

# Using Hubble Parameter Measurements to Find Constraints on Dark Energy Based on Different Cosmological Models

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## Abstract

In this paper, Hubble parameter versus redshift data, collected from multiple resources, is used to place constraints on the parameters of two current Cosmological dark energy models. The first dark energy model considered is the Standard Model of cosmology, also known as  $\Lambda$ CDM with spatial curvature, which is primarily based on Einstein's General Theory of Relativity with a spatially homogeneous time-independent cosmological constant,  $\Lambda$ . The second is the XCDM model which parameterize dark energy as a fluid whose density can vary with time. The  $H(z)$  data collected through different experimental sources was used to put constraints on the parameters of these models. The constraints obtained are then compared with the previously obtained constraints using different probes like type-1a supernovae, distance modulus, CMB anisotropy, and baryonic acoustic oscillations peak length scale. The results of analyzing the Hubble parameter vs redshift data is consistent with previous conclusions that we live in an approximately flat, accelerating Universe. However, in order to deduct tighter constraints on cosmological models' parameter, like the geometry of the Universe, more and better-quality data will be needed.

## Keywords

Hubble parameter, Dark Energy, Cosmological Models

## 1. Introduction

Since the early 1900s, it was widely accepted in the scientific community that the Universe was expanding. The idea was first introduced by Georges Lemaître in 1927, who suggested an expanding Universe by solving Einstein's field equations. Then, in 1929, using redshift observations, Hubble was able to measure this value, realizing that galaxies were receding from Earth (Li et al). This recession caused redshift in each galaxy's light spectrum, and Hubble expressed this redshift mathematically as:

$$v = H_0 * d \quad (1)$$

where  $v$  is the radial velocity of the galaxy,  $d$  is the galaxy's distance, and  $H_0$  is a constant of proportionality that was later coined Hubble's constant. Over the decades, Hubble's constant has been refined by many new and improved probes in the cosmos (Liddle). In this paper, two different constants were used on both dark energy models.  $H_0 = 68 \pm 2.8 \text{ kms}^{-1} \text{ Mpc}^{-1}$ , from statistically analyzing 553 measurements of the constant, and  $H_0 = 73.8 \pm 2.4 \text{ kms}^{-1} \text{ Mpc}^{-1}$ , based on recent measurements by the Hubble Space Telescope.

It wasn't until 1998-99, however, that two independent research groups discovered that the Universe was actually accelerating (Li et al). Before this, it was thought that the expansion of the Universe was decelerating under the gravitational attraction present between galaxies of the Universe, so this discovery came as a big surprise. Since then, multiple models have been proposed to try and explain this phenomenon. This research attempts to fit empirical data to two of these models. The first is the  $\Lambda$ CDM model and stems from Albert Einstein's General Theory of Relativity. In this, Einstein introduced a time-independent energy density (the cosmological constant,  $\Lambda$ ), which was also considered spatially homogeneous. He included this in his field equations to help support the idea that the universe was static. He later retracted this term, calling it his "biggest blunder". The fact is, it wasn't a blunder at all, Einstein was just looking at the universe as static instead of expanding (Liddle). The second is the XCDM model. This model instead considers the dark energy density to be time dependent or changing as time goes on (Farooq et al, 2013). The data used in this research is the measurements of the Hubble parameter as a function of redshift,  $H(z)$ . Specifically, 38  $H(z)$  measurements were used to constrain and analyze the  $\Lambda$ CDM and XCDM models. Traditionally, constraints on dark energy models were made with data from SNIa, CMB anisotropy, and baryonic acoustic oscillations, so using these Hubble parameter vs redshift measurements offer us a new approach when considering what dark energy model best fits our cosmos. The results gathered further support the idea that we live in an accelerating Universe.

Some fairly common questions arise when discussing dark energy and the cosmos. For instance, what is the shape or curvature of our Universe? Curvature,  $K$ , shows up mathematically in the Friedmann equation (3) and is very important when discussing the past, present, and future of our Universe. There are three distinct possibilities of  $K$ , it can either have a closed, open or flat geometry. A closed Universe has a spherical ( $K>0$ ) curvature and has a defined surface area, while still keeping homogeneity and isotropy. An open Universe is considered to be hyperbolic ( $K<0$ ) and is infinite in all directions in order to keep the homogeneity and isotropy. A Universe with a flat ( $K=0$ ) geometry has no curvature to it and is similar to the open Universe in the sense that is infinite in all directions while being homogeneous and isotropic (Peebles).

This paper uses Hubble parameter vs redshift data to find constraints on dark energy based on two different cosmological models. Previous studies have introduced the idea of using this type of data to constrain the parameters of the dark energy models. However, this research used a new updated set of 38  $H(z)$  data, compiled in Farooq et al, 2017 to reexamine the estimates of the parameters of the two commonly known models of dark energy. In section 2, we discuss the dark energy models in detail and also mention the parameters used in these models. Section 3 lays out the  $H(z)$  data used in this research and how the data was collected. Section 4 discusses in detail the statistics used to analyze this data and how it can be interpreted in each model. Finally, the last section concludes the research.

## 2. Dark Energy Models and Parameters

In this paper, two dark energy models were studied, the  $\Lambda$ CDM and XCDM models. To determine how the  $H(z)$  influences these two models, we start with the Einstein equation of general relativity:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu}, \quad (2)$$

where  $g_{\mu\nu}$  is the metric tensor,  $R_{\mu\nu}$  is the Ricci tensor,  $R$  is the Ricci scalar, and  $T_{\mu\nu}$  is the energy-momentum tensor of any matter present.  $\Lambda$  is the cosmological constant and  $G$  is Newtonian's gravitational constant. In a homogeneous space, Einstein's equation simplifies to two independent Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{K}{a^2}, \quad (3)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \quad (4)$$

where  $a(t)$  and  $K$  is the time-dependent scale factor and curvature of the Universe, respectively. When considering the

evolution of the scale factor and matter densities of the Universe, it is also convenient to use the equation of state parameter:

$$p = p(\rho) = \omega\rho, \tag{5}$$

where  $p$  is the pressure of the fluid,  $\rho$  is the energy density, and  $\omega$  is the dimensionless equation-of-state parameter. Taking the derivative of Equation (3) with respect to time and combining it with Equation (4) and (5) provides the energy conservation equation:

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p) = -3\rho\left(\frac{\dot{a}}{a}\right)(1 + \omega). \tag{6}$$

For non-relativistic matter,  $\omega = \omega_m = 0$  and  $\rho_m$  is proportional to  $a^{-3}$ , and for a cosmological constant,  $\omega = \omega_\Lambda = -1$  and  $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ , with  $\dot{\rho}_\Lambda = 0$ . Setting up and solving Equation (6) gives us a time-dependent energy density:

$$\rho(t) = \rho_0\left(\frac{a_0}{a}\right)^{3(1+\omega)}, \tag{7}$$

where  $\rho_0$  and  $a_0$  are the current energy density of a particular type of energy and scale factor values, respectively. In the  $\Lambda$ CDM model, rewriting Equation (3) using the present density parameter values:

$$\Omega_{m_0} = \frac{8\pi G\rho_0}{3H_0^2}, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad \Omega_{K_0} = -\frac{K}{(H_0 a_0)^2}, \tag{8}$$

where  $\Omega_{m_0}$  is the non-relativistic matter energy density parameter,  $\Omega_\Lambda$  is the dark energy density parameter, and  $\Omega_{K_0}$  is the curvature density parameter. These parameters, along with the redshift equation:

$$z = \frac{a_0}{a} - 1, \tag{9}$$

gives us:

$$H^2(z; H_0, p) = H_0^2[\Omega_{m_0}(1+z)^3 + \Omega_\Lambda + (1 - \Omega_{m_0} - \Omega_\Lambda)(1+z)^2], \tag{10}$$

where the model parameters are  $p = (\Omega_{m_0}, \Omega_\Lambda)$ ,  $H(z)$  is the Hubble parameter, also expressed  $\frac{\dot{a}(t)}{a(t)}$ , and  $H_0$  is the present value of the Hubble parameter, also known as the Hubble constant.

Here, we used the Friedmann equation of the  $\Lambda$ CDM model with special curvature:

$$\Omega_{K_0} = 1 - \Omega_{m_0} - \Omega_\Lambda. \tag{11}$$

With the XCDM model, we treat dark energy as time-varying and spatially homogenous, with an equation of state parameter:

$$\omega_x < -\frac{1}{3}, \tag{12}$$

and so the Friedmann equation becomes:

$$H^2(z, H_0, p) = H_0^2[\Omega_{m_0}(1+z)^3 + (1 - \Omega_{m_0})(1+z)^{3(1+\omega_x)}], \tag{13}$$

where the model parameters for this model is  $p = (\Omega_{m_0}, \omega_x)$ . We only consider the XCDM model in flat spatial hypersurfaces, as it breaks down in the evolution of energy density inhomogeneities (Farooq et al, 2013).

### 3. H(z) Data

The data collected used to place the constraints on these cosmological models were collected from Farooq et al, 2017. These 38 measurements of the Hubble parameter  $H(z)$  between redshifts were compiled using multiple references and are used in current work to place constraints on the  $\Lambda$ CDM and XCDM models.

**Table 1.** Hubble Parameter versus Redshift Data

$z$	$H(z)(kms^{-1}Mpc^{-1})$	$\sigma_H (kms^{-1}Mpc^{-1})$
0.07	69	19.6
0.09	69	12
0.12	68.6	26.2
0.17	83	8
0.179	75	4
0.199	75	5
0.2	72.9	29.6
0.27	77	14
0.28	88.8	36.6
0.352	83	14
0.38	81.5	1.9
0.3802	83	13.5
0.4	95	17
0.4004	77	10.2
0.4247	87.1	11.2
0.44	82.6	7.8
0.4497	92.8	12.9
0.4783	80.9	9
0.48	97	62
0.51	90.4	1.9
0.593	104	13
0.6	87.9	6.1
0.61	97.3	2.1
0.68	92	8
0.73	97.3	7
0.781	105	12
0.875	125	17
0.88	90	40
0.9	117	23
1.037	154	20
1.3	168	17
1.363	160	33.6
1.43	177	18
1.53	140	14
1.75	202	40
1.965	186.5	50.4
2.34	222	7
2.36	226	8

### 4. Statistical Analysis of Models

In this research, 38  $H(z)$  data points were used to constrain the dark energy model parameters. Each data point contained three pieces of information, the Hubble parameter,  $H_{obs}(z_i)$ , at the redshift of  $z_i$ , and the corresponding one standard deviation of uncertainty,  $\sigma_i$  in the Hubble parameter value at a given redshift  $z$ . We have considered that the data points are independent.

To place constraints on each models' parameters, we used a statistical analysis, computing  $\chi^2_H$  function:

$$\chi^2_H(H_0, p) = \sum_{i=1}^{38} \frac{[H_{th}(z_i; H_0, p) - H_{obs}(z_i)]^2}{\sigma_i^2}, \tag{14}$$

where  $H_{th}(z_i; H_0, p)$  is the theoretical Hubble parameter value. Keeping in mind that  $H_{th}(z_i; H_0, p) = H_0 E(z; p)$ , equation (14) becomes:

$$\chi_H^2(H_0, p) = H_0^2 \sum_{i=1}^{38} \frac{E^2(z_i, p)}{\sigma_i^2} - 2H_0 \sum_{i=1}^{38} \frac{H_{\text{obs}}(z_i)E(z_i, p)}{\sigma_i^2} + \sum_{i=1}^{38} \frac{H_{\text{obs}}^2(z_i)}{\sigma_i^2}. \tag{15}$$

We know that  $\chi_H^2$  depends on the parameters  $p$  and  $H_0$ , but  $H_0$  is tricky since the value is uncertain. To fix this, we assume that the distribution of  $H_0$  is a Gaussian with one standard deviation width  $\sigma_{H_0}$  and mean  $\bar{H}_0$ . Assuming this allows us to create the posterior likelihood function:

$$\mathcal{L}_H(p) = \frac{1}{\sqrt{2\pi\sigma_{H_0}^2}} \int_0^\infty e^{-\chi_H^2(H_0, p)/2} e^{-(H_0 - \bar{H}_0)^2 / (2\sigma_{H_0}^2)} dH_0. \tag{16}$$

Simplifying the integral and defining:

$$\alpha = \frac{1}{\sigma_{H_0}^2} + \sum_{i=1}^{38} \frac{E^2(z_i, p)}{\sigma_i^2}, \quad \beta = \frac{\bar{H}_0}{\sigma_{H_0}^2} + \sum_{i=1}^{38} \frac{H_{\text{obs}}(z_i)E(z_i, p)}{\sigma_i^2},$$

$$\gamma = \frac{\bar{H}_0^2}{\sigma_{H_0}^2} + \sum_{i=1}^{38} \frac{H_{\text{obs}}^2(z_i)}{\sigma_i^2},$$

gives us the integral:

$$\mathcal{L}_H(p) = \frac{1}{\sqrt{2\pi\sigma_{H_0}^2}} e^{-\frac{1}{2}\gamma} \int_0^\infty e^{-\frac{1}{2}[\alpha H_0^2 - 2\beta H_0]} dH_0 \tag{17}$$

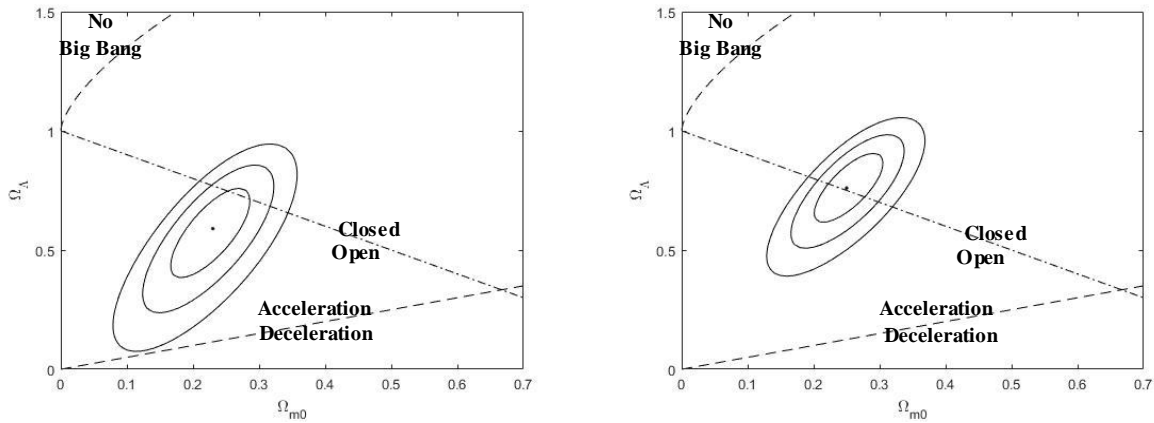
Finally, completing the square in the exponent inside the integral and simplifying, we get:

$$\mathcal{L}_H(p) = \frac{1}{2\sqrt{\alpha\sigma_{H_0}^2}} e^{-\frac{1}{2}[\gamma - \frac{\beta^2}{\alpha}]} \left[ 1 + \text{erf}\left(\frac{\beta}{2\alpha}\right) \right] \tag{18}$$

where  $\text{erf}(x) = \left(\frac{2}{\sqrt{\pi}}\right) \int_0^x e^{-t^2} dt$ . Since the likelihood function and the  $\chi_H^2$  function are related by the equation  $\chi_H^2(p) = -2\ln\mathcal{L}_H(p)$ , maximizing the likelihood function means finding the minimum  $\chi_H^2$  value, with respect to the parameters  $p$  to find the best-fit parameter values  $p_0$ . Just like in any statistics, we also need to define standard deviations, or in this case,  $1\sigma$ ,  $2\sigma$ , and  $3\sigma$  confidence contours. So, we set these confidence intervals as two-dimensional parameter sets bounded by  $\chi_H^2(p) = \chi_H^2(p_0) + 2.3$ ,  $\chi_H^2(p) = \chi_H^2(p_0) + 6.17$ , and  $\chi_H^2(p) = \chi_H^2(p_0) + 11.8$ .

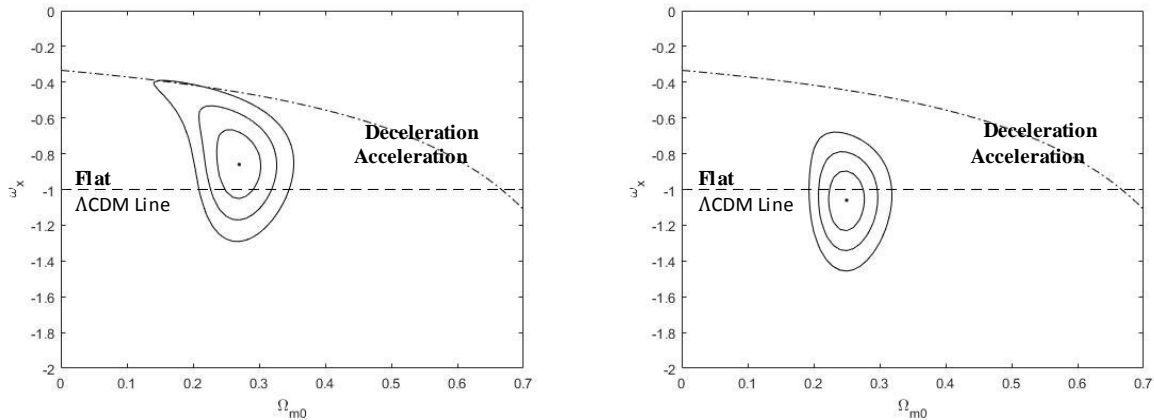
As mentioned earlier, two different Hubble constant measurements were used for both models, with the higher value more locally obtained by the Hubble Space Telescope and the lower value obtained by median statistics analysis in 2001. Furthermore, the uncertainties in the Hubble constant still greatly affect the parameter estimations (Farooq et al, 2013). Still, the Hubble parameter vs redshift data was comparable to other methods (SNIa, CMB anisotropy, and baryonic acoustic oscillation measurements) when placing constraints on these dark energy models and can be a very helpful in future experiments involving joint constraints (Huterer et al).

The plots of the  $\Lambda$ CDM model in figure 1, when compared to the results obtained by Farooq et al, 2013, shows tighter constraints on the parameters of models by the shrinking of the sigma intervals. We can now say that within 3 sigma values, using the new  $H(z)$  data set, that we live in an accelerating Universe. Figure 1 also gives a better insight into whether we live in an open or closed geometry. With  $H_0 = 68 \pm 2.8 \text{ kms}^{-1} \text{ Mpc}^{-1}$ , we can deduct that within 1 sigma value, that we live in an Universe that has an open geometry.



**Results for  $\Lambda$ CDM Model**

**Figure 1:** Contour plot for the  $\Lambda$ CDM model from the H(z) data. Three ellipses represent  $1\sigma$ ,  $2\sigma$ , and  $3\sigma$  confidence intervals surrounding the minimum  $\chi^2_H$  value. The left contour plot is with the Hubble parameter,  $H_0 = 68 \pm 2.8 \text{ km s}^{-1} \text{ Mpc}^{-1}$  and the right plot is with the Hubble parameter,  $H_0 = 73.8 \pm 2.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ .



**Results for XCDM Model**

**Figure 2:** Contour plot for the XCDM model from the H(z) data. Three ellipses represent  $1\sigma$ ,  $2\sigma$ , and  $3\sigma$  confidence intervals surrounding the minimum  $\chi^2_H$  value. The left contour plot is with the Hubble parameter,  $H_0 = 68 \pm 2.8 \text{ km s}^{-1} \text{ Mpc}^{-1}$  and the right plot is with the Hubble parameter,  $H_0 = 73.8 \pm 2.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ .

The plots of the XCDM model in figure 2 also show a tightening of the sigma intervals when compared with the results by Farooq et al, 2013. These plots both point to an accelerating Universe to within 3 sigma values. Also, both minimum  $\chi^2_H$  values are very close to the flat  $\Lambda$ CDM line, which further strengthens the idea that we are in an accelerating, flat Universe with an open geometry. In order to deduce more information from the plots, however, more H(z) data with better precision will be needed in the future.

## 5. Conclusion

In summary, the results of statistically analyzing the  $H(z)$  data is consistent with an accelerating Universe. The  $\Lambda$ CDM model with a Hubble constant,  $H_0$ , of  $68 \pm 2.8 \text{ kms}^{-1}\text{Mpc}^{-1}$  shows that, within  $1\sigma$  value, or with about 68% accuracy, we live in an open, accelerating Universe. In the XCDM model, both Hubble constant values point towards a flat, accelerating cosmos. The cosmological Principle states that the Universe, as a whole, is homogeneous and isotropic. So, in order for this to be true, while still maintains a flat geometry, space must be infinite in all directions. In order to deduct more results, better quality data will be needed to reduce the size of the sigma intervals and narrow down the uncertainty in the model parameters.

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