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A New View on Solving the Limits of Functions and Estimating a Point of Functions

Hongfei Fang

Department of Mathematics and Applied Mathematics, Shandong University, Jinan 250100, Shandong, China.

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***Corresponding author:** Hongfei Fang, Department of Mathematics and Applied Mathematics, Shandong University, Jinan 250100, Shandong, China.

Abstract

Usually in the study of limit problems, we will encounter more complex problems. This paper discusses how to use the concept of equivalent infinitesimal to better limit operation, so as to achieve the purpose of simplification. At the same time, in the course of the study, we re-explored the proof process of Taylor's formula, and found that some functions have a similar expansion form to Taylor's formula, that is, 'fractional expansion'. At the same time, it is found that after the linear combination of Taylor expansion and fractional expansion, the obtained expansion is more accurate, which helps us to better understand the approximation of function expansion, so as to more accurately estimate the value of some functions and reduce the relative calculation amount. The universality of the method and some phenomena in it are also discussed.

Keywords

Equivalent infinitesimal; L 'Hopital's rule; Taylor expansion; linear combination

1. Introduction

We can replace the infinitesimal term in a function with other equivalent infinitesimal terms to simplify and solve the limit. This method is called equivalent infinitesimal replacement [1].

In solving the limit problem, the concept of infinitesimal substitution is widely used. For example, Huang Yumei and Li Na mentioned the application of infinitesimal substitution in solving limit problems in their research and emphasized the importance of this method in teaching [2]. Through the understanding and application of infinitesimal substitution, we can well deal with some common limit problems, including the derivation, the area and the calculation of the function value of some specific points.

In addition to the application in limit operation, the concept of infinitesimal substitution has been widely studied in other fields. For example, C Fang et al. studied the use of infinitesimal substitution to solve the indefinite integral problem [3], and Z Peng et al. studied the application of infinitesimal substitution to solve the positive solution problem in Kirchhoff-Schrödinger-Poisson system [4].

At the same time, Taylor expansion is a method of approximating the function with infinite series, which has been widely used in the field of mathematics. The history of Taylor expansion can be traced back to the 17th century, which was discovered and developed by Scottish mathematician James Gregg.

By expanding the function into an infinite sum at a certain point, the function can be approximated by the sum of a series of infinite series, so as to realize the approximate calculation of the function.

Firstly, Taylor expansion is of great significance in numerical calculation. According to the research results of Reference [5], Taylor expansion can be used to calculate the approximation of higher order functions, so as to improve the accuracy and efficiency of calculation.

Secondly, Taylor expansion is widely used in function approximation. According to the research results of Reference [6], Taylor expansion can be used to approximate complex functions into series forms, so as to better understand and analyze the properties of functions [7, 8].

In practical problems, we often need to fit and predict the existing data, and Taylor expansion provides an effective approximation method. By selecting the appropriate expansion point and series truncation method, a series expression similar to or close to the original function can be obtained, so as to realize the approximation and prediction of the function.

Therefore, it is necessary for us to conduct in-depth research on equivalent infinitesimal substitution and function approximation.

2. Theorem

Definition 1 [9] is set in the same change process of the independent variable x, f（x）→ 0，g（x）→ 0， and g（x） ≠ 0 . If $\lim \frac{f(x)}{g(x)} = 1$, Let f (x) be the equivalent infinitesimal of g (x), denoted by f (x)~g (x).

Theorem 1 [10]: Let f (x), g (x), h (x) be defined on U ° ($\boldsymbol{x_0}$), and there is f(x)~g(x) (x→$\boldsymbol{x_0}$).

(1) if $\lim_{x \to x_0} f(x)h(x) = A$, $then \lim_{x \to x_0} g(x)h(x) = A$;

(2) if $\lim_{x \to x_0} \frac{h(x)}{f(x)} = B, then = B.$

Example 1: Find the limit $\lim_{x \to 0^+} (x^x - 1)lnx.$

Solution: Because $\lim_{x \to 0^+} xlnx = 0$, so the original formula = $\lim_{x \to 0^+} (e^{xlnx} - 1)lnx = \lim_{x \to 0^+} x(lnx)^2 = \lim_{x \to 0^+} \frac{(lnx)^2}{x^{-1}} = \lim_{x \to 0^+} \frac{2lnx}{-x^{-1}} = 0.$

From this example, it can be seen that the equivalent infinitesimal has important applications, which can simplify the operation of the limit to a certain extent, but in Theorem 1, only the substitution of multiplication and division is involved, and in more cases, the operation of addition and subtraction is also involved.

Example 2 [11]: It is given in the book " Higher Mathematics Exercise Course (Engineering)" edited by Gong Manqi published by Science Press. Such a wrong example:

$$\lim_{x \to 0} \frac{tanx - sinx}{x^2 sinx} = \lim_{x \to 0} \frac{x - x}{x^3}$$

Then the author comments: here only for the addition and subtraction factor equivalent infinitesimal substitution may be wrong but why what is wrong.

Example 3 [12]: Economic Mathematics: Calculus Division, edited by Liu Guiru and Sun Yonghua, published by Nankai University Press in 2002. It is also said in a book that 'the substitution of infinitesimal can only be used in the multiplication (division) method and cannot be used in the addition (subtraction) method'.

Theorem 2: $If\ \alpha \sim \alpha', \beta \sim \beta', and \lim \frac{\alpha}{\beta} = A \neq -1,\ then\ (\alpha + \beta) \sim (\alpha' + \beta');$

$$If\ \alpha \sim \alpha', \beta \sim \beta', and\ lim\ \frac{\alpha}{\beta} = A \neq 1, then\ \alpha - \beta \sim \alpha' - \beta'.$$

Proof: $lim \frac{\alpha + \beta}{\alpha' + \beta'} = lim \frac{\frac{\alpha}{\beta} + 1}{\frac{\alpha'}{\beta} + \frac{\beta'}{\beta}} = \frac{A + 1}{A + 1} = 1$ so α + β ~ α ' + β ', the same can be proved subtraction.

Theorem 3 [13]: If the function f (z) has n-order derivative at x, then there is a neighborhood of z_0, for any z in the neighborhood$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n$.

3. Results

3.1 Construct Equivalent Infinitesimal

3.1.1 The method of constructing equivalent infinitesimal

We often encounter problems related to the limit, such as:

$$\lim_{x \to 0} \frac{(1 + x)^x - cos\, 2\, x}{(tan\, 2x - sin\, x)(sin\, 2\, x - x)}$$

We only use infinitesimal replacement is difficult to solve, or use Taylor 's Formula to solve the problem but often requires high precision, or use L 'Hospital rule but sometimes more complex, or use the method of equivalent infinitesimal replacement. However, the amount of equivalent infinitesimal we know is relatively small, and the situation encountered is more complicated. That is to say, in order to deal with this situation, we start from the concept of infinitesimal order and construct a new equivalent infinitesimal.

From the above example questions

$$(\sin 2x - x)' = 2cos2x - 1=1 \text{ (when x approaches 0)}$$

We continue to derive the function when the function is 0 until the result of the derivation is not 0, and continuously integrate the final result (the number of integrations is equal to the number of derivations) without adding constant values. On the upper style

$$\int 1\, dx = x$$

Then we get the first equivalent infinitesimal (sin2x-x) ~ x. The principle we do is the L 'Obida rule. The equivalent infinitesimal obtained by simultaneous derivation of the upper and lower sides is calculated according to the above method.

$$(tan2x - sinx) \sim x$$

$$[(1+x)^x - cos2x] \sim 3x^2$$

$$[(1+x)^x - 1] \sim x^2$$

$$(1 - cos2x) \sim 2x^2$$

So we can solve the problem of limit.

And from the equivalent infinitesimal, we can split the required formula. As long as the equivalent infinitesimal replacement of the split formula is satisfied, the numerator or denominator will not be 0, thus simplifying the calculation. Its essence is that the remaining expansions of addition and subtraction at different orders can be omitted, while the addition and subtraction at the same order cannot, because only the first expansion is the same, does not mean that the rest are the same. (It's like looking up a dictionary, one at a time, one at a time)

3.1.2 The principle of constructing an equivalent infinitesimal replacement

Firstly, we generalize the universality, but we only need to consider the limit case where the molecular denominator is 0 (the case where the molecular denominator is infinite can be transformed into the case of 0 by constructing the reciprocal). If it is not 0 at the same time, the result can be obtained by using the addition and subtraction of the limit. The following is an equivalent infinitesimal substitution based on the order of infinitesimal: $\lim_{x \to x_0} f(x) = 0$, Truze structure $\lim_{x \to x_0} \frac{f(x)}{a(x-x_0)^n} = 1$.

The premise is that f (x) n (n is a finite number) is derivable, and the n-order derivative is not 0. We use the L 'Hopital 's rule if n times after derivation, $f^{(n)}(x) = C$,

$$\lim_{x \to x_0} \frac{f(x)}{a(x-x_0)^n} = 1, \text{ we can get } a = \frac{c}{n!}.$$

3.2 Taylor Expansion and Fraction Expansion

Note: There is no discussion of the remaining items in this discussion (the remaining items are no longer written in all expansions).

Since we have solved the limit of 0, it is easy to think of Lobida 's law for n times. Considering that the function f (x) can be derived n times (i.e., derivable of order n), a new limit is constructed as

$$\lim_{x \to x_0} \frac{f(x) - \sum_0^{n-1} f^i(x_0) \frac{(x-x_0)^i}{i!}}{\frac{f^n(x_0)(x-x_0)^n}{n!}} = 1,$$

the denominator is multiplied to the right side to obtain the Taylor expansion:

$$f(z) = \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

However, we deal with some special functions in another way, as follows:

We do not assign values to the final result, but transform it into the idea of solving the equation. Then we get a class of functions that can be reproduced by the three derivative primitive functions such as e^x, sinx and cosx. So we discuss a new infinitesimal substitution method and a new infinite expansion similar to Taylor's Formula, which is the fractional expansion.

3.2.1 e^x

We first consider the simple case as follows: we first discuss the initial example of exponential function $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0} e^x$ (obtained by L 'Hospital 's rule), then solve the equation about e^x, and get the expansion at 0 is $e^x = \frac{1}{1-x}$. The image is shown in Figure 1, and the error is shown in Figure 2.

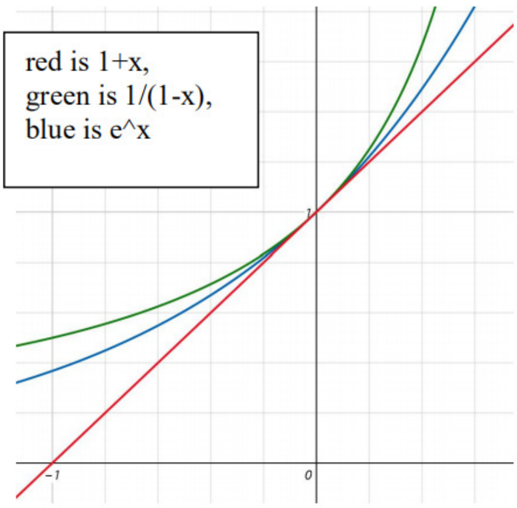

Figure 1. Two first-order expansion forms and primitive functions.

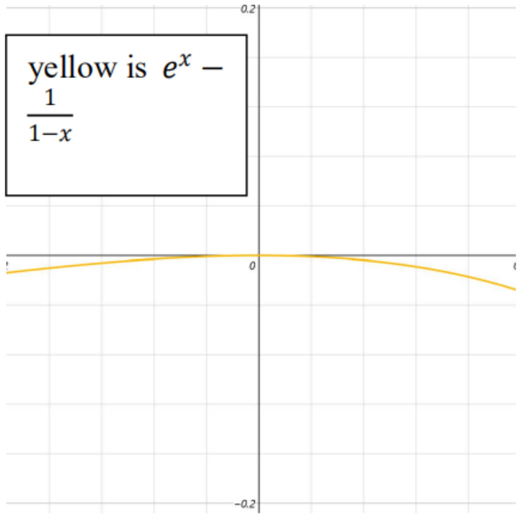

Figure 2. The difference between the first-order fractional expansion and the original function.

We consider the higher order case: First, we derive the following limit for the following problem

$$\lim_{x\to 0}\frac{e^x-(\sum_0^n \frac{x^n}{n!})}{\frac{x^{n+1}}{(n+1)!}}$$

Using the Lobida 's law to find the limit of the above formula, we finally get

$$\lim_{x\to 0}\frac{e^x-(\sum_0^n \frac{x^n}{n!})}{\frac{x^{n+1}}{(n+1)!}} = \lim_{x\to 0} e^x$$

Then we solve the equation and get

$$\lim_{x\to 0} e^x = \lim_{x\to 0}\frac{(n+1)!\,(\sum_0^n \frac{x^n}{n!})}{(n+1)!-x^{n+1}}$$

This formula has a similar form to the Taylor expansion from − 1 to − (n + 1). (Not only can be used at 0 and in other places can still only use the substitution method to solve).

3.2.2 A linear combination of two expansions

We have found two expansion methods. Next, we discuss whether the combination of the two expansion methods will improve the accuracy. First, we think of a further approximation, that is, a linear combination of the two expansions (the reason for not using other forms of combination is that the linear combination is simple and clear and can ensure that the function at 0 is still 1). The method is as follows:

$$\lim_{x\to 0} e^x = a \lim_{x\to 0} \frac{(n+1)!\,(\sum_0^n \frac{x^n}{n!})}{(n+1)! - x^{n+1}} + (1-a)\,(\sum_0^{n+1} \frac{x^{n+1}}{(n+1)!})\ \ ,a \in R$$

Selecting a special value will make the degree of approximation in a certain range very high. Here are the data and images for one of these cases (Figures 3 and 4):

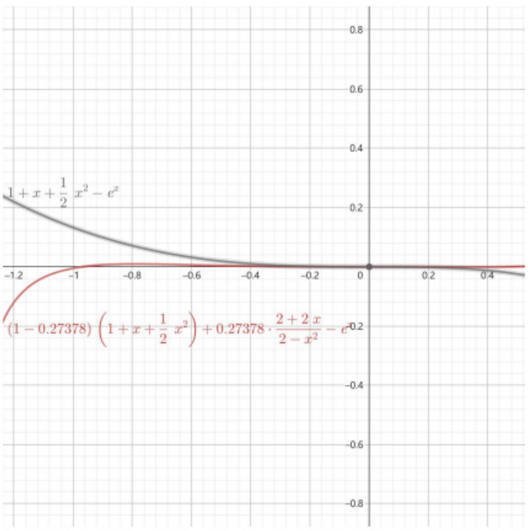

Figure 3. The degree of approximation of Taylor expansion and the degree of change after linear combination with fractional expansion.

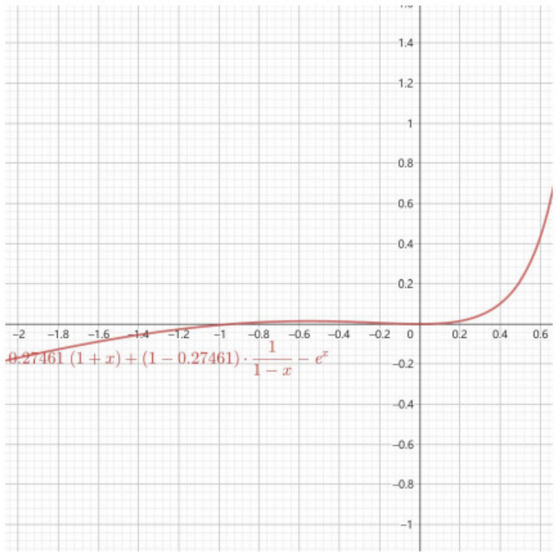

Figure 4. The degree of approximation of a linearcombination.

We know that the estimated value of e is 2.718281828459045. When we are constantly changing the value of a, we find that the closer a is to 0.1e, the result of the expansion approximation after linear combination can be accurately expressed in a large range.

Note: At present, we are only guessing, and we have not been able to give an exact proof (at least in the case of a small number of items).

3.2.3 sinx (cosx can also be compared to the results)

First of all, there are two general situations to discuss:

$$\lim_{x\to 0} \frac{sinx}{x} = \lim_{x\to 0} cosx = \lim_{x\to 0} \sqrt{1-\sin^2(x)} \qquad (1)$$

$$\lim_{x\to 0} \frac{sinx - x}{x^2} = \lim_{x\to 0} \frac{cosx - 1}{2x} = \lim_{x\to 0} -\frac{1}{2} sinx \qquad (2)$$

For the approximation of the fractional results obtained from (1), the difficulty of calculation is generally relatively large, so we discuss the following situations. (2) The following formula is obtained:

$$\lim_{x\to 0} \frac{sinx - \sum_1^{2n-1} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}}{\frac{(-1)^n x^{2n}}{(2n)!}} = \lim_{x\to 0} sinx$$

The following results can be obtained by solving the above equations (in the case of x approaching 0):

$$sinx = \frac{(2n)! - (-1)^n x^{2n}}{(2n)!} \left(\sum_1^{2n-1} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \right)$$

Although the two functions sinx and e^x we study can perform similar operations, the difference between sinx and e^x

is that sinx is periodic and has periodicity, and the effect of fractional expansion is better than Taylor expansion. Because the fraction is bounded, it can show some special properties on the image (Figures 5 and 6):

It is not difficult to find another similar phenomenon here, that is, when a is close to 0.1π, the approximation result is the closest. These two results are our intuitive feelings, rather than the conclusion that is really being proved, so they need to be verified and considered.

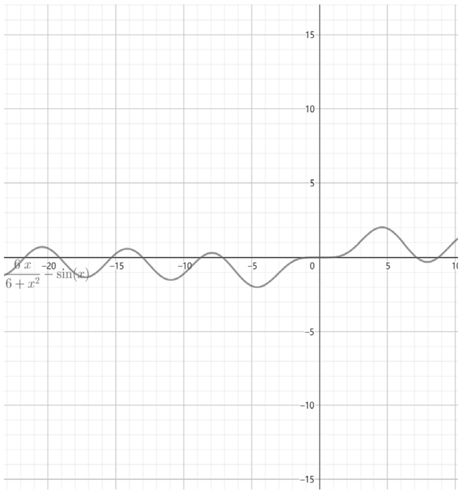

Figure 5. sinx (The degree of approximation of a fractional expansion).

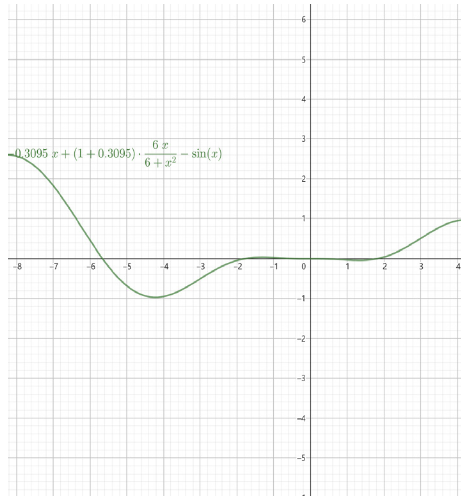

Figure 6. sinx (The degree of approximation of a linear combination).

3.2.4 Ordinary circumstances

For the case that the original function is the same as the n-order derivative function or the difference is constant times, the corresponding fractional expansion can be obtained by the above method. This article here is just a lot of overstatement. Although this restriction condition most of the functions cannot meet, we can still use the derivative function to represent (or expand) the original function in the limit case.

3.2.5 Remainder term

We know that the Taylor expansion is an approximate method, so the remainder will be generated. Then we will discuss the relevant fractional expansions here, and there will also be remainders. We will discuss these remainders now. The analysis of this remainder can first be compared with the Peano remainder, and similar results can also be similar to the specific remainder of Taylor expansion.

4. Discussion

In the study of infinitesimal substitution, we deeply analyze the relevant literature and sort out some relevant findings. Infinitesimal substitution is an important method in limit operation. These research results have certain guiding significance for us to deeply understand the principle and application of infinitesimal substitution. We discuss how to obtain the equivalent infinitesimal and calculate it. When the equivalent infinitesimal replacement is carried out in the limit operation, there is a case of error analysis. How to better improve the system of equivalent infinitesimal replacement can be widely used is a problem.

Moreover, there are limitations on the scope of application of the fractional expansion in our discussion. First, it is not extended to universal functions, and more items are needed to ensure accuracy. However, as the number of supplementary terms increases, the amount of calculation will also increase significantly, resulting in a decrease in computational efficiency. Therefore, in some cases, we need to find more efficient and accurate expansion methods. In view of the above problems, we look forward to the future development direction. On the one hand, other mathematical tools can be combined to improve the applicability of Taylor expansion. On the other hand, more efficient and accurate approximation methods can be explored to improve the approximation effect of non-smooth functions. For example, wavelet analysis and Taylor expansion can be combined to improve the accuracy of expansion by using the multi-scale characteristics of wavelet analysis, so as to better approximate non-smooth functions.

5. Conclusions

The core method of this paper is the process of using L 'Hopital's law to find the limit. We do not extend the fractional expansion to the case of general functions, which is a question worth considering. Therefore, we should study the expansion of functions more deeply, so as to better understand the relevant conclusions and better estimate the accuracy of functions. The numerical proportion of the determined linear combination is not obtained so that the approximation degree is greater. We still need to continue to study whether the coefficients of the linear combination are related to e or π, and why they are related.

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