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On the Geometric Boundary of Some Convex Sets

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Abstract

Since the convex set plays an important role in many mathematical branches such as convex optimization and computer aided geometry design, it is of great significance to investigate the properties of convex sets, especially the geometric structures of convex sets. In a relatively general setting, this paper investigates the geometric boundary of some convex sets defined in terms of two given symmetric positive semi-definite matrices A and B when these two matrices have some special forms. Especially, when A and B are both diagonal or when they commute each other, several interesting theorems are established and the corresponding geometric boundaries are characterized. For a more general setting when A is symmetric positive semi-definite and B is diagonal, although general theorems are not obtained, an interesting example is studied in detail. The graph of the corresponding boundary is plotted to inspire interested readers to gain a deeper insight into this problem.

Keywords

Symmetric positive semi-definite matrices, Convex set, Geometric boundary

1. Introduction

Convex sets are important research objects in many mathematical branches such as optimal recovery [1-5], computer aided geometric design [6] and so on. Since the convex set is so important, many researchers work on the properties of convex sets. For example, Gustin [7] first introduced and studied the interior of the convex hull of a set in Euclidean space. Endou and his collaborators studied the convex sets and convex combinations in [8]. Kleiner and Leeb [9] investigated the rigidity of invariant convex sets in some symmetric spaces. Among these properties, the geometric structures of different kinds of convex sets are especially interested. For instance, Edelman and Jamison [10] studied the theory of convex geometries. Phelps established a representation theorem for bounded convex sets in [11]. Fuller et al. [12] defined the notion of a compact rectangular matrix convex set and proved the natural analogues of the Krein-Milman and bipolar theorems, as well as the notion of boundary representation for an operator space. Reitzner investigated random polytopes with vertices chosen according to a density function concentrated on the boundary of the given convex body [13]. Moreover, convex sets have attractive localization properties [14] that can be used to find upper bounds on the number of geometric permutations. Also, Dorfler [15] studied the approximation problem of unbounded convex sets by polyhedra.

In this paper, we investigate the geometric boundary of some convex sets. Specifically, for two given symmetric positive semi-definite matrices $A, B \in \mathbb{R}^{n\times n}$, we define the set

$$S = \{(v^T Av, v^T Bv): v = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n, \sum_{i\in\mathbb{N}_n} {v_i}^2 = 1\}$$

where $\mathbb{N}_n = \{1, 2, ..., n\}$. In [1], the authors proved that, in a more general setting, the set S has a convex profile. Furthermore, in [2], the author proved that the set S is a convex subset of $\mathbb{R}_+^2 := \{(x, y) : x, y > 0\}$. Can we characterize the geometric boundary of set S in more detail? This is the main purpose of this paper.

The paper is organized as follows. In Section 2, we present main results for some basic theorems and lemma of *conv E* and boundary convex set. In Section 3, we determine the lower and upper boundary of set S for the geometric structure. Section 4 will be devoted to drawing a conclusion.

2. Main results

2.1 The case of both A and B are diagonal

We first consider the simplest case when both A and B are diagonal matrices. In this case, we write

$$A=\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix} \quad and \quad B=\begin{bmatrix}\mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n\end{bmatrix}. \tag{2.1}$$

If the convex hull of the set $E:=\{(\lambda_i,\mu_i): i\in\mathbb{N}_n\}$ is denoted by $convE$ then we have the following lemma.

Lemma 2.1. If both A and B are diagonal matrices with the form (2.1) then

$$S=convE.$$

Proof: If the matrices A and B have the form (2.1) then for any point $(\lambda,\mu)\in S$, there exists $v=(v_1,\ldots,v_n)\in\mathbb{R}^n$ satisfying $\sum_{i\in\mathbb{N}_n} v_i^2=1$ such that $\lambda=v^T Av=\sum_{i\in\mathbb{N}_n}\lambda_i v_i^2$ and $\mu=v^T Bv=\sum_{i\in\mathbb{N}_n}\mu_i v_i^2$. Thus, there exist nonnegative $\theta_i=v_i^2, i\in\mathbb{N}_n$, satisfying $\sum_{i\in\mathbb{N}_n}\theta_i=1$ such that

$$(\lambda,\mu)=\textstyle\sum_{i\in\mathbb{N}_n}\theta_i(\lambda_i,\mu_i). \tag{2.2}$$

By the definition of convex hull of a set, for example, [16], page 24, we have that $(\lambda,\mu)\in convE$.

Conversely, for any $(\lambda,\mu)\in convE$, there exist $\theta_1,\ldots,\theta_n$ satisfying $\theta_i\ge 0, i\in\mathbb{N}_n$ and $\sum_{i\in\mathbb{N}_n}\theta_i=1$ such that equation (2.2) holds. By writing $v_i=\sqrt{\theta_i}, i\in\mathbb{N}_n$, we have that

$$\lambda=\sum_{i\in\mathbb{N}_n}\lambda_i v_i^2 \quad and \quad \mu=\sum_{i\in\mathbb{N}_n}\mu_i v_i^2.$$

That is to say, $(\lambda,\mu)\in S$ with A,B having the form (2.1). The proof is completed.

Lemma 2.1 reduces this problem to describing the boundary of the $convE$. To this end, we need to distinguish the points in the set E, which are interior to $convE$, which are on the boundary of it. For the interior points of $convE$, we have the following lemma.

Lemma 2.2. If $(\lambda,\mu)\in E$, is interior to $convE$ then there always exist four different points $(\lambda_i,\mu_i),(\lambda_j,\mu_j),(\lambda_k,\mu_k),\ (\lambda_l,\mu_l)\in E$ such that

$$\lambda\begin{pmatrix}\mu_i-\mu_j\\ \mu_j-\mu_k\\ \mu_k-\mu_l\\ \mu_l-\mu_i\end{pmatrix}+\mu\begin{pmatrix}\lambda_j-\lambda_i\\ \lambda_k-\lambda_j\\ \lambda_l-\lambda_k\\ \lambda_i-\lambda_l\end{pmatrix}>\begin{pmatrix}\lambda_j\mu_i-\lambda_i\mu_j\\ \lambda_k\mu_j-\lambda_j\mu_k\\ \lambda_l\mu_k-\lambda_k\mu_l\\ \lambda_i\mu_l-\lambda_l\mu_i\end{pmatrix}. \tag{2.3}$$

Proof: The Theorem Δ_n in [5] states that any point interior to the convex hull of a set E in an n-dimensional Euclidean space is interior to the convex hull of some subset of E containing at most $2n$ points. In the problem, we are considering, $2=n$. Thus, if $(\lambda,\mu)\in E$ is interior to $convE$ then there always exist four different points $(\lambda_i,\mu_i),(\lambda_j,\mu_j),\ (\lambda_k,\mu_k),(\lambda_l,\mu_l)\in E$ such that (λ,μ) is interior to the convex hull of these four points. Therefore, (λ,μ) is enclosed by four lines, each of which passes through two of these four points sequently. Direct computation shows that this fact is equivalent to the condition given by (2.3).

In Lemma 2.2, we characterize the interior points of $convE$ in terms of several linear inequalities. For more information about the relation between convex hull and linear inequalities, please refer to [17] and [18]. Since $convE$ is closed and the boundary of a closed set is the difference between the set itself and the interior of this set, we immediately have the following corollary.

Corollary 2.3. For the point $(\lambda,\mu)\in E$ if there do not exist four different points $(\lambda_i,\mu_i),(\lambda_j,\mu_j),(\lambda_k,\mu_k),(\lambda_l,\mu_l)\in E$ such that condition (2.3) holds then (λ,μ) is a boundary point of $convE$. Furthermore, (λ,μ) is either a corner or on a line segment of the boundary of (λ,μ).

Now let us define a subset F of E by

$$F=\{(\lambda_i,\mu_i)\in E:\ i\in\mathbb{N}_m,\qquad m\le n,\qquad (\lambda_i,\mu_i)\ are\ boundary\ points\ of\ E.\}$$

It is clear that the elements in the set F can be arranged, for example, in a clockwise order. The ordered set F is denoted by $\vec{F}$. Now we are ready to state the following theorem.

Theorem 2.4. The boundary of $convE$ is a polygon with the line segments in the form

$$x(\mu_i-\mu_j)+y(\lambda_j-\lambda_i)=\lambda_j\mu_i-\lambda_i\mu_j,\ x\ is\ between\ \lambda_i\ and\ \lambda_j \tag{2.4}$$

where (λ_i,μ_i) and (λ_j,μ_j) are neighboring points in $\vec{F}$ in an anticlockwise order.

Proof: By the supporting hyperplane theorem, every point on the boundary of $convE$ is defined by a supporting hyperplane. In this case, the supporting hyperplane is a supporting line, which can be written in the form (2.4) by direct computation.

2.2 The case of $AB=BA$

Next, we are going to consider the case when A and B commute with each other. That is to say, $AB=BA$. In this case, we first have the following lemma.

Lemma 2.5. Let A and B be two real symmetric matrices. Then A and B can be simultaneously diagonalized by an orthogonal matrix U if and only if $AB=BA$.

Proof: Suppose A and B can be simultaneously diagonalized by an orthogonal matrix U, that is to say, $UAU^T=\Lambda$ and $UBU^T=\Gamma$, where Λ and Γ are diagonal matrices corresponding to A and B respectively. Since all diagonal matrices commute, we have that $\Lambda\Gamma=\Gamma\Lambda$.Thus, if $UAU^T=\Lambda$ and $UBU^T=\Gamma$ then $\left(UAU^T\right)\left(UBU^T\right)=\left(UBU^T\right)\left(UAU^T\right)$, which means $UABU^T=UBAU^T$. Therefore, $AB=BA$

Conversely, suppose that $AB=BA$, where A and B are real symmetric matrices. Then there exists an orthogonal matrix V such that $VAV^T=\Lambda$, where Λ is diagonal. Using the same matrix V, we shall define $\tilde{B}=VBV^T$. Notice that since B is really symmetric, it follows that $\tilde{B}$ is really symmetric as well:

$$\tilde{B}^T=\left(VBV^T\right)^T=\left(V^T\right)^T B^TV^T=VBV^T=\tilde{B}.$$

The relation $AB=BA$ imposes a strong constraint on the form of $\tilde{B}$. Noticing that

$$\Lambda\tilde{B}=\left(VAV^T\right)\left(VBV^T\right)=VABV^T=VBAV^T=VBV^TVAV^T=\tilde{B}\Lambda,$$

we have that $\Lambda\tilde{B}-\tilde{B}\Lambda=0$, which means $\left(\lambda_i-\lambda_j\right)\tilde{b}_{ij}=0$ for $i,j=1,\dots,n,$ where $\lambda_i's$ are eigenvalues of A. If all the λ_i were distinct then we would be able to conclude that $\tilde{b}_{ij}=0$ for $i\ne j$. That is, $\tilde{B}$ is diagonal. Now, let us examine carefully what happens if some of the diagonal elements are equal. In this case, we can order the columns of V so that the repeated eigenvalues are contiguous along the diagonal. Henceforth, we assume this to be the case and we also conclude that $\tilde{b}_{ij}=0$ if $\lambda_i\ne\lambda_j$. One can write Λ and $\tilde{B}$ in block matrix form:

$$\Lambda=\begin{pmatrix}\lambda_1I_1 & 0 & \cdots & 0\\ 0 & \lambda_2I_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_kI_k\end{pmatrix},\qquad \tilde{B}=\begin{pmatrix}\tilde{B}_1 & 0 & \cdots & 0\\ 0 & \tilde{B}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \tilde{B}_k\end{pmatrix},$$

assuming that A possesses k distinct eigenvalues. I_j indicates the identity matrix whose dimension is equal to the multiplicity of the corresponding eigenvalue λ_j. The corresponding $\tilde{B}_j$ is a symmetric matrix with the same dimension as I_j. Since $\tilde{B}_j$ is symmetric there exists an orthogonal matrix $\tilde{V}_j$ such that $\tilde{V}_j\tilde{B}_j\tilde{V}_j^T$ is diagonal. Furthermore, we can find an orthogonal matrix $\tilde{V}$ of the form

$$\tilde{B}=\begin{pmatrix}\tilde{V}_1 & 0 & \cdots & 0\\ 0 & \tilde{V}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \tilde{V}_k\end{pmatrix}$$

such that $\tilde{V}\tilde{B}\tilde{V}^T=\Gamma$ is diagonal. By explicit multiplication it is easy to see that $\tilde{V}\Lambda\tilde{V}^T=\Lambda.$ Hence, we have succeeded in finding an orthogonal matrix $U=V\tilde{V}$ such that

$$UAU^T=\Lambda \quad \textit{and} \quad UBU^T=\Gamma.$$

That is, A and B are simultaneously diagonalizable by an orthogonal matrix U. The columns of U are the simultaneous eigenvectors of A and B.

With Lemma 2.5, we have the following theorem.

Theorem 2.6. If A and B are two real symmetric matrices that commute then the boundary of the set S is a polygon.

Proof: Suppose A and B are two real symmetric matrices such that $AB=BA$. By Lemma2.5, there exists an orthogonal matrix U such that $UAU^T=\Lambda$ and $UBU^T=\Gamma$, where $\Lambda=diag(\lambda_1,\dots,\lambda_n),\Gamma=diag(\mu_1,\dots,\mu_n)$, and the $i-th$ column vector u_i of U is the eigenvector corresponding to the eigenvalues λ_i and u_i respectively. That is to say, $Au_i=\lambda_i u_i$ and $Bu_i=\mu_i u_i$. Therefore, for any vector $v\in\mathbb{R}^n$ satisfying $\|v\|_2^2=1$ we have that

$$v^TAv=v^TU^T\Lambda Uv=y^T\Lambda y \quad \textit{and} \quad v^TBv=v^TU^T\Gamma Uv=y^T\Gamma y,$$

where $y=Uv\in\mathbb{R}^n$ satisfies $\|y\|_2^2=1$. Since U is an orthogonal matrix, it can be also treated as an orthogonal transformation on $\mathbb{R}^n$, which is an isomorphism mapping on $\mathbb{R}^n$. Thus,

$$\left\{\left(v^TAv,v^TBv\right):\|v\|_2^2=1\right\}=\left\{\left(y^T\Lambda y,y^T\Gamma y\right):y=Uv,U\ \textit{is an orthogonal matrix}\right\}.$$

Thus, by Lemma 2.1 and Theorem 2.4, the boundary of the set S is a polygon.

3. Examples

Define the set $S=\left\{\left(v^TAv,v^TBv\right):v=(v_1,v_2,v_3)^T\in\mathbb{R}^3,v_1^2+v_2^2+v_3^2=1\right\}$, where

$$A=\begin{pmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{pmatrix}, \qquad B=\begin{pmatrix}6 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 10\end{pmatrix}.$$

The problem is to find the boundary of the set S. To this end, we first notice that $A=4I_3-J_3$, where I_3 is the 3×3 identity matrix and J_3 is the 3×3 all-ones matrix. Thus, $v^TAv=4-(v_1+v_2+v_3)^2$ and we have for all $v\in\mathbb{R}^3$ that $v^TAv\le4.$ At the same time, a direct computation yields that the lowest eigenvalue of A is 1 so we have that $v^TAv\ge1$ for all $v\in\mathbb{R}^3$. For the second coordinate v^TBv, we obtain that

$$6=6\left(v_1^2+v_2^2+v_3^2\right)\le v^TBv=6v_1^2+8v_2^2+10v_3^2\le10\left(v_1^2+v_2^2+v_3^2\right)=10.$$

Therefore, the boundary of the set is contained in the box $[1,4]\times[6,10]$.

Now, we are going to find the right boundary of the set S, that is, when $v^TAv=4$. Since

$v^T Av = 4-(v_1+v_2+v_3)^2$, for the points on the right boundary of the set, we must have that $v_1+v_2+v_3=0$. Hence, the problem is reduced to investigate the following maximum and minimum problems

$$\max_{v\in\mathbb{R}^3}\{6v_1^2+8v_2^2+10v_3^2 : v_1+v_2+v_3=0, v_1^2+v_2^2+v_3^2=1\}$$

and

$$\min_{v\in\mathbb{R}^3}\{6v_1^2+8v_2^2+10v_3^2 : v_1+v_2+v_3=0, v_1^2+v_2^2+v_3^2=1\}$$

The idea is to express v_1 and v_2 in terms of v_3 and treat $6v_1^2+8v_2^2+10v_3^2$ as a function of v_3. For this purpose, we solve the following equations:

$$\begin{cases} v_1+v_2+v_3=0, \\ v_1^2+v_2^2+v_3^2=1, \end{cases}$$

where v_1 and v_2 are viewed as unknowns. From the first equation, we obtain that $v_1=-v_2-v_3$. Substituting it into the second equation yields that $2v_2^2+2v_2v_3-1+2v_3^2=0$. Thus,

$$v_2=\frac{-v_3\pm\sqrt{2-3v_3^2}}{2}.$$

Substituting it into the first equation $v_1+v_2+v_3=0$ we have that

$$v_1=\frac{-v_3\mp\sqrt{2-3v_3^2}}{2}.$$

Therefore,

$$(v_1,v_2,v_3)=\left(\frac{-v_3-\sqrt{2-3v_3^2}}{2},\frac{-v_3+\sqrt{2-3v_3^2}}{2},v_3\right),$$

or

$$(v_1,v_2,v_3)=\left(\frac{-v_3+\sqrt{2-3v_3^2}}{2},\frac{-v_3-\sqrt{2-3v_3^2}}{2},v_3\right).$$

If $(v_1,v_2,v_3)=\left(\left(-v_3-\sqrt{2-3v_3^2}\right)/2,(-v_3+\sqrt{2-3v_3^2})/2,v_3\right)$ then

$$6v_1^2+8v_2^2+10v_3^2=7+3v_3^2-v_3\sqrt{2-3v_3^2}.$$

Set $f(v_3)=7+3v_3^2-v_3\sqrt{2-3v_3^2}$ and let $f'(v_3)=0$ then we have that

$$6v_3-\sqrt{2-3v_3^2}+\frac{3v_3^2}{\sqrt{2-3v_3^2}}=0.$$

Solving this equation yields that $v_3^2=\left(2\pm\sqrt{3}\right)/6$. Direct calculation yields that $f(\sqrt{(2+\sqrt{3})/6})=8+\sqrt{3}/3$, $f(-\sqrt{(2+\sqrt{3})/6})=8+2\sqrt{3}/3$, $f(\sqrt{(2-\sqrt{3})/6})=8-2\sqrt{3}/3$ and $f(-\sqrt{(2-\sqrt{3})/6})=8-\sqrt{3}/3$. Thus, in this case, we have that $8-\frac{2}{3}\sqrt{3}\le 6v_1^2+8v_2^2+10v_3^2\le 8+\frac{2}{3}\sqrt{3}$.

In the case of $(v_1,v_2,v_3)=\left(\left(-v_3+\sqrt{2-3v_3^2}\right)/2,(-v_3-\sqrt{2-3v_3^2})/2,v_3\right)$, similar analysis yields that $8-\frac{2}{3}\sqrt{3}\le 6v_1^2+8v_2^2+10v_3^2\le 8+\frac{2}{3}\sqrt{3}$. Therefore, we conclude that the right boundary of the set is the vertical line segment

$$\left\{(x,y): x=4, 8-\frac{2}{3}\sqrt{3}\le y\le 8+\frac{2}{3}\sqrt{3}\right\}.$$

Now, we are going to investigate the left boundary of the set S. In this case, on the one hand, we have that $v^T Av=1$, which means that $4-(v_1+v_2+v_3)^2=1$. Thus, $(v_1+v_2+v_3)^2=3$. On the other hand, by the Cauchy-Schwarz inequality, we have that $(v_1+v_2+v_3)^2\le(v_1^2+v_2^2+v_3^2)\cdot(1^2+1^2+1^2)=3.$

The equality holds if and only if $\alpha v_i+\beta=0, i=1,2,3$ for some nonzero constants α and β. Thus, $v_i=-\beta/\alpha, i=1,2,3$ Hence, $v_i^2=1/3$ for $i = 1,2,3$. This gives that $6v_1^2+8v_2^2+10v_3^2=8$.

Therefore, the left boundary of the set *S* consists of the unique point (1,8).

Now, we are ready to explore the boundary of the set *S* between the left boundary (1,8) and the right boundary $\left\{(x,y): x=4, 8-\frac{2}{3}\sqrt{3}\le y\le 8+\frac{2}{3}\sqrt{3}\right\}$. To this end, we fix the first coordinate. That is, let $v^T Av = t, 1 \le t \le 4$ and we have $v^T Av=4-(v_1+v_2+v_3)^2=t$. Thus, $(v_1+v_2+v_3)^2=4-t:=\sigma, 0\le\sigma\le 3.$

It is easy to see that $\sigma=0$ corresponds to the right boundary and $\sigma=3$ corresponds to the left boundary of the set S. Now we want to probe into the maximal problem

$$\max_{v\in\mathbb{R}^3}\{6v_1^2+8v_2^2+10v_3^2: (v_1+v_2+v_3)^2=0, v_1^2+v_2^2+v_3^2=1\}$$

for a fixed $0<\sigma<3$ to get the upper boundary and the minimal problem

$$\min_{v\in\mathbb{R}^3}\{6v_1^2+8v_2^2+10v_3^2: (v_1+v_2+v_3)^2=0, v_1^2+v_2^2+v_3^2=1\}$$

for a fixed $0<\sigma<3$ to get the lower boundary.

We first consider the following case

$$\begin{cases} v_1+v_2+v_3=\sqrt{\sigma}, \\ v_1^2+v_2^2+v_3^2=1. \end{cases}$$

From the first equation, we have that $v_1=(\sqrt{\sigma}-v_3)-v_2$. Substituting it into the second equation yields that

$$2v_2^2-2v_2(\sqrt{\sigma}-v_3)+v_3^2-1+(\sqrt{\sigma}-v_3)^2=0.$$

Thus, we have that

$$v_2=\frac{(\sqrt{\sigma}-v_3)\pm\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}}{2}.$$

Hence, $v_1=\left[(\sqrt{\sigma}-v_3)\mp\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}\right]/2$ and the points either have the coordinates

$$\left(\frac{(\sqrt{\sigma}-v_3)-\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}}{2},\frac{(\sqrt{\sigma}-v_3)+\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}}{2},v_3\right)$$

or

$$\left(\frac{(\sqrt{\sigma}-v_3)+\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}}{2},\frac{(\sqrt{\sigma}-v_3)-\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}}{2},v_3\right).$$

If $(v_1,v_2,v_3)=\left(\left[\left(\sqrt{\sigma}-v_3\right)-\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}\right]/2,\left[\left(\sqrt{\sigma}-v_3\right)+\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}\right]/2,v_3\right)$ then

$$6v_1^2+8v_2^2+10v_3^2=7+3v_3^2+\left(\sqrt{\sigma}-v_3\right)\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}.$$

Set $f_\sigma(v_3)=7+3v_3^2+\left(\sqrt{\sigma}-v_3\right)\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}$ and let $f_\sigma'(v_3)=0$ we have that

$$3v_3^2-3\left(\sqrt{2-\sigma+2v_3\sqrt{\sigma}-v_3^2}-\sqrt{\sigma}\right)v_3+\sigma-1=0.$$

Solving this equation yields that

$$v_{31}=\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{\sigma}}{4}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}$$

and

$$v_{32}=\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{\sigma}}{4}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}.$$

Direct computation presents that

$$f_\sigma(v_{31})=8-\frac{\sigma}{4}+\frac{\sqrt{3\sigma(4-\sigma)}}{4}-\frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\left(\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}\right)}{4\sqrt{2}}$$

$$+\left(\frac{3\sqrt{\sigma}}{4}-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right)\times\sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}.$$

and

$$f_\sigma(v_{32})=8-\frac{\sigma}{4}+\frac{\sqrt{3\sigma(4-\sigma)}}{4}+\frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\left(\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}\right)}{4\sqrt{2}}$$

$$+\left(\frac{3\sqrt{\sigma}}{4}-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right)\times\sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}-\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}.$$

If $(v_1,v_2,v_3)=\left(\left[\left(\sqrt{\sigma}-v_3\right)+\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}\right]/2,\left[\left(\sqrt{\sigma}-v_3\right)-\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}\right]/2,v_3\right)$

then

$$6v_1^2+8v_2^2+10v_3^2=7+3v_3^2-\left(\sqrt{\sigma}-v_3\right)\sqrt{2-3v_3^2-\sigma+2v_3\sqrt{\sigma}}.$$

Set $g_\sigma(v_3)=7+3v_3^2-\left(\sqrt{\sigma}-v_3\right)\sqrt{2-3v_3^2-\sigma+2\sqrt{\sigma}v_3}$ and let $g_\sigma'(v_3)=0$ we have that

$$3v_3^2-3\left(\sqrt{2-\sigma+2v_3\sqrt{\sigma}-v_3^2}+\sqrt{\sigma}\right)v_3+\sigma-1=0.$$

Solving this equation also yields that

$$v_{31}=\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{\sigma}}{4}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\quad\text{and}\quad v_{32}=\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{\sigma}}{4}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}.$$

Direct computation presents that

$$g_\sigma(v_{31}) = 8 - \frac{\sigma}{4} + \frac{\sqrt{3\sigma(4-\sigma)}}{4} - \frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$-\left(\frac{3\sqrt{\sigma}}{4} - \frac{\sqrt{4-\sigma}}{4\sqrt{3}} + \frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right) \times \sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}$$

and

$$g_\sigma(v_{32}) = 8 - \frac{\sigma}{4} + \frac{\sqrt{3\sigma(4-\sigma)}}{4} + \frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$-\left(\frac{3\sqrt{\sigma}}{4} - \frac{\sqrt{4-\sigma}}{4\sqrt{3}} - \frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right) \times \sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}-\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}.$$

By comparing $f_\sigma(v_{31})$ and $f_\sigma(v_{32})$ with $g_\sigma(v_{31})$ and $g_\sigma(v_{32})$, we have for $0 \le \sigma \le 1$ that

$$g_\sigma(v_{31}) \le f_\sigma(v_{31}) \le f_\sigma(v_{32}) \le g_\sigma(v_{32})$$

and for $1 \le \sigma \le 3$ that

$$g_\sigma(v_{31}) \le f_\sigma(v_{31}) \le g_\sigma(v_{32}) \le f_\sigma(v_{32}).$$

The figure is shown in Figure 1. Therefore, the lower boundary of the set S is given as for $0 \le \sigma \le 3$ that

$$L(\sigma) = 8 - \frac{\sigma}{4} + \frac{\sqrt{3\sigma(4-\sigma)}}{4} - \frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$-\left(\frac{3\sqrt{\sigma}}{4} - \frac{\sqrt{4-\sigma}}{4\sqrt{3}} + \frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right) \times \sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}$$

and the upper boundary of the set S is given as a piecewise function. That is, for $0 \le \sigma \le 1$ we have that

$$U(\sigma) = 8 - \frac{\sigma}{4} + \frac{\sqrt{3\sigma(4-\sigma)}}{4} + \frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$-\left(\frac{3\sqrt{\sigma}}{4} - \frac{\sqrt{4-\sigma}}{4\sqrt{3}} - \frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right) \times \sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}-\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}$$

and for $0 \le \sigma \le 3$ we have that

$$U(\sigma) = 8 - \frac{\sigma}{4} + \frac{\sqrt{3\sigma(4-\sigma)}}{4} + \frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$+\left(\frac{3\sqrt{\sigma}}{4} - \frac{\sqrt{4-\sigma}}{4\sqrt{3}} - \frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right) \times \sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}-\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}.$$

Now let us consider the case of

$$\begin{cases} v_1+v_2+v_3=-\sqrt{\sigma}, \\ v_1^2+v_2^2+v_3^2=1. \end{cases}$$

In this case, the points must have either the form

$$\left(\frac{-\left(\sqrt{\sigma}+v_3\right)-\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}}{2},\frac{-\left(\sqrt{\sigma}+v_3\right)+\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}}{2},v_3\right)$$

or

$$\left(\frac{-\left(\sqrt{\sigma}+v_3\right)+\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}}{2},\frac{-\left(\sqrt{\sigma}+v_3\right)-\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}}{2},v_3\right).$$

If $(v_1,v_2,v_3)=\left(\left[-\left(\sqrt{\sigma}+v_3\right)-\sqrt{2-3v_3^2-\sigma-2v_3\sqrt{\sigma}}\right]/2,\left[-\left(\sqrt{\sigma}+v_3\right)+\sqrt{2-3v_3^2-\sigma-2v_3\sqrt{\sigma}}\right]/2,v_3\right)$

then

$$6v_1^2+8v_2^2+10v_3^2=7+3v_3^2-\left(\sqrt{\sigma}+v_3\right)\sqrt{2-3v_3^2-\sigma-2v_3\sqrt{\sigma}}.$$

Set $h_\sigma(v_3)=7+3v_3^2-\left(\sqrt{\sigma}+v_3\right)\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}$ and let $h_\sigma'(v_3)=0$ we have that

$$3v_3^2+3\left(\sqrt{2-\sigma-2v_3\sqrt{\sigma}-3v_3^2}+\sqrt{\sigma}\right)v_3+\sigma-1=0.$$

Solving this equation yields that

$$v_{31}=-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{\sigma}}{4}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}$$

and

$$v_{32}=-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{\sigma}}{4}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}.$$

Direct computation presents that

$$h_\sigma(v_{31})=8-\frac{\sigma}{4}+\frac{\sqrt{3\sigma(4-\sigma)}}{4}+\frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$-\left(\frac{3\sqrt{\sigma}}{4}-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right)\times\sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}$$

and

$$h_\sigma(v_{32})=8-\frac{\sigma}{4}+\frac{\sqrt{3\sigma(4-\sigma)}}{4}-\frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$

$$-\left(\frac{3\sqrt{\sigma}}{4}-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right)\times\sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}.$$

If $(v_1, v_2, v_3) = \left(\left[-\left(\sqrt{\sigma}+v_3\right)+\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}\right]\Big/2, \left[-\left(\sqrt{\sigma}+v_3\right)-\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}\right]\Big/2, v_3\right)$

then

$$6v_1^2+8v_2^2+10v_3^2=7+3v_3^2+\left(\sqrt{\sigma}+v_3\right)\sqrt{2-3v_3^2-\sigma-2v_3\sqrt{\sigma}}.$$

Set $i_\sigma(v_3)=7+3v_3^2+\left(\sqrt{\sigma}+v_3\right)\sqrt{2-3v_3^2-\sigma-2\sqrt{\sigma}v_3}$ and let $i'_\sigma(v_3)=0$, we have that

$$3v_3^2+3\left(\sqrt{2-\sigma-2v_3\sqrt{\sigma}-3v_3^2}-\sqrt{\sigma}\right)v_3+\sigma-1=0.$$

Solving this equation also yields that

$$v_{31}=-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{\sigma}}{4}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}$$

and

$$v_{32}=-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{\sigma}}{4}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}.$$

Direct computation presents that

$$i_\sigma(v_{31})=8-\frac{\sigma}{4}+\frac{\sqrt{3\sigma(4-\sigma)}}{4}+\frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$
$$+\left(\frac{3\sqrt{\sigma}}{4}-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}-\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right)\times\sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}$$

and

$$i_\sigma(v_{32})=8-\frac{\sigma}{4}+\frac{\sqrt{3\sigma(4-\sigma)}}{4}-\frac{\left(\sqrt{4-\sigma}+\sqrt{3\sigma}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{4\sqrt{2}}$$
$$+\left(\frac{3\sqrt{\sigma}}{4}-\frac{\sqrt{4-\sigma}}{4\sqrt{3}}+\frac{\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}{2\sqrt{6}}\right)\times\sqrt{1-\frac{\sigma}{4}-\frac{\sqrt{3\sigma(4-\sigma)}}{12}+\left(\frac{\sqrt{8-2\sigma}}{8}-\frac{\sqrt{6\sigma}}{24}\right)\sqrt{6-3\sigma+\sqrt{3\sigma(4-\sigma)}}}.$$

By comparing $h_\sigma(v_{31})$ and $h_\sigma(v_{32})$ with $i_\sigma(v_{31})$ and $i_\sigma(v_{32})$, we get the same results as discussed in the first case. The figure is shown in Figure 2.

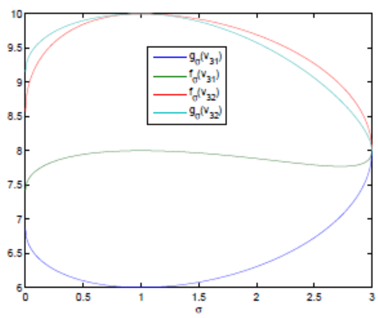

Figure 1. The graph of the functions $f(\sigma)$ and $g(\sigma)$.

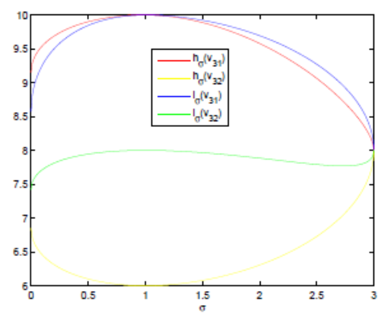

Figure 2. The graph of the functions $h(\sigma)$ and $i(\sigma)$.

4. Conclusion

Inspired by the optimal recovery problem researched in [1] and [2], the geometric boundary of certain convex sets which are defined by two given symmetric positive semi-definite matrices *A* and *B* are characterized in two special cases. That is to say, when *A* and *B* are both diagonal or when *A* and *B* commute with each other. An interesting example is studied in detail and the corresponding geometric boundaries are plotted.

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