note that for $m = 3$, Λ_T is exactly the Cantor Middle-Thirds set. In Figure 7 the red graph shows T_3 and the green graph shows T_3^2. The intervals from the tops that form the tops of these graphs contain all the points that are mapped out of I after one or two iterations, respectively.

We can calculate the set of points that leave I after the first iteration exactly:

$$T(x) > 1$$

$$3x > 1 \text{ and } 3(1-x) > 1$$

$$x > \frac{1}{3} \quad \text{and } x < \frac{2}{3}$$

So, the points that do not leave I after the first iteration equals $C_0 = I - \left(\frac{1}{3}, \frac{2}{3}\right)$ which is indeed the same as in the Cantor Middle-thirds set. The dotted lines in Figure show the endpoints of the interval. A point on the dotted line is mapped exactly to 1. For example, $T_3\left(\frac{1}{3}\right) = 1$ and $T_3^2\left(\frac{1}{9}\right) = 1$. The endpoints never leave I as the next iteration maps these points to the fixed point $x = 0$. Therefore, C_0 is a closed set.

We can check that the points that leave I after the second iteration also correspond to the Cantor middle third set. Indeed, T_3^2 maps all endpoints $\frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$ to 1.

7. Conclusion

This paper has focused on a discrete-time dynamical system and investigated some dynamical and chaotic behavior of the parameterized tent maps for some values of its control parameter and has pointed out the MATLAB and MATHEMATICA implementations of the dynamical behavior such as fixed points and their stability, period orbits, graphics behavior (cobweb plot) and bifurcation diagrams. The bifurcation analysis was considered and the presence of chaotic behavior of the discrete dynamical system T_m was solved by investigating the sensitive dependence on the initial condition.

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