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The Euler-like Sums Involving Harmonic Numbers Obtained from Integral Transformations

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Abstract

In this note, we employ two identities from symbolic calculations and combine several known series expansions for specific special functions. By utilizing these tools, we can generate numerous Euler-like sums that connect harmonic numbers, pi and zeta values etc. These results are aesthetically pleasing and widely used in Euler-like sums fields. The main step is to calculate some new Euler-like sums of the harmonic number by integration transformations. The definite integrals of some elementary functions are first calculated by symbol software Mathematica. On the other hand, we can write these elementary functions in the form of Maclaurin series and then integrate them term by term and finally get the series containing harmonic numbers. In the process, we need to deal with the series, integrals, differentiating and other operations skillfully. We can continue to use the integral transform in this paper to expand, or we can find another new integral transform to get more valuable results by using similar methods.

Keywords

Harmonic numbers; Euler-like sums; Integral transformations

Introduction

Campbell and Sofo have investigate many identities of Euler-like sums involving harmonic numbers and they obtained a lots of beautiful results. These results play very important roles in integral calculations and related fields. In this note, we study Euler-like sums containing binomial coefficients and harmonic numbers, which like

$$\sum_{n=0}^{\infty}\frac{\mathrm{H}'_{2n}C_n^2}{16^n}=\frac{24+16\ln 2-16G}{\pi}+8\ln 2-12\,,\quad \sum_{n=0}^{\infty}\frac{\mathrm{H}_{2n}}{\left(4n^2-1\right)^2}=\frac{7\zeta(3)}{16}-\frac{\ln 2}{2}\,.$$

where

$$\mathrm{H}_n^{(r)}:=1+\frac{1}{2^r}+\frac{1}{3^r}+\cdots+\frac{1}{n^r},h_n^{(r)}:=1+\frac{1}{3^r}+\frac{1}{5^r}+\cdots+\frac{1}{(2n-1)^r},$$

$$\mathrm{H}_n^r:=\left(\sum_{k=1}^{n}\frac{1}{k}\right)^r,\mathrm{H}_n^{'(r)}:=1-\frac{1}{2^r}+\frac{1}{3^r}-\frac{1}{4^r}+\cdots+\frac{(-1)^{n+1}}{n^r}.$$

$$G=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\approx 0.91597.$$

By introducing an integral transformation [1],

$$\int_0^1 x^n \arcsin(x)\ln(x)dx = \frac{-\sqrt{\pi}(2\Gamma(\frac{n}{2}+1)((-1)^n(n+1)(\mathrm{H}_{\lfloor (n+1)/2\rfloor}-\mathrm{H}_{n+1}+\ln 2)-1)+\sqrt{\pi}(n+1)\Gamma(\frac{n+1}{2}))}{2(n+1)^3\Gamma(\frac{n+1}{2})},$$

Campbell and Sofo obtained a series of identities of the Euler-like sums with the alternating harmonic numbers. It is wondering that whether some Euler-like sums can be obtained by this method. In this paper, by introducing two integral transformations

$$\int_0^1 x^n\left(1-x^2\right)^{\frac{1}{2}}\log\left(1-x^2\right)dx, \int_0^1 x^n\left(1+x^2\right)\log\left(1-x^2\right)dx,$$

we get a series of new Euler-like sums identity.

1. Some Lemmas

Lemma 1 [3].

$$\left(\arcsin x\right)^p = \sum_{n=0}^{\infty} x^{2n+p}\left[\prod_{k=1}^{p-1}\left\{\sum_{n_k=0}^{n_{k-1}}\frac{\binom{2n_{k-1}-2n_k}{n_{k-1}-n_k}}{2^{2n_{k-1}-2n_k}\left(2n_{k-1}-2n_k+1\right)}\right\}\frac{\binom{2n_{p-1}}{n_{p-1}}}{2^{2n_{p-1}}\left(2n_{p-1}+1\right)}\right].$$

Lemma 2.

$$\int_0^1 x^n(1-x^2)^{\frac{1}{2}}\log(1-x^2)dx = \frac{-\sqrt{\pi}\Gamma(\frac{n+1}{2})(-2+H_{\frac{n}{2}+1}+2\log 2)}{4\Gamma(\frac{n}{2}+2)}.$$

Proof. By Mathematica.

Lemma 3 [2].

$$\int_0^1 \frac{x^n\log(1-x^2)}{\sqrt{1-x^2}}dx = -\frac{n\pi\Gamma(n)(H_{\frac{n}{2}}+2\log 2)}{2^{n+1}\Gamma^2(\frac{n}{2}+1)}.$$

Lemma 3.

$$\int_0^1 x^n(1+x^2)\log(1-x^2)dx = \frac{2(-1-\frac{(n+2)(n+3)H_{\frac{n+1}{2}}}{n+1})}{(n+3)^2} = -\frac{2}{(n+3)^2}-\frac{2(n+2)H_{\frac{n+1}{2}}}{(n+1)(n+3)}.$$

Proof. By Mathematica.

2. Euler-like sums from $\int_0^1 f(x)(1-x^2)^{\frac{1}{2}}\log(1-x^2)dx$

Theorem 1.1.

$$\sum_{n=1}^{\infty}\frac{H_{n+\frac{1}{2}}}{(2n-1)^2(2n+1)} = -1+\frac{\pi^2}{16}+\frac{\log 2}{2}+\frac{7\zeta(3)}{16}. \tag{1}$$

Proof. It can be seen from Lemma 1 that $\arcsin x=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{2n}}\frac{x^{2n+1}}{2n+1}$, let $f(x)=\arcsin x$,we obtain that from Lemma 2,

$$\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)\arcsin x dx=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{2n}(2n+1)}\cdot\frac{-\sqrt{\pi}\Gamma(n+1)(-2+H_{n+\frac{3}{2}}+2\log 2)}{4\Gamma(n+\frac{5}{2})}$$

$$=\sum_{n=0}^{\infty}\frac{2-2\log 2}{(2n+1)^2(2n+3)}-\sum_{n=0}^{\infty}\frac{H_{n+\frac{3}{2}}}{(2n+1)^2(2n+3)}.$$

Since $\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)\arcsin x dx=\frac{1}{2}+\frac{\pi^2}{16}-\frac{\pi^2\log 2}{8}-\frac{7\zeta(3)}{16}$ and

$\sum_{n=0}^{\infty}\frac{2-2\log 2}{(2n+1)^2(2n+3)}=-\frac{1}{2}+\frac{\log 2}{2}+\frac{\pi^2}{8}-\frac{\pi^2\log 2}{8}$, we get (1).

Theorem 1.2.

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+1)}=-\frac{\pi^2}{6}+2\zeta(3). \tag{2}$$

Proof. From Lemma 1, $(\arcsin x)^2=\sum_{n=0}^{\infty}\frac{2^{2n}x^{2n+2}}{(n+1)(2n+1)\binom{2n}{n}}$, let $f(x)=(\arcsin x)^2$ and by Lemma 2,

$$\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)(\arcsin x)^2dx=\sum_{n=0}^{\infty}\frac{2^{2n}}{(n+1)(2n+1)\binom{2n}{n}}\cdot\frac{-\sqrt{\pi}\Gamma(n+\frac{3}{2})(-2+H_{n+2}+2\log 2)}{4\Gamma(n+3)}$$

$$=\sum_{n=0}^{\infty}\frac{\pi-\pi\log 2}{4(n+1)^2(n+2)}-\sum_{n=0}^{\infty}\frac{\pi H_{n+2}}{8(n+1)^2(n+2)}.$$

Since $\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)(\arcsin x)^2dx=\frac{\pi^3}{48}-\frac{\pi^3\log 2}{24}+\frac{\pi}{8}+\frac{\pi\log 2}{4}-\frac{\pi\zeta(3)}{4}$

and $\sum_{n=0}^{\infty}\frac{\pi-\pi\log 2}{4(n+1)^2(n+2)}=-\frac{\pi}{4}+\frac{\pi\log 2}{4}+\frac{\pi^3}{24}-\frac{\pi^3\log 2}{24}$.

so $\sum_{n=0}^{\infty}\frac{H_{n+2}}{(n+1)^2(n+2)}=-3+\frac{\pi^2}{6}+2\zeta(3)$.

Since $\sum_{n=0}^{\infty}\frac{H_{n+2}}{(n+1)^2(n+2)}=\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+1)}+\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^2}=-3+\frac{\pi^3}{3}$, so we get the equation (2).

Theorem 1.3.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n}{16^n(2n-1)^2(n+1)}=\frac{80}{9\pi}-\frac{128\log 2}{9\pi}+\frac{4}{9}. \tag{3}$$

Proof. Integrating $\arcsin x$: $\sqrt{1-x^2}+x\arcsin x-1=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}x^{2n+2}}{2^{2n}(2n+1)(2n+2)}$ (4)

one has $\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)(\sqrt{1-x^2}+x\arcsin x-1)dx=-\frac{44}{27}+\frac{16\log 2}{9}-\frac{\pi}{4}+\frac{\pi\log 2}{2}$.

By Lemma 2,

$\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)(\sqrt{1-x^2}+x\arcsin x-1)dx=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2\pi(1-\log 2)}{8\times 16^n(n+1)^2(n+2)}-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2\pi H_{n+2}}{16^{n+1}(n+1)^2(n+2)}$, where

$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2\pi(1-\log 2)}{8\times 16^n(n+1)^2(n+2)}=\frac{16}{9}-\frac{16\log 2}{9}-\frac{\pi}{2}+\frac{\pi\log 2}{2}$.

So $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2 H_{n+2}}{16^{n+1}(n+1)^2(n+2)}=\frac{92}{27\pi}-\frac{32\log 2}{9\pi}-\frac{1}{4}$.

Since $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2 H_{n+2}}{16^{n+1}(n+1)^2(n+2)}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_{n+1}}{4\times 16^n(2n-1)^2(n+1)}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_{n}}{4\times 16^n(2n-1)^2(n+1)}+\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{4\times 16^n(2n-1)^2(n+1)^2}$

where $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{4\times 16^n(2n-1)^2(n+1)^2}=\frac{32}{27\pi}-\frac{13}{36}$, we get the equation (3).

Theorem 1.4.

$$\sum_{n=1}^{\infty}\frac{H_{n+\frac{1}{2}}}{(n-1)^2(2n+1)^2(2n+3)}=-\frac{95}{192}+\frac{5\log 2}{12}+\frac{13\pi^2}{256}-\frac{3\pi^2\log 2}{16}-\frac{21\zeta(3)}{128}. \quad (5)$$

Proof. Integrating the formula (4):

$$-x+\frac{3}{4}x\sqrt{1-x^2}+\frac{\arcsin x}{4}+\frac{1}{2}x^2\arcsin x=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}x^{2n+3}}{2^{2n}(2n+1)(2n+2)(2n+3)}.$$

Let $f(x)=-x+\frac{3}{4}x\sqrt{1-x^2}+\frac{\arcsin x}{4}+\frac{1}{2}x^2\arcsin x$, then

$\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)(-x+\frac{3}{4}x\sqrt{1-x^2}+\frac{\arcsin x}{4}+\frac{1}{2}x^2\arcsin x)dx=\frac{173}{576}+\frac{5\pi^2}{256}-\frac{9\pi^2\log 2}{64}-\frac{21\zeta(3)}{128}$ from Lemma 2,

$\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)(-x+\frac{3}{4}x\sqrt{1-x^2}+\frac{\arcsin x}{4}+\frac{1}{2}x^2\arcsin x)dx=\sum_{n=0}^{\infty}\frac{2-2\log 2}{(2n+1)^2(2n+3)^2(2n+5)}$

$-\sum_{n=0}^{\infty}\frac{H_{n+\frac{5}{2}}}{(2n+1)^2(2n+3)^2(2n+5)}$.

Where $\sum_{n=0}^{\infty}\frac{2-2\log 2}{(2n+1)^2(2n+3)^2(2n+5)}=-\frac{5}{12}+\frac{5\log 2}{12}+\frac{3\pi^2}{64}-\frac{3\pi^2\log 2}{64}$.

So $\sum_{n=0}^{\infty}\frac{H_{n+\frac{5}{2}}}{(2n+1)^2(2n+3)^2(2n+5)}=-\frac{413}{576}+\frac{5\log 2}{12}+\frac{7\pi^2}{256}-\frac{3\pi^2\log 2}{16}-\frac{21\zeta(3)}{128}$.

Since $\sum_{n=0}^{\infty}\frac{H_{n+\frac{5}{2}}}{(2n+1)^2(2n+3)^2(2n+5)}=\sum_{n=1}^{\infty}\frac{H_{n+\frac{1}{2}}}{(2n-1)^2(2n+1)^2(2n+3)}+\sum_{n=1}^{\infty}\frac{2}{(2n-1)^2(2n+1)^2(2n+3)^2}$,

where $\sum_{n=1}^{\infty}\frac{2}{(2n-1)^2(2n+1)^2(2n+3)^2}=-\frac{128}{576}+\frac{3\pi^2}{128}$, so we get the formula (5).

Theorem 1.5.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}H_n}{2^{2n}n(n+1)}=\frac{\pi^2}{3}-4\log 2. \tag{6}$$

Proof. Let $f(x)=\log(1-x^2)=\sum_{n=1}^{\infty}(-1)\frac{x^{2n}}{n}$, then $\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)^2\,dx=-\frac{\pi}{2}+\frac{\pi^3}{12}-\pi\log 2+\pi(\log 2)^2$.

From Lemma 2, $\int_0^1(1-x^2)^{\frac{1}{2}}\log(1-x^2)^2\,dx=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}\pi H_{n+1}}{4\cdot 2^{2n}n(n+1)}-\sum_{n=1}^{\infty}\frac{\binom{2n}{n}\pi(2-2\log 2)}{4\cdot 2^{2n}n(n+1)}$.

Where $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}\pi(2-2\log 2)}{4\cdot 2^{2n}n(n+1)}=-\frac{\pi}{2}+\frac{3\pi\log 2}{2}-\pi(\log 2)^2$, so $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}H_{n+1}}{2^{2n}n(n+1)}=-4+\frac{\pi^2}{3}+2\log 2$.

Since $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}H_{n+1}}{2^{2n}n(n+1)}=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}H_n}{2^{2n}n(n+1)}+\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{2^{2n}n(n+1)^2}$, where $\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{2^{2n}n(n+1)^2}=-4+6\log 2$, we get equation (6).

Similarly, let $f(x)$ be $(\arcsin x)^3$, $(\arcsin x)^4$, $(\arcsin x)^5$, $(\arcsin x)^6$, $(\arcsin x)^7$, $(\arcsin x)^8$ respectively, it can be seen from Lemma 1, $(\arcsin x)^3=3\sum_{n=1}^{\infty}\frac{\binom{2n}{n}h_n^{(2)}x^{2n+1}}{2^{2n-1}(2n+1)}$, $(\arcsin x)^4=\frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n^{(2)}(2x)^{2n+2}}{(n+1)(2n+1)\binom{2n}{n}}$,

$(\arcsin x)^5=5!\sum_{n=2}^{\infty}\frac{\binom{2n}{n}\mathfrak{h}_2(n)x^{2n+1}}{2^{2n}(2n+1)}$, $(\arcsin x)^6=6!\sum_{n=2}^{\infty}\frac{2^{2n}\mathcal{H}_2(n)}{(2n+1)\binom{2n}{n}}\frac{x^{2n+2}}{2n+2}$, $(\arcsin x)^7=7!\sum_{n=3}^{\infty}\frac{\binom{2n}{n}\mathfrak{h}_3(n)x^{2n+1}}{2^{2n}(2n+1)}$,

$(\arcsin x)^8=8!\sum_{n=3}^{\infty}\frac{2^{2n}\mathcal{H}_3(n)}{(2n+1)\binom{2n}{n}}\frac{x^{2n+2}}{2n+2}$, where $\mathfrak{h}_p(n)=\sum_{n\ge n_1>\cdots>n_p>0}\frac{1}{(2n_1-1)^2\cdots(2n_p-1)^2}$,

$\mathcal{H}_p(n)=\frac{1}{2^{2p}}\sum_{n\ge n_1>\cdots>n_p>0}\frac{1}{n_1^2\cdots n_p^2}$ [3]. We can draw the following corollaries.

Corollary 1.6

$$\sum_{n=1}^{\infty}\frac{h_n^{(2)}(h_n-1)}{(2n+1)^2(2n+3)}=\frac{1}{8}-\frac{\pi^4}{1536}+\frac{\pi^4\log 2}{768}-\frac{\pi^2}{128}-\frac{\pi^2\log 2}{64}+\frac{\pi^4\zeta(3)}{64}-\frac{7\zeta(3)}{128}-\frac{31\zeta(5)}{512}, \tag{7}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}\left(H_{n+2}-2+2\log 2\right)}{(n+1)(n+2)^2}=-\frac{\pi^4}{120}+\frac{\pi^4\log 2}{60}-\frac{\pi^2}{6}-\frac{\pi^2\log 2}{3}+\frac{\pi^2\zeta(3)}{3}+3+2\log 2 -2\zeta(3)-2\zeta(5), \tag{8}$$

$$\sum_{n=2}^{\infty}\frac{\mathfrak{h}_2(n)\left(h_n-1\right)}{(2n+1)^2(2n+3)}=-\frac{\pi^6}{184320}+\frac{\pi^6\log 2}{92160}-\frac{\pi^4}{6144}+\frac{\pi^4\log 2}{3072}+\frac{\pi^4\zeta(3)}{3072}+\frac{3\pi^2}{512}+\frac{\pi^2\log 2}{256} -\frac{\pi^2\zeta(3)}{256}-\frac{\pi^2\zeta(5)}{256}-\frac{3}{64}+\frac{7\zeta(3)}{512}+\frac{31\zeta(5)}{2048}+\frac{127\zeta(7)}{8192}, \tag{9}$$

$$\sum_{n=2}^{\infty}\frac{\mathcal{H}_2(n)\left(H_{n+2}-2+2\log 2\right)}{(n+1)^2(n+2)}=-\frac{\pi^6}{80640}+\frac{\pi^6\log 2}{40320}-\frac{\pi^4}{1920}-\frac{\pi^4\log 2}{960}+\frac{\pi^4\zeta(3)}{960}+\frac{\pi^2}{32} +\frac{\pi^2\log 2}{48}-\frac{\pi^2\zeta(3)}{48}-\frac{\pi^2\zeta(5)}{48}-\frac{5}{16}-\frac{\log 2}{8}+\frac{\zeta(3)}{8}+\frac{\zeta(5)}{8}+\frac{\zeta(7)}{8}, \tag{10}$$

$$\sum_{n=3}^{\infty}\frac{\mathfrak{h}_3(n)\left(h_n-1\right)}{(2n+1)^2(2n+3)}=-\frac{\pi^8}{41287680}+\frac{\pi^8\log 2}{20643840}-\frac{\pi^6}{737280}-\frac{\pi^6\log 2}{368640}+\frac{\pi^6\zeta(3)}{368640}+\frac{\pi^4}{8192} +\frac{\pi^4\log 2}{12288}-\frac{\pi^4\zeta(3)}{12288}-\frac{\pi^4\zeta(5)}{12288}-\frac{5\pi^2}{2048}-\frac{\pi^2\log 2}{1024}+\frac{\pi^2\zeta(3)}{1024}+\frac{\pi^2\zeta(5)}{1024}+\frac{\pi^2\zeta(7)}{1024}+\frac{1}{64} -\frac{7\zeta(3)}{2048}-\frac{31\zeta(5)}{8192}-\frac{127\zeta(7)}{32768}-\frac{511\zeta(9)}{131072}, \tag{11}$$

$$\sum_{n=3}^{\infty}\frac{\mathcal{H}_3(n)\left(H_{n+2}-2+2\log 2\right)}{(n+1)^2(n+2)}=-\frac{\pi^8}{233224320}+\frac{\pi^8\log 2}{11612160}-\frac{\pi^6}{322560}-\frac{\pi^6\log 2}{161280}+\frac{\pi^6\zeta(3)}{161280} +\frac{\pi^4}{2560}+\frac{\pi^4\log 2}{3840}-\frac{\pi^4\zeta(3)}{3840}-\frac{\pi^4\zeta(5)}{3840}-\frac{5\pi^2}{384}-\frac{\pi^2\log 2}{192}+\frac{\pi^2\zeta(3)}{192}+\frac{\pi^2\zeta(5)}{192}+\frac{\pi^2\zeta(7)}{192} +\frac{7}{64}+\frac{\log 2}{32}-\frac{\zeta(3)}{32}-\frac{\zeta(5)}{32}-\frac{\zeta(7)}{32}-\frac{\zeta(9)}{32}. \tag{12}$$

3. Euler-like sums from $\int_0^1 f(x)\frac{\log(1-x^2)}{\sqrt{1-x^2}}dx$

Theorem 2.1.

$$\sum_{n=1}^{\infty}\frac{H_{n+\frac{1}{2}}}{(2n+1)^2}=\frac{7\zeta(3)}{8}. \tag{13}$$

Proof. Let $f(x)=\arcsin x$, then $\int_0^1\frac{\log(1-x^2)\arcsin x}{\sqrt{1-x^2}}dx=-\frac{\pi^2\log 2}{4}-\frac{7\zeta(3)}{8}$.

From Lemma 3 $\int_0^1\frac{\log(1-x^2)\arcsin x}{\sqrt{1-x^2}}dx=-\sum_{n=1}^{\infty}\frac{H_{n+\frac{1}{2}}}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{2\log 2}{(2n+1)^2}$, where $\sum_{n=1}^{\infty}\frac{2\log 2}{(2n+1)^2}=\frac{\pi^2\log 2}{4}$,

we get the formula (13).

Theorem 2.2.

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3). \tag{14}$$

Proof. Let $f(x)=(\arcsin x)^2$, then $\int_0^1 \frac{\log(1-x^2)(\arcsin x)^2}{\sqrt{1-x^2}}dx=-\frac{\pi^3\log 2}{12}-\frac{\pi\zeta(3)}{2}$.

From Lemma 3, $\int_0^1 \frac{\log(1-x^2)(\arcsin x)^2}{\sqrt{1-x^2}}dx=-\sum_{n=0}^{\infty}\frac{\pi H_{n+1}}{4(n+1)^2}-\sum_{n=0}^{\infty}\frac{2\pi\log 2}{4(n+1)^2}$, where $\sum_{n=0}^{\infty}\frac{2\pi\log 2}{4(n+1)^2}=\frac{\pi^3\log 2}{12}$, we get the formula (14).

Theorem 2.3.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}H_n}{2^{2n}n}=\frac{\pi^2}{3}+2(\log 2)^2. \tag{15}$$

Proof. Let $f(x)=\log(1-x^2)$, then $\int_0^1 \frac{\log(1-x^2)^2}{\sqrt{1-x^2}}dx=\frac{\pi^3}{6}+2\pi(\log 2)^2$. By Lemma 3,

$\int_0^1 \frac{\log(1-x^2)^2}{\sqrt{1-x^2}}dx=\sum_{n=1}^{\infty}\frac{\binom{2n}{n}\pi H_n}{2^{2n+1}n}+\sum_{n=1}^{\infty}\frac{2\binom{2n}{n}\pi\log 2}{2^{2n+1}n}$, where $\sum_{n=1}^{\infty}\frac{2\binom{2n}{n}\pi\log 2}{2^{2n+1}n}=\pi(\log 2)^2$, so we get the formula (15).

We can obtain the following theorems in the same way.

Theorem 2.4.

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2 H_n}{16^n(n+1)}=4-\frac{16\log 2}{\pi}. \tag{16}$$

Theorem 2.5.

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}H_n}{2^{2n}(n+1)}=-4\log 2-4\sqrt{2}\log 2-4\sqrt{2}\log\left(2-\sqrt{2}\right)-4\log\left(\sqrt{2}-1\right). \tag{17}$$

4. Euler-like sums from $\int_0^1 f(x)(1+x^2)\log(1-x^2)dx$

Theorem 3.1.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}(2n+1)H_n}{2^{2n}(n+1)(2n-1)^2}=-\frac{44}{9}+\frac{16G}{3}+\frac{4\pi}{3}-\frac{4\pi\log 2}{3}-\frac{4\log 2}{9}.$$

Proof. Let $f(x)=\arcsin x$, the theorem can be proven.

Theorem 3.2.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}(2n+3)H_n}{2^{2n}(n+1)(n+2)(2n+1)(2n-1)^2}=-\frac{5911}{675}+\frac{24G}{5}+\frac{8\pi}{15}-\frac{6\pi\log 2}{5}-\frac{12\log 2}{5}.$$

Proof. Let $f(x)=-x+\frac{3}{4}x\sqrt{1-x^2}+\frac{\arcsin x}{4}+\frac{1}{2}x^2\arcsin x$, the theorem can be proven.

In the same way, we can prove the following theorems.

Theorem 3.3.

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}H_n}{2^{2n}n(n+1)}=\frac{\pi^2}{2}-4\log 2\,.$$

Theorem 3.4.

$$\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)H_n}{n(n+1)(2n-1)}=-\frac{8G}{3}-\frac{4\pi}{3}+\frac{\pi^2}{12}+\frac{8\log 2}{3}-\frac{\log^2 2}{3}+\frac{4\pi\log 2}{3}\,.$$

Theorem 3.5.

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_n}{n(n+1)}=\log^2 2+\frac{\pi^2}{12}\,.$$

Theorem 3.6.

$$\sum_{n=1}^{\infty}\frac{(n+1)H_{n+\frac{1}{2}}}{n(2n+1)(2n+3)}=\frac{22}{9}-\frac{5\pi^2}{72}-\frac{25\log 2}{9}-\frac{4\log^2 2}{3}\,.$$

Theorem 3.7.

$$\sum_{n=1}^{\infty}\frac{(2n+1)H_n}{n^2(n+1)}=2\zeta(3)+\frac{\pi^2}{6}\,.$$

Theorem 3.8.

$$\sum_{n=1}^{\infty}\frac{H_{n-\frac{1}{2}}}{(2n-1)(2n+1)}=\frac{\pi^2}{8}-\log 2\,.$$

Theorem 3.9.

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}(n+1)H_{n+\frac{1}{2}}}{2^{2n}(2n+1)(2n+3)}=\frac{\pi}{8}+\frac{\pi\log 2}{4}\,.$$

5. Conclusion

In this note, using integral transformations, we established connections between the harmonic numbers and some special functions. Our methods is writing these elementary functions in the form of Maclaurin series and using integration of them to connect the harmonic numbers. Our methods can produce a series of new identities involving special constants and the harmonic numbers.

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