Journal of Applied Mathematics and Computation, 2024, 8(3), 272-279
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

Direct Solution of Maxwell's Equations

Savely Rabinovich[1], **Victor Malyutin**[2,*]

[1]Nephrology Department, Ichilov Medical Center, Tel Aviv 6423906, Israel.
[2]Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk 220072, Belarus.

How to cite this paper: Savely Rabinovich, Victor Malyutin. (2024) Direct Solution of Maxwell's Equations. *Journal of Applied Mathematics and Computation*, 8(3), 272-279.
DOI: 10.26855/jamc.2024.09.008

Received: August 23, 2024
Accepted: September 19, 2024
Published: October 17, 2024

***Corresponding author:** Victor Malyutin, Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk 220072, Belarus.

Abstract

A new solution to Maxwell's differential equations is proposed. A new approach for writing solutions to these equations under consideration uses quaternions. The equations are written as a kind of generalization of the Cauchy-Riemann equations and have a form of partial differential equation of first order. The Green's function was found for direct (without potentials) solutions of Maxwell's equations. To calculate Green's function, we use factorization of the d'Alembert operator and the fact that Green's function for the d'Alembert operator is known. Three examples of determining the electromagnetic field strength were considered. This is an example of finding the strength of the electromagnetic field created by the charge q moving with constant speed v along the axis x_1. The example of finding the electric field strength created by a uniformly charged thin rod at a point perpendicular to the rod at a distance R from the rod. The example of finding the electric field strength created by a dipole at a point located perpendicular to the middle of the dipole at a distance R from the middle of the dipole.

Keywords

Maxwell's equations; Green's function; quaternions

1. Introduction

Maxwell's equations for the electromagnetic field

$$div\underline{E} = 4\pi i_0 \,,\quad div\underline{H} = 0 \,,\quad \frac{1}{c}\partial_0\underline{E} - rot\underline{H} = -\frac{4\pi}{c}\underline{I} \,,\quad \frac{1}{c}\partial_0\underline{H} + rot\underline{E} = 0 \,,$$

where $\underline{E} = (E_1, E_2, E_3)$, $\underline{H} = (H_1, H_2, H_3)$, $\underline{I} = (i_1, i_2, i_3)$, are the most famous equations of physics.

Over the course of a century and a half, many approaches and refinements have been proposed for solutions to these equations.

In this note, we propose a new approach for writing solutions to these equations, which does not use potentials. The approach under consideration uses quaternions, which are widely used in physics and mathematics [1-5]. We will show how non-conventional notation leads directly to finding Green's function for these equations.

The starting point for this approach is Kuni Imaeda's paper Imaeda [6]

$$DF\ (X) = -4\pi I \,, \tag{1}$$

where $D = \frac{1}{c}\partial_0 - e_1\partial_1 - e_2\partial_2 - e_3\partial_3$, $\partial_j = \frac{\partial}{\partial x_j}$, $F(x) = \sum_{j=0}^{3} e_j (E_j + iH_j)$, $E_0 = H_0 = 0$, $I = \sum_{j=0}^{3} e_j i_j$, $e_0 = 1$, $e_j^2 = 1$, $e_1 e_2 = -e_2 e_1 = ie_3$ and so on.

When the right side is zero, this entry represents a condition similar to the condition Cauchy-Riemann for a complexvalued function of a complex variable $Df(z)=0$, where $D=\partial_x+i\partial_y$, $z=x+iy$, $f=u(x,y)+iv(x,y)$. In our case, we will talk about a bivector of quaternions $E+iH=\sum_{j=1}^{3}e_j(E_j+iH_j)$ (classical Hamiltonian quaternions with complex coefficients) over a set constructed from the same quaternions with real coefficients $x=\sum_{j=0}^{3}x_je_j$. Given this choice of x, its norm is equal to $\sqrt{(x_0+\sum_{j=0}^{3}x_je_j)(x_0-\sum_{j=0}^{3}x_je_j)}=\sqrt{x_0^2-\sum_{j=0}^{3}x_j^2}$, which coincides with the relativistic interval and therefore automatically ensures relativistic invariance.

Then this formal notation (1) breaks down into four equations, which are exactly Maxwell's equations. The right side of these equations, namely, charges and currents represents the phenomenological part of Maxwell's equations. Equations with the right side equal zero (that represents Maxwell equations in vacuum) are an exact analogue of Cauchy-Riemann Equations. It is known that Caucy-Riemann equations are the definition of analytic or regular functions. Solutions for such equations are written in the form of Further polynomials [7, 8].

Instead of breaking equation (1) into four notations, we will use a single notation of this hypercomplex system. Then, it is clear that this is a partial differential equation of first order.

2. Calculation of Green's function

As is customary in such cases, we will look for a solution to the equation for the Green's function

$$DG(x)=\delta(x)$$

with the right side in the form of a delta function.

These equations can be solved in a completely standard way. By Fourier-transform analysis and corresponding integration. However, we offer another solution by noting that the operator D factorizes the d'Alembert operator $\Box$ $=-\frac{1}{c^2}\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$. Exactly, $-DD^*=-D^*D=\Box$,

where

$$D^*=\frac{1}{c}\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+e_3\frac{\partial}{\partial x_3}.$$

This exactly corresponds to the factorization of the Klein-Gordon operator by the Dirac Hamiltonian.

Since, by definition, the Green's function $G_\Box$ is a solution to the equation

$$-DD^*\ G_\Box=\delta(x),$$

then for $G=-D^*\ G_\Box$ is true $DG(x)=\delta(x)$. Thus, the solution to Maxwell's equations will take the form

$$E-iH=4\pi\int D^*\ G_\Box(x-x')I(x')dx'. \tag{2}$$

After multiplication and integration, the result is organized by the imaginary unit to obtain a solution for electric E and magnetic H fields. The answer very much depends on the specific form of the right side of equation I.

Green's function $G_\Box$ is known. However, it should be noted that if the kernel of the operator is non-trivial, then Green's function is not unique. However, in practice, using the principle of symmetry, boundary conditions, or other additional conditions allows you to determine a specific Green's function.

In general, the function $G_\Box$ can be chosen as a linear combination

$$G_\Box=-c_1\frac{\delta(cx_0+r)}{4\pi r}-c_2\frac{\delta(-cx_0+r)}{4\pi r},$$

where $r=\sqrt{x_1^2+x_2^2+x_3^2}$, $c_1+c_2=1$.

By calculating the operator D^* from the function $-G_\Box$ we obtain the Green's function G for our problem.

To calculate we use the following equality

$$e_j\partial_j \frac{\delta(\pm cx_0+r)}{4\pi r} = -\frac{\delta(\pm cx_0+r)}{r^2}\frac{e_jx_j}{r} + e_j\frac{\delta'(\pm cx_0+r)}{4\pi r}.$$

We get

$$G = -D^* \ G_\square = c_1\frac{\delta'(cx_0+r)}{4\pi r} - c_2\frac{\delta'(-cx_0+r)}{4\pi r} +$$

$$\left[-c_1\frac{\delta(cx_0+r)}{4\pi r^2} - c_2\frac{\delta(-cx_0+r)}{4\pi r^2} + c_1\frac{\delta'(cx_0+r)}{4\pi r} + c_2\frac{\delta'(-cx_0+r)}{4\pi r}\right]\frac{x_1e_1+x_2e_2+x_3e_3}{r}. \quad (3)$$

So far, we proved the following:

Statement 1. *The solution to Maxwell equation (1) can be formulated as equation (2) where* $D^*\ G_\square$ *is determined by formula (3).*

Now, after calculating the Green's function, we are able to consider three examples of determining the electromagnetic field strength using formula (2) and the Green's function G defined by equation (3).

3. Example 1

Let us consider the problem of finding the strength of the electromagnetic field created by the charge q moving with constant speed v along the axis x_1. In this case, function I can be written

$$I(x_0,x_1,x_2,x_3) = q\delta(x_1 - vx_0)\delta(x_2-0)\delta(x_3-0)\left(1+\frac{v}{c}e_1\right). \quad (4)$$

The electromagnetic field strength is found using formula (2), where I is given by equation (4).

To find the electromagnetic field strength, we first calculate

$$\int \frac{\delta(c(x_0-x_0')+r-r')}{4\pi(r-r')^2} I(x_0',x_1',x_2',x_3')dx_0'dx_1'dx_2'dx_3',$$

where $r-r' = \sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}$.

For simplicity, we will consider calculating the electromagnetic field strength at a point on the x_1 axis. Therefore, after integration over $dx_1'dx_2'dx_3'$ we obtain the expression

$$qe_1\left(1+\frac{v}{c}e_1\right)\int \frac{\delta(c(x_0-x_0')+x_1-vx_0')}{4\pi(x_1-vx_0')^2}dx_0'. \quad (5)$$

To calculate the integral over x_0' of a function in expression (5), we find x_0', at which the delta function argument is zero.

From the condition $c(x_0-x_0')+x_1-vx_0' = 0$, we get $x_1+cx_0 = x_0'(v+c)$ or $x_0' = \frac{x_1+cx_0}{v+c}$.

Then, allow us to obtain the equation

$$\int \frac{\delta(c(x_0-x_0')+r-r')}{4\pi(r-r')^2} I(x_0',x_1',x_2',x_3')dx_0'dx_1'dx_2'dx_3' =$$

$$qe_1\left(1+\frac{v}{c}e_1\right)\frac{1}{4\pi\left(x_1 - v\frac{x_1+cx_0}{v+c}\right)^2} = qe_1\left(1+\frac{v}{c}e_1\right)\frac{\left(1+\frac{v}{c}\right)^2}{4\pi(x_1-vx_0)^2}. \quad (6)$$

Let us now calculate

$$\int \frac{\delta(-c(x_0 - x_0^{'}) + r - r^{'})}{4\pi(r - r^{'})^2} I(x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) dx_0^{'} dx_1^{'} dx_2^{'} dx_3^{'} .$$

After integration over $dx_1^{'} dx_2^{'} dx_3^{'}$ we obtain the equation

$$qe_l \left(1 + \frac{v}{c} e_1\right) \int \frac{\delta(-c(x_0 - x_0^{'}) + x_1 - vx_0^{'})}{4\pi(x_1 - vx_0^{'})^2} dx_0^{'} . \tag{7}$$

To calculate the integral over $x_0^{'}$ of a function in expression (7) we find $x_0^{'}$, at which the delta function argument is zero, namely $x_0^{'} = \frac{x_1 - cx_0}{v - c}$. Thus, we obtain the following equation

$$\int \frac{\delta(-c(x_0 - x_0^{'}) + r - r^{'})}{4\pi(r - r^{'})^2} I(x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) dx_0^{'} dx_1^{'} dx_2^{'} dx_3^{'} =$$

$$qe_l \left(1 + \frac{v}{c} e_1\right) \frac{1}{4\pi\left(x_1 - v\frac{x_1 - cx_0}{v - c}\right)^2} = qe_l \left(1 + \frac{v}{c} e_1\right) \frac{\left(1 - \frac{v}{c}\right)^2}{4\pi(x_1 - vx_0)^2} . \tag{8}$$

Let us now calculate

$$-\int \frac{\delta^{'}(c(x_0 - x_0^{'}) + r - r^{'})}{4\pi(r - r^{'})^2} I(x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) dx_0^{'} dx_1^{'} dx_2^{'} dx_3^{'} .$$

After integration over $dx_1^{'} dx_2^{'} dx_3^{'}$ we obtain the expression

$$qe_l \left(1 + \frac{v}{c} e_1\right) \int \frac{\delta^{'}_{x_0^{'}}(c(x_0 - x_0^{'}) + x_1 - vx_0^{'})}{4\pi(v + c)(x_1 - vx_0^{'})^2} dx_0^{'} .$$

By analogy with calculating the integral over $x_0^{'}$ in expression (5) we get

$$-\int \frac{\delta^{'}(c(x_0 - x_0^{'}) + r - r^{'})}{4\pi(r - r^{'})^2} I(x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) dx_0^{'} dx_1^{'} dx_2^{'} dx_3^{'} =$$

$$\frac{vqe_l \left(1 + \frac{v}{c} e_1\right)}{v + c} \frac{\left(1 + \frac{v}{c}\right)^2}{4\pi(x_1 - vx_0)^2} . \tag{9}$$

Similarly, it can be shown that

$$-\int \frac{\delta^{'}(-c(x_0 - x_0^{'}) + r - r^{'})}{4\pi(r - r^{'})^2} I(x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) dx_0^{'} dx_1^{'} dx_2^{'} dx_3^{'} =$$

$$\frac{vqe_l \left(1 + \frac{v}{c} e_1\right)}{v - c} \frac{\left(1 - \frac{v}{c}\right)^2}{4\pi(x_1 - vx_0)^2} . \tag{10}$$

From equations (6), (8) and (9), (10) we obtain that

$$\int G(x_0 - x_0^{'}, x_1 - x_1^{'}, x_2 - x_2^{'}, x_3 - x_3^{'}) I(x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) dx_0^{'} dx_1^{'} dx_2^{'} dx_3^{'} =$$

$$-c_1\frac{vqe_I\left(1+\frac{v}{c}e_1\right)}{v+c}\frac{\left(1+\frac{v}{c}\right)^2}{4\pi(x_1-vx_0)^2}+c_2\frac{vqe_I\left(1+\frac{v}{c}e_1\right)}{v-c}\frac{\left(1-\frac{v}{c}\right)^2}{4\pi(x_1-vx_0)^2}+$$

$$\left[-c_1\left(1+\frac{v}{v+c}\right)\frac{\left(1+\frac{v}{c}\right)^2}{4\pi(x_1-vx_0)^2}-c_2\left(1+\frac{v}{v-c}\right)\frac{\left(1-\frac{v}{c}\right)^2}{4\pi(x_1-vx_0)^2}\right]qe_I\left(1+\frac{v}{c}e_1\right)=$$

$$\left[-c_1\frac{v}{v+c}\left(1+\frac{v}{c}\right)^2+c_2\frac{v}{v-c}\left(1-\frac{v}{c}\right)^2-c_1\left(1+\frac{v}{v+c}\right)\left(1+\frac{v}{c}\right)^2-c_2\left(1+\frac{v}{v-c}\right)\left(1-\frac{v}{c}\right)^2\right]\times$$

$$\frac{qe_I\left(1+\frac{v}{c}e_1\right)}{4\pi(x_1-vx_0)^2}. \tag{11}$$

When $c_1=\frac{1}{2}(1-\frac{v}{c})$, $c_2=\frac{1}{2}(1+\frac{v}{c})$, expression (11) will be written in the form

$$-4\pi\int G(x_0-x_0',x_1-x_1',x_2-x_2',x_3-x_3')I(x_0',x_1',x_2',x_3')dx_0'dx_1'dx_2'dx_3'=$$

$$\frac{qe_I\left(1+\frac{v}{c}e_1\right)}{(x_1-vx_0)^2}\left(1-\frac{v^2}{c^2}\right)\left[\frac{v}{c}+1\right]. \tag{12}$$

At $\frac{v}{c}<<1$ coefficient for e_1 is equal to $\frac{q}{(x_1-vx_0)^2}$ and gives the value of the electric field along the axis x_1. In this case, the value of the magnetic field is zero.

4. Example 2

Let us consider the problem of finding the electric field strength created by a uniformly charged thin rod at a point perpendicular to the rod at a distance R from the rod.

Let us assume that the rod is directed along the x_3 axis, the linear charge density in the rod is equal to q and the considered point at the coordinates $x_2=0$, $x_3=0$. In this case, the function I looks like

$$I(x_0,x_1,x_2,x_3)=q\delta(x_1-0)\delta(x_2-0). \tag{13}$$

The electric field strength is found using formula (2), where G and I are determined by equations (3) and (13), respectively.

In a cylindrical coordinate system

$$G=c_1\frac{\delta'(cx_0+\sqrt{\rho^2+z^2})}{4\pi\sqrt{\rho^2+z^2}}-c_2\frac{\delta'(-cx_0+\sqrt{\rho^2+z^2})}{4\pi\sqrt{\rho^2+z^2}}+$$

$$(-c_1\frac{\delta(cx_0+\sqrt{\rho^2+z^2})}{4\pi(\rho^2+z^2)}-c_2\frac{\delta(-cx_0+\sqrt{\rho^2+z^2})}{4\pi(\rho^2+z^2)}+$$

$$c_1\frac{\delta'(cx_0+\sqrt{\rho^2+z^2})}{4\pi\sqrt{\rho^2+z^2}}+c_2\frac{\delta'(cx_0+\sqrt{\rho^2+z^2})}{4\pi\sqrt{\rho^2+z^2}})\times$$

$$\frac{\rho\cos(\varphi)e_1+\rho\sin(\varphi)e_2+ze_3}{4\pi\sqrt{\rho^2+z^2}}. \tag{14}$$

$$I(x_0,\rho,\varphi,z)=q\delta(\rho-0)\delta(\varphi-0). \tag{15}$$

From formulas (2), (14), and (15) after integration over x_0',ρ',φ', we obtain that

$$F\ (x_0,\rho,\varphi,z)=(c_1+c_2)\int_{-\infty}^{\infty}\frac{q}{\rho^2+(z-z')^2}\frac{\rho\cos(\varphi)e_1+\rho\sin(\varphi)e_2+(z-z')e_3}{\sqrt{\rho^2+(z-z')^2}}dz'.$$

Since the point under consideration has $z=0$, we obtain that

$$F\ (x_0,\rho,\varphi,0)=(c_1+c_2)\int_{-\infty}^{\infty}\frac{q}{\rho^2+(z')^2}\frac{\rho\cos(\varphi)e_1+\rho\sin(\varphi)e_2-z'e_3}{\sqrt{\rho^2+(z')^2}}dz'.$$

Because

$$\int_{-\infty}^{\infty}\frac{z}{(\rho^2+z^2)^{\frac{3}{2}}}dz=0,\quad \int_{-\infty}^{\infty}\frac{dz}{(\rho^2+z^2)^{\frac{3}{2}}}=\frac{z}{\rho^2(\rho^2+z^2)^{\frac{1}{2}}}\to\frac{1}{\rho^2}\quad \text{as}\quad z\to\infty$$

we get that

$$F\ (x_0,\rho,\varphi,0)=\frac{2(c_1+c_2)q}{\rho^2}(\rho\cos(\varphi)e_1+\rho\sin(\varphi)e_2).$$

For $c_1+c_2=1$, we get that

$$F\ (x_0,\rho,\varphi,0)=\frac{2q}{\rho^2}(\rho\cos(\varphi)e_1+\rho\sin(\varphi)e_2).$$

If the point is on the x_1 axis, then $\varphi=0,\rho=R$ and

$$F\ (x_0,R,0,0)=\frac{2q}{R}e_1,$$

that is we get the value of the electric field along the x_1 axis is equal to $\frac{2q}{R}$.

If the point is on the x_2 axis then $\varphi=\frac{\pi}{2},\rho=R$ and

$$F\ (x_0,R,\frac{\pi}{2},0)=\frac{2q}{R}e_2,$$

that is we get the value of the electric field along the x_2 axis is equal to $\frac{2q}{R}$.

5. Example 3

Let's consider the problem of finding the electric field strength created by a dipole at a point located perpendicular to the middle of the dipole at a distance R from the middle of the dipole. Let us assume that a dipole with a distance l between charges q and $-q$ directed along axis x_3, and the point under consideration lies on the axis x_1.

In this case, the function I has the form

$$I(x_0,x_1,x_2,x_3)=q\delta(x_1-0)\delta(x_2-0)\left(\delta(x_3+\frac{l}{2})-\delta(x_3-\frac{l}{2})\right). \tag{16}$$

The electric field strength is found using formula (2), where G and I are determined by equalities (3) and (16), respectively.

In a cylindrical coordinate system, G is determined by equation (14).

$$I(x_0,\rho,\varphi,z)=q\delta(\rho-0)\delta(\varphi-0)\left(\delta(z+\frac{l}{2})-\delta(z-\frac{l}{2})\right). \quad (17)$$

From formulas (2), (14), and (17) after integration over x_0',ρ',φ' we obtain that

$$F\ (x_0,\rho,\varphi,z)=(c_1+c_2)\int_{-\infty}^{\infty}\frac{q\left(\delta(z'+\frac{l}{2})-\delta(z'-\frac{l}{2})\right)}{\rho^2+(z-z')^2}\frac{\rho\cos(\varphi)e_1+\rho\sin(\varphi)e_2+(z-z')e_3}{\sqrt{\rho^2+(z-z')^2}}dz'z=0.$$

Since the point under consideration lies on the x_1 axis, then for it $\varphi=0,\rho=R,$. Then after integration over dz' we obtain that

$$F^*(x_0,R,0,0)=(c_1+c_2)\frac{q(e_1R+\frac{l}{2}e_3-e_1R+\frac{l}{2}e_3)}{\left(R^2+\left(\frac{l}{2}\right)^2\right)^{\frac{3}{2}}}.$$

For $c_1+c_2=1$, we get that

$$F^*(x_0,R,0,0)=\frac{qle_3}{\left(R^2+\left(\frac{l}{2}\right)^2\right)^{\frac{3}{2}}}.$$

For a point dipole (the distance l between charges is significantly less than the distance R to the point where the field created by these charges is considered), we obtain

$$F\ (x_0,R,0,0)=\frac{qle_3}{R^3},$$

that is, the value of the electric field along the x_3 axis is $\frac{ql}{R^3}$.

6. Conclusion

In this note, we looked at a new approach to writing solutions to Maxwell's equations and showed how an unconventional notation leads directly to finding Green's functions for these equations. It should be noted that this is not just a decorative difference from previously used methods, which, generally speaking, required the calculation of potentials, scalar and vector. Here, the calculation of the fields E,H is direct and simultaneous.

There is, however, one more extremely important difference. Previously electric and magnetic fields were transferred from the physical reality and equations were compiled for them. Here the situation is fundamentally different since the bivector $E+iH$ *is absolutely arbitrary*, just like function under Cauchy-Riemann conditions. This makes equation (1) a statement containing not physical but geometric aspect. Moreover, the fact that hypergeometric numbers from which we take “derivatives” do not form a group hints at various possible generalizations. Since this corresponds to “physical realities” unknown to us and other “Maxwell's equations” we decided to leave the discussion of this outside the scope of this work.

References

[1] Yefremov, A.P. (2004). Quaternions: algebra, geometry and physical theories. Hypercomplex Numbers in Geometry and Physics, 1, 111-127.

[2] De Leo, S., Rodrigues, W.A. (1997). Quantummechanics: from complex to complexified quaternions. Int. J. Theor. Phys., 36, 2725-2757.

[3] Girard, P.R. (2007). Quaternions, Clifford Algebras and Relativistic Physics. Basel.

[4] Girard, P.R., Pujol, R., Clarysse, P., Delachartre, P. (2023). Hyperquaternions and Physics. SciPost Phys. Proc., 14, 030.

[5] Berezin, A.V., Kurochkin, Yu.A., Tolkachev, E.A. (2003). Quaternions in relativistic physics. Moscow.

[6] Imaeda, K. (1976). Quaternionic formulation of classical electrodynamics and theory of functions of a biquaternion variable. Nuovo Cimento, 32B, 138-162.

[7] Fueter, R. (1934-1935). Comm. Math. Helv., B7, 307-330.

[8] Fueter, R. (1936-1937). Comm. Math. Helv., 8, 371-378.