Journal of Applied Mathematics and Computation (JAMC), 2018, 2(10), 483-495
https://www.hillpublisher.com/journals/JAMC/
ISSN Online:2576-0645
ISSN Print:2576-0653

Proofs of Logic Consistency of A Formal Axiomatic Epistemology Theory Ξ, and Demonstrations of Improvability of The Formulae (Kq → q) and (□q → q) in It

V.O. Lobovikov

Laboratory for Applied System Research, Ural Federal University, Yekaterinburg

How to cite this paper: Lobovikov, V.O. (2018) Proofs of Logic Consistency of A Formal Axiomatic Epistemology Theory Ξ, and Demonstrations of Improvability of The Formulae (Kq → q) and (□q → q) in It. *Journal of Applied Mathematics and Computation*, **2(10)**, 483-495.
DOI: 10.26855/jamc.2018.10.004
***Corresponding author**: V.O. Lobovikov, Laboratory for Applied System Research, Ural Federal University, Yekaterinburg.
Email: vlobovikov@mail.ru

Abstract

Synthesizing some normal and non-normal modal logic systems by the formal axiomatic epistemology theory Ξ is under investigation. New proofs of logic non-contradictoriness of the theory Ξ are constructed. For the first time several proofs of improvability of (Kq → q) and (□q → q) in Ξ are submitted. (Here "Kq" stands for "person knows that q"; "□q" stands for "it is necessary that q", and "q" stands for a proposition.) The logically formalized axiomatic epistemology system Ξ is considered as a response to the critique of the classical epistemic modal logic by the empiricist-minded philosophers and representatives of the evolutionary epistemology. Some aspects of the system under discussion are graphically represented by the square and hexagon of conceptual opposition.

Keywords

epistemic-modality; knowledge; a-priori; a-posteriori; axiomatic-epistemology-system -Ξ; formal-theory; interpretation; model; consistency; improvability

1. Introduction.

In this paper the mathematical method is applied to philosophy of knowledge. In general philosophical epistemology and in philosophical logic (modal one) the word "knowledge" is exploited by different writers in different meanings. For instance, in relation to usage of the word "knowledge" there is a significant discrepancy between the epistemic modal logic [Hintikka 1962; 1974; 1989] and the *evolutionary* epistemology [Campbell 1970; 1974; 1987; 1990; Feyerabend 1975; Kuhn 2012; Popper 1992; 2002; 1979; Ruse 1985; Tulmin 1967; Wuketits 1990]. Some logicians recognize this discrepancy and affirm that there is a logic contradiction between the two [Kostyuk 1978]. Especially, the contradiction between the theorem (Kq → q) of epistemic modal logic and the idea of knowledge *evolution* is meant. The critique of the classical epistemic modal logic by the *empiricist*-minded philosophers and representatives of the *evolutionary* epistemology inspired me to construct such a logically formalized axiomatic theory of knowledge, in which (Kq → q) is not a theorem. Finally, my constructing has resulted in the below-defined theory Ξ [Lobovikov 2017b; 2018a; 2018b], which deals not only with Kq, but also with Aq (person a-priori knows that q) and Eq (person a-posteriori knows that q). Meanings of Aq and Eq are precisely defined by the axiomatic system.

During the oral discussion of my presentation of the logically formalized axiomatic theory Ξ at the 6th World Congress on Universal logic in Vichy, France (June 16 to 26, 2018), Alexei Muravitsky asked me an important question about existence of proof of consistency of Ξ. Also, during that discussion Marek Nasieniewsky expressed a guess that, probably, Ξ is inconsistent. Thus, the presented axiomatization of philosophical epistemology had been problematized in relation to its logic consistency. Finishing the discussion, I promised to think over the problem. In this relation the present paper publication is justified as a definite and direct response to questioning the indicated aspect of Ξ. (I am grateful to Alexei and Marek for their questions and remarks which have stimulated my developing the theory.) Moreover, in [Lobovikov 2017b; 2018a] it has been affirmed several times that the formulae $(Kq \to q)$ and $(\Box q \to q)$ are not provable in Ξ. However, up to the present time rigorous proofs of/for these nontrivial affirmations have not been submitted. The affirmations have been made on the basis of some philosophical-logic intuition. Therefore, the present paper publication is justified as a first clarification, explication and formal representation of that nontrivial philosophical-logic intuition by means of a set of rigorous proofs of improvability of the formulae $(Kq \to q)$ and $(\Box q \to q)$ in Ξ. The formal axiomatic theory under investigation is defined as follows.

2. Definition of Ξ

The paragraph 2 of this paper is aimed at making the reader acquainted with the rigorous formulation of Ξ. According to the definition, the logically formalized axiomatic epistemology system Ξ contains all symbols (of the alphabet), expressions, formulae, axioms, and inference-rules of the classical propositional logic. Symbols q, p, d, … (called propositional letters) are *elementary* formulae of Ξ. Symbols α, β, ω, π, … (belonging to meta-language) stand for any formulae of Ξ. In general, the notion "formulae of Ξ" is defined as follows.

1) All propositional letters q, p, d, … are formulae of Ξ.
2) If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are formulae of Ξ, then all such expressions of the object-language of Ξ, which possess logic forms $\neg\alpha$, $(\alpha \to \beta)$, $(\alpha \leftrightarrow \beta)$, $(\alpha \,\&\, \beta)$, $(\alpha \vee \beta)$, are formulae of Ξ as well.
3) If $\boldsymbol{\alpha}$ is a formula of Ξ, then $\boldsymbol{\Psi\alpha}$ is a formula of Ξ as well.
4) Successions of symbols (belonging to the alphabet of the object-language of Ξ) are formulae of Ξ, only if this is so owing to the above-given items 1) – 3) of the present definition.

The symbol Ψ belonging to meta-language stands for any element of the set of modalities {$\Box$, K, A, E, S, F, T, P, Z, G, O, B, U, Y}. Symbol $\Box$ stands for the alethic modality "necessary". Symbols K, A, E, S, T, P, Z, respectively, stand for modalities "agent *knows* that…", "agent *a-priori knows* that…", "agent *a-posteriori knows* that…", "under some conditions in some space-and-time a person (immediately or by means of some tools) *sensually perceives* (has *sensual verification*) that…", "agent *believes* that…", "it is *true* that…", "it is *provable* that…", "there is *an algorithm* (a machine could be constructed) *for deciding* that…".

Symbols G, O, B, U, Y, respectively, stand for modalities "it is *(morally) good* that…", "it is *obligatory* that …", "it is *beautiful* that …", "it is *useful* that …", "it is *pleasant* that …". Meanings of the mentioned symbols are defined by the following schemes of own-axioms of epistemology system Ξ which axioms are added to the axioms of classical propositional logic. Schemes of axioms and inference rules of the classical propositional logic are applicable to all formulae of Ξ (including the ones constructed by the item 3 of the definition).

Axiom scheme AX-1: $A\alpha \to (\Box\beta \to \beta)$.

Axiom scheme AX-2: $A\alpha \to (\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))$.

Axiom scheme AX-3: $A\alpha \leftrightarrow (K\alpha \,\&\, (\Box\alpha \,\&\, \Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)))$.

Axiom scheme AX-4: $E\alpha \leftrightarrow (K\alpha \,\&\, (\neg\Box\alpha \vee \neg\Box\neg S\alpha \vee \neg\Box(\beta \leftrightarrow \Omega\beta)))$.

In AX-3 and AX-4, the symbol Ω (belonging to the meta-language) stands for any element of the set $\Re$ = {$\Box$, K, F, T, P, Z, G, O, B, U, Y}. Let elements of $\Re$ are called "*perfection*-modalities" or simply "perfections".

3. Graphic modeling logical relations among different epistemic modalities by means of the square and hexagon of opposition

The above-given axiomatic definition of meanings of Aα and Eα may be visualized owing to the below-located fig.1 which shows that Ξ unites some well-known normal and not-normal modal logics in one conceptual scheme. The normal modal logics are somehow connected with Aα; the not-normal modal logics are somehow linked with Eα. Of normal modal logics see [Kripke 1963; Bull and Segerberg 1984]; of not-normal ones – [Kripke 1965; Priest 1992; 2008].

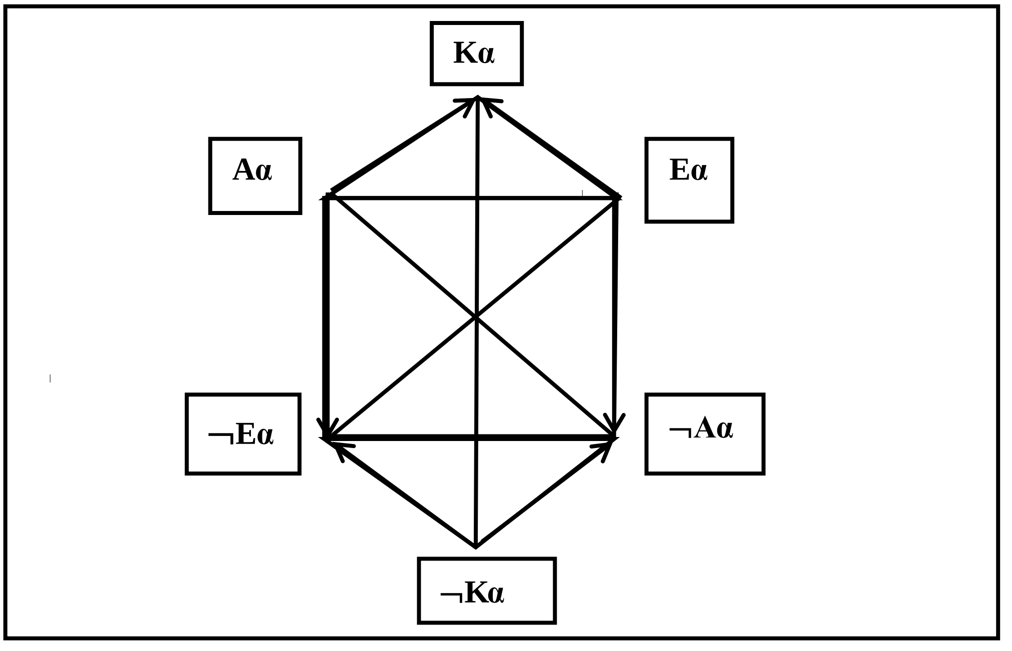

Fig.1. Synthesizing a-priori-ism and empiricism in one conceptual scheme of general philosophical epistemology

In this hexagon: the *contrariety* relation between **Aα** and **Eα** is modeled by the upper horizontal line; the *sub-contrariety* relation between **¬Aα** and **¬Eα** is modeled by the bottom horizontal line; the *contradictoriness* relations between elements of the couples: < **Aα**, **¬Aα** >; < **Eα**, **¬Eα** >; < **Kα**, **¬Kα** > are modeled by the lines crossing the square. The relations of *logic consequence* (entailment) are modeled by arrows. Submitting this interpretation of hexagon [Lobovikov 2016b] was inspired by works of [Béziau 2012a; 2012b] and [Blanché 1966] universalizing methodological significance of the square and hexagon of opposition for modeling logical structures of conceptual systems.

4. Proofs of the consistency of formal theory Ξ, and demonstrations of improvability of (Kq → q) and (□q → q) in it

Above the axioms of Ξ were defined by the axiom-schemes. Now first of all it is relevant to depart from the meta-language to the object-language, i.e. to move hereafter from the above axiom-schemes to the axioms, respectively. In this paper such interpretations of/for Ξ are considered in which all the axioms of Ξ are true. Now everything is prepared for defining and discussing interpretation-functions to be used for constructing demonstrations of the logic consistency of Ξ, and demonstrations of improvability of (Kq → q) and (□q → q) in Ξ.

4.1. Interpretation-function ©

Let the meta-language symbols α and β be substituted by the object-language symbol q. Also let the meta-language symbol Ω be substituted by the object-language symbol O (It is *obligatory* that). In this particular case the system of ax-

iom-schemes of Ξ is represented by the following axioms, respectively.
Axiom AX-1*: $Aq \to (\Box q \to q)$.
Axiom AX-2*: $Aq \to (\Box(q \to q) \to (\Box q \to \Box q))$.
Axiom AX-3*: $Aq \leftrightarrow (Kq \,\&\, (\Box q \,\&\, \Box\neg Sq \,\&\, \Box(q \leftrightarrow Oq)))$.
Axiom AX-4*: $Eq \leftrightarrow (Kq \,\&\, (\neg\Box q \vee \neg\Box\neg Sq \vee \neg\Box(q \leftrightarrow Oq)))$.
The interpretation-function © is defined as follows. (It is implied here that ω and π stand for any formulae belonging to Ξ.)

1) $\copyright\neg\omega = \neg\copyright\omega$ for any formulae ω.
2) $\copyright(\omega \oplus \pi) = (\copyright\omega \oplus \copyright\pi)$ for any formulae ω and π, and for any classical-logic binary-connective $\oplus$.
3) $\copyright q$ = false.
4) $\copyright Aq$ = false.
5) $\copyright Kq$ = true.
6) $\copyright Eq$ = true.
7) $\copyright\Box q$ = true.
8) $\copyright\Box\neg Sq$ = true.
9) $\copyright\Box(q \to q)$ = true.
10) $\copyright Oq$ = true.
11) $\copyright\Box(q \leftrightarrow Oq)$ = false (according to "Hume's Guillotine" [Hume 1874]). Also, substantial philosophical-content-analysis of $\Box(q \leftrightarrow Oq)$ making up solid grounds for affirming falsity of $\Box(q \leftrightarrow Oq)$ is given in [Adler 1997].

In the interpretation ©, all the axioms of Ξ are true, consequently, Ξ has a model, hence Ξ is consistent. Moreover, in the interpretation ©, the formulae $\neg(Kq \to q)$, $\neg(Eq \to q)$, and $\neg(\Box q \to q)$ are true as well. Consequently, $(Kq \to q)$, $(Eq \to q)$, and $(\Box q \to q)$ are not provable in Ξ.

4.2. Interpretation ®

Let the meta-language symbols α and β be substituted by the object-language symbol q. Also let the meta-language symbol Ω be substituted by the object-language symbol G (It is *good* that). In this particular case the system of axiom-schemes of Ξ is represented by the following axioms, respectively.
Axiom AX-1**: $Aq \to (\Box q \to q)$.
Axiom AX-2**: $Aq \to (\Box(q \to q) \to (\Box q \to \Box q))$.
Axiom AX-3**: $Aq \leftrightarrow (Kq \,\&\, (\Box q \,\&\, \Box\neg Sq \,\&\, \Box(q \leftrightarrow Gq)))$.
Axiom AX-4**: $Eq \leftrightarrow (Kq \,\&\, (\neg\Box q \vee \neg\Box\neg Sq \vee \neg\Box(q \leftrightarrow Gq)))$.
The interpretation ® is defined as follows.

3) $\circledR\neg\omega = \neg\circledR\omega$ for any formulae ω.
4) $\circledR(\omega \oplus \pi) = (\circledR\omega \oplus \circledR\pi)$ for any formulae ω and π, and for any classical binary connective $\oplus$.
5) $\circledR q$ = false.
6) $\circledR Aq$ = false.
7) $\circledR Kq$ = true.
8) $\circledR Eq$ = true.
9) $\circledR\Box q$ = true.
10) $\circledR\Box\neg Sq$ = true.
11) $\circledR\Box(q \to q)$ = true.
12) $\circledR Gq$ = true.
13) $\circledR\Box(q \leftrightarrow Gq)$ = false (according to Moore's doctrine of naturalistic fallacies in ethics [Moore 1903] and

> Hume's extremely *empirical* doctrine of moral philosophy [Hume 1874]). Also, such a fundamental content analysis of $\Box(q \leftrightarrow Gq)$, which makes up solid basis for affirming that $\Box(q \leftrightarrow Gq)$ is false, can be found in [Adler 1997].

In the interpretation ®, all the axioms of Ξ are true, consequently, Ξ has a model, hence Ξ is consistent. Moreover, in the interpretation ®, the formulae $\neg(Kq \rightarrow q)$, $\neg(Eq \rightarrow q)$, and $\neg(\Box q \rightarrow q)$ are true as well. Consequently, $(Kq \rightarrow q)$, $(Eq \rightarrow q)$, and $(\Box q \rightarrow q)$ are not provable in Ξ.

4.3. Interpretation $\Im$

Let the meta-language symbols α and β be substituted by the object-language symbol q. Also let the meta-language symbol Ω be substituted by the object-language symbol $\Box$ (It is *necessary* that). In this particular case the system of axiom-schemes of Ξ is represented by the following axioms, respectively.

Axiom AX-1***: $Aq \rightarrow (\Box q \rightarrow q)$.

Axiom AX-2***: $Aq \rightarrow (\Box(q \rightarrow q) \rightarrow (\Box q \rightarrow \Box q))$.

Axiom AX-3***: $Aq \leftrightarrow (Kq \,\&\, (\Box q \,\&\, \Box\neg Sq \,\&\, \Box(q \leftrightarrow \Box q)))$.

Axiom AX-4***: $Eq \leftrightarrow (Kq \,\&\, (\neg\Box q \vee \neg\Box\neg Sq \vee \neg\Box(q \leftrightarrow \Box q)))$.

The interpretation $\Im$ is defined as follows.

1) $\Im\neg\omega = \neg\Im\omega$ for any formulae ω.
2) $\Im(\omega \oplus \pi) = (\Im\omega \oplus \Im\pi)$ for any formulae ω and π, and for any classical binary connective $\oplus$.
3) $\Im q$ = false.
4) $\Im Aq$ = false.
5) $\Im Eq$ = true.
6) $\Im\Box q$ = true.
7) $\Im Kq$ = true.
8) $\Im\Box\neg Sq$ = true.
9) $\Im\Box(q \rightarrow q)$ = true.
10) $\Im\Box(q \leftrightarrow \Box q)$ = false (according to the *empiricism* doctrine [Hume 1994; Locke 1994; Mach 1914; Feyerabend 1975; Kuhn 2012; Popper 1979; 1992; 2002; Wittgenstein 1992] denying Spinoza's metaphysical *necessitarianism* [Spinoza 1994]).

About Spinoza's *necessitarianism* see also [Curley and Walski 1999; Koistinen 2003; Miller 2001; Newlands 2010; 2013a; 2013b]. The formula $\Box(q \leftrightarrow \Box q)$ represents absolute fatalism affirming that everything is necessarily necessary. According to $\Im$, the representation of absolute fatalism is evaluated as false.

In the interpretation $\Im$, all the axioms of Ξ are true, consequently, Ξ has a model, hence Ξ is consistent. Moreover, in the interpretation $\Im$, the formulae $\neg(Eq \rightarrow q)$, $\neg(\Box q \rightarrow q)$, and $\neg(Kq \rightarrow q)$ are true. Consequently, $(Eq \rightarrow q)$, $(\Box q \rightarrow q)$, and $(Kq \rightarrow q)$ are not provable in Ξ.

5. The rule of elimination of $\Box$, and the necessitation rule by Gödel

The logic underlying the system Ξ is not a "normal modal logic" in that meaning of the term which has been used by [Kripke 1963; 1965; Priest 1992; 2008; Bull, Segerberg 1984]. In general, the inference-rule of elimination of $\Box$ does not belong to the set of inference-rules of Ξ. Nevertheless, *under the condition*, that **Aα** (but *not in general*) the following inference-rule of $\Box$-elimination is valid: "If **A$\alpha \vdash \Box\beta$**, then **A$\alpha \vdash \beta$**". It is easy to demonstrate this *limited* inference-rule by using the axiom scheme AX-3 and *modus ponens*.

Gödel's necessitation rule does not belong to the set of inference rules of Ξ. Nevertheless, it is easy to demonstrate in Ξ that *under the condition* that **Aα** (but *not in general*), the following (limited) inference-rule of *necessitation* is valid: "If

$\mathbf{A\alpha} \mid\!\!— \boldsymbol{\beta}$, then $\mathbf{A\alpha} \mid\!\!— \Box\boldsymbol{\beta}$". The following inference is a demonstration of this rule.

1. $\mathbf{A\alpha \leftrightarrow (K\alpha \;\&\; (\Box\alpha \;\&\; \Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta))}$: axiom scheme AX-3.
2. $\mathbf{A\alpha}$**:** assumption.
3. $\mathbf{K\alpha \;\&\; \Box\alpha \;\&\; \neg\Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta)}$: from 1 and 2 by propositional logic.
4. $\mathbf{\Box(\beta \leftrightarrow \Omega\beta)}$: from 3 by the rule of &–elimination.
5. $\mathbf{(\beta \leftrightarrow \Omega\beta)}$: from 4 by the (limited) rule of $\Box$–elimination.
6. $\mathbf{A\alpha} \mid\!\!— \mathbf{(\beta \leftrightarrow \Omega\beta)}$: by 1—5.
7. $\mathbf{A\alpha} \mid\!\!— \mathbf{(\beta \leftrightarrow \Box\beta)}$: from 6 by substituting $\Box$ for $\mathbf{\Omega}$.
8. $\mathbf{A\alpha} \mid\!\!— \boldsymbol{\beta}$ is given.
9. $\mathbf{A\alpha} \mid\!\!— \Box\boldsymbol{\beta}$ from 7 and 8 by propositional logic.
10. If $\mathbf{A\alpha} \mid\!\!— \boldsymbol{\beta}$ then $\mathbf{A\alpha} \mid\!\!— \Box\boldsymbol{\beta}$: by 1—9.

Here you are.

6. Proving the scheme of formulae $(\mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)})$ in Ξ, and indicating some philosophically interesting examples of this scheme

For any $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, it is provable in Ξ that $(\mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)})$, where the symbols $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ (belonging to the meta-language) stand for any elements of the set $\mathfrak{R} = \{\Box, \mathbf{K, F, T, P, Z, G, O, B, U, Y}\}$. (Elements of $\mathfrak{R}$ are called *perfection*-modalities.) The following succession of schemes of formulae is a scheme of proofs of/for $(\mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)})$ in Ξ.

1) $\mathbf{A\alpha \leftrightarrow (K\alpha \;\&\; (\Box\alpha \;\&\; \Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta)))}$**:** axiom scheme AX-3.
2) $\mathbf{A\alpha \to (K\alpha \;\&\; (\Box\alpha \;\&\; \Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta))}$: from 1 by the rule of elimination of $\leftrightarrow$.
3) $\mathbf{A\alpha}$: assumption.
4) $\mathbf{(K\alpha \;\&\; (\Box\alpha \;\&\; \Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta))}$: from 2 and 3 by *modus ponens*.
5) $\mathbf{\Box(\beta \leftrightarrow \Omega\beta)}$: from 4 by the rule of elimination of **&**.
6) $\mathbf{(\beta \leftrightarrow \Omega\beta)}$: from 5 by the rule of elimination of $\Box$.
7) $\mathbf{(\alpha \leftrightarrow \Sigma\alpha)}$: from 6 by substituting ($\boldsymbol{\alpha}$ for $\boldsymbol{\beta}$, and $\mathbf{\Sigma}$ for $\mathbf{\Omega}$).
8) $\mathbf{(\alpha \leftrightarrow \Omega\alpha)}$: from 6 by substituting ($\boldsymbol{\alpha}$ for $\boldsymbol{\beta}$).
9) $\mathbf{(\Sigma\alpha \leftrightarrow \alpha)}$: from 7 by commutativity of $\leftrightarrow$.
10) $\mathbf{(\Sigma\alpha \leftrightarrow \Omega\alpha)}$: from 9 and 8 by transitivity of $\leftrightarrow$.
11) $\mathbf{A\alpha} \mid\!\!— \mathbf{(\Sigma\alpha \leftrightarrow \Omega\alpha)}$: by 1—10.
12) $\mid\!\!— \mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)}$: from 11 by the rule of introduction of $\to$.

From the viewpoint of purely mathematical technique, the proof of $(\mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)})$ is not interesting (too simple). But from the viewpoint of proper philosophy contents, the statement $(\mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)})$ is very interesting and important. Various concrete philosophical interpretations (particular cases) of that statement are well-known as fundamental philosophical principles of the rationalism (a-priori-ism). For example, the following specific philosophical interpretations of the theorem-scheme $(\mathbf{A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha)})$ are worth mentioning.

- **a)** $\mathbf{A\alpha \to (\Box\alpha \leftrightarrow G\alpha)}$: the rationalistic principle of equivalence between necessary being and (universal) goodness. This principle was expressed by Aristotle, some outstanding creators of Ancient-Roman-Law, for example, Ulpian, and some great theologians, for example, St. Tomas Aquinas. In the *rationalistic jurisprudence* the mentioned principle is *axiological* option of the *natural law* doctrine.
- **b)** $\mathbf{(A\alpha \to (O\alpha \leftrightarrow \Box\alpha))}$: the *rationalistic jurisprudence* principle (*normative* option of the *natural law* doctrine) by Cicero, G.W. Leibniz [1971], H. Kelsen (Also I. Kant's idea of *prescribing a-priori laws to nature* is relevant to this case).

c) $\mathbf{(A\alpha \to (O\alpha \leftrightarrow G\alpha))}$: the *rationalistic jurisprudence* principle of equivalence of normative and evaluative options of formulating the natural law system. (It follows logically from a) and b.)

d) $\mathbf{A\alpha \to (G\alpha \leftrightarrow T\alpha)}$: the rationalistic principle of *optimism in ethics* by N. Malebranche and G.W. Leibniz [1952].

e) $\mathbf{A\alpha \to (T\alpha \leftrightarrow P\alpha)}$: the rationalistic principle of *optimism in epistemology* by G.W. Leibniz [1903] and D. Hilbert. About modeling this principle see [Lobovikov 2016a; 2016c].

f) $\mathbf{A\alpha \to (P\alpha \leftrightarrow Z\alpha)}$: the rationalistic principle of *mechanistic (algorithmic) optimism in epistemology* by R. Llull (Lullus), G.W. Leibniz [1903; 1981], and A.A. Lovelace (Augusta Ada King-Noel, Countess of Lovelace).

g) $\mathbf{A\alpha \to (G\alpha \leftrightarrow B\alpha)}$: the principle of *kalokagathia* (Socrates, Xenophon, Plato, Aristotle);

h) $\mathbf{A\alpha \to (G\alpha \leftrightarrow U\alpha)}$: the principle of *utilitarianism* ethics (J. Bentham, J.-St. Mill). About modeling this principle in Ξ, see [Lobovikov 2017a].

i) $\mathbf{A\alpha \to (G\alpha \leftrightarrow Y\alpha)}$: the principle of *hedonism* ethics (Aristippus, Epicurus). Modeling this principle in Ξ is discussed in [Lobovikov 2017a].

j) $\mathbf{A\alpha \to (B\alpha \leftrightarrow Y\alpha)}$: the principle of *hedonism in aesthetics*;

k) $\mathbf{A\alpha \to (B\alpha \leftrightarrow U\alpha)}$: the principle of *beauty of useful* (and *usefulness of beauty*).

l) $\mathbf{A\alpha \to (T\alpha \leftrightarrow U\alpha)}$: the principle of *pragmatism* in theory of truth (J. Dewey, W. James, C.S. Peirce).

m) $\mathbf{A\alpha \to (T\alpha \leftrightarrow B\alpha)}$: the principle of *beauty as criterion of truth*. (W. Blake, P.A.M. Dirac).

n) $\mathbf{A\alpha \to (P\alpha \leftrightarrow B\alpha)}$: the principle of *beauty as criterion of proof* (S.S. Averincev).

7. Theorem-schemes $(A\alpha \to \Omega\alpha)$, $(A\alpha \to (\Box\alpha \leftrightarrow \Omega\alpha))$, $(A\alpha \to (\Omega\alpha \,\&\, (\Box\alpha \leftrightarrow \Omega\alpha)))$, and their exemplifications by $(A\alpha \to O\alpha)$, $(A\alpha \to (\Box\alpha \leftrightarrow O\alpha))$, $(A\alpha \to (O\alpha \,\&\, (\Box\alpha \leftrightarrow O\alpha)))$, respectively

The theorem-scheme $\mathbf{(A\alpha \to (\Box\alpha \leftrightarrow \Omega\alpha))}$, is derived in Ξ from the above-proved theorem-scheme $\mathbf{(A\alpha \to (\Sigma\alpha \leftrightarrow \Omega\alpha))}$ by substitution of $\Box$ for $\mathbf{\Sigma}$. The theorem-scheme $\mathbf{(A\alpha \to \Omega\alpha)}$, is derived in Ξ according to the below-given succession of formula-schemes.

1) $\mathbf{(A\alpha \to (\Box\alpha \leftrightarrow \Omega\alpha))}$: the theorem-scheme.
2) $\mathbf{A\alpha}$: assumption.
3) $\mathbf{(\Box\alpha \leftrightarrow \Omega\alpha)}$: from 1 and 2 by modus ponens.
4) $\mathbf{A\alpha \leftrightarrow (K\alpha \,\&\, (\Box\alpha \,\&\, \Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)):}$ axiom scheme AX-3.
5) $\mathbf{A\alpha \to (K\alpha \,\&\, (\Box\alpha \,\&\, \Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)):}$ from 4 by the rule of elimination of $\leftrightarrow$.
6) $\mathbf{K\alpha \,\&\, \Box\alpha \,\&\, \neg\Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)}$: from 2 and 5 by modus ponens.
7) $\mathbf{\Box\alpha}$**:** from 6 by the rule of &–elimination.
8) $\mathbf{(\Box\alpha \to \Omega\alpha))}$: from 3 by the rule of elimination of $\leftrightarrow$.
9) $\mathbf{\Omega\alpha}$: from 7 and 8 by modus ponens.
10) $\mathbf{A\alpha \vdash \Omega\alpha}$**:** by 1—9.
11) $\vdash \mathbf{(A\alpha \to \Omega\alpha)}$: from 10 by the rule of introduction of $\to$.
12) $\mathbf{A\alpha \vdash (\Box\alpha \leftrightarrow \Omega\alpha)}$: by 1—3.
13) $\mathbf{A\alpha \vdash (\Omega\alpha \,\&\, (\Box\alpha \leftrightarrow \Omega\alpha))}$: from 10 and 12 by the rule of introduction of **&**.
14) $\vdash \mathbf{(A\alpha \to (\Omega\alpha \,\&\, (\Box\alpha \leftrightarrow \Omega\alpha)))}$: from 13 by the rule of introduction of $\to$.

Here you are.

By substituting $O\alpha$ for $\Omega\alpha$ in the above-proved theorem-schemes $(A\alpha \to \Omega\alpha)$, $(A\alpha \to (\Box\alpha \leftrightarrow \Omega\alpha))$, $(A\alpha \to (\Omega\alpha \,\&\, (\Box\alpha \leftrightarrow \Omega\alpha)))$, it is possible to derive the theorem-scheme-exemplifications $(A\alpha \to O\alpha)$, $(A\alpha \to (\Box\alpha \leftrightarrow O\alpha))$, $(A\alpha \to (O\alpha \,\&$

$(\Box\alpha \leftrightarrow O\alpha)))$, respectively.

Systematical thinking about the theorem-scheme $(A\alpha \rightarrow (O\alpha \,\&\, (\Box\alpha \leftrightarrow O\alpha)))$ results in making a significant distinction among three different deontic modalities "obligatory". The three are introduced and defined in the following paragraph.

8. The nontrivial problem of equivalence of corresponding alethic and deontic modalities from the viewpoint of graphic modeling logical relations among three different deontic modalities "obligatory" by means of the square and hexagon of opposition

The modal logic of norms is represented by plenty of somewhat different axiomatizations and moral-legal-philosophy doctrines [Hilpinen 1971; Kalinowski 1972]. The syntactic differences expose corresponding semantic ones based on qualitatively different philosophical intuitions. Below I am to consider logic interconnections among three substantially different kinds of "obligatory". In XX century the famous jurisprudential intuition of fundamental correlation among corresponding Aristotelian (alethic) and juridical (deontic) modalities [Leibniz 1971] was interpreted as their *similarity (analogy) but not an equivalence* [Wright 1983]. Similarity and equivalence are logically different relations: the last is transitive but the first is not. From the literal viewpoint it is worth noting that Leibniz himself did not use the proper logic term "equivalence" in his manifestations of the jurisprudential intuition of fundamental correlation among corresponding Aristotelian and juridical modalities [Leibniz 1971]; his relevant expressions are ambiguous. Consequently, Wright's interpreting Leibniz text is *correcting* it by substituting quite definite term "analogy (similarity)" for the relevant ambiguous expressions in "Elementa Juris Naturalis" [Leibniz 1971]. But is such correcting by eliminating ambiguity correct from the historicism principle viewpoint? Leibniz legal-philosophy writings belonged to the natural-law tradition which dominated in XVII century. Wright's normative logic discourse was in accordance with the legal positivism of XX century. Correcting the natural-law-philosophy by the legal positivism is a nontrivial problem. I do not think that Wright's "correction" of Leibniz is correct in general. The ambiguity in question may be eliminated in different ways, but Wright has realized only one of them. An important alternative has been missed or ignored by him. Generally speaking, it would be just to say that in one *concrete* relation (namely, in respect to *empiricism*) Wright's interpretation of Leibniz is quite correct, but in some other *specific* relation (namely, in respect to *a-priori-ism metaphysics*) it is not. To clarify the ambiguous situation and to formulate the nontrivial problem precisely, in addition to the classical deontic modality O (obligatory) let us introduce and define two substantially different kinds of nonclassical deontic modality "obligatory", namely, $O^1\alpha$ and $O^2\alpha$ by the below equivalences.

1: $O^1\alpha \leftrightarrow (O\alpha \,\&\, (\Box\alpha \leftrightarrow O\alpha))$.

2: $O^2\alpha \leftrightarrow (O\alpha \,\&\, \neg(\Box\alpha \leftrightarrow O\alpha))$.

3: $O\alpha \leftrightarrow (O^1\alpha \vee O^2\alpha)$.

The logical interconnections among the deontic modalities $O\alpha$, $O^1\alpha$ and $O^2\alpha$ may be visualized owing to the below-placed fig. 2.

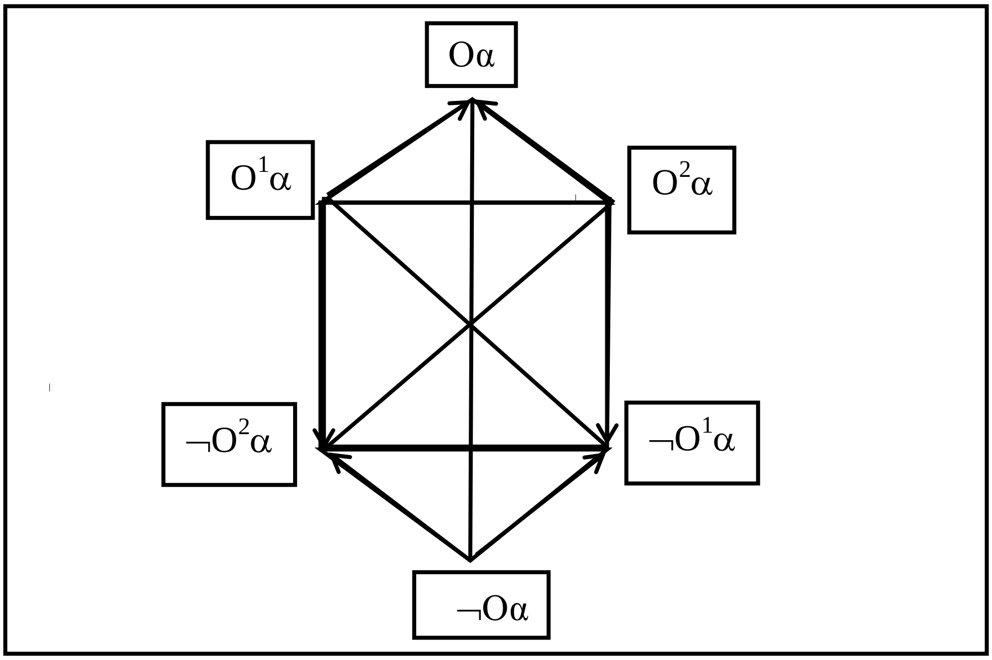

Fig. 2. Square and hexagon of opposition of three different kinds of deontic modality "obligatory"

In this hexagon: the *contrariety* relation between $O^1\alpha$ and $O^2\alpha$ is modeled by the upper horizontal line; the *sub-contrariety* relation between $\neg O^1\alpha$ and $\neg O^2\alpha$ is modeled by the bottom horizontal line; the *contradictoriness* relations between elements of the couples: $< O^1\alpha, \neg O^1\alpha >$; $< O^2\alpha, \neg O^2\alpha >$; $< O\alpha, \neg O\alpha >$ are modeled by the lines crossing the square. The relations of *logic consequence* (entailment) are modeled by arrows.

9. Theorem-scheme $(A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$

In addition to the above-said it is worth mentioning that the following succession of formula-schemes is a scheme of proofs (in Ξ) of the philosophically interesting theorem-schemes $(A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$, where Ω takes values from the set $\mathfrak{R}$.

1) $A\alpha \leftrightarrow (K\alpha \;\&\; (\Box\alpha \;\&\; \Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta))$: axiom scheme AX-3.
2) $A\alpha \rightarrow (K\alpha \;\&\; (\Box\alpha \;\&\; \Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta))$: from 1 by the rule of elimination of $\leftrightarrow$.
3) $A\alpha$: assumption.
4) $K\alpha \;\&\; \Box\alpha \;\&\; \neg\Box\neg S\alpha \;\&\; \Box(\beta \leftrightarrow \Omega\beta)$: from 2 and 3 by modus ponens.
5) $\Box(\beta \leftrightarrow \Omega\beta)$: from 4 by the rule of &–elimination.
6) $\Box(\alpha \leftrightarrow \Omega\alpha)$: from 5 by substituting α for β.
7) $A\alpha \rightarrow (\Box(\alpha \leftrightarrow \beta) \rightarrow (\Box\alpha \leftrightarrow \Box\beta))$: theorem-scheme.
8) $A\alpha \rightarrow (\Box(\alpha \leftrightarrow \Omega\alpha) \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$: from 7 by substituting $\Omega\alpha$ for β.
9) $\Box(\alpha \leftrightarrow \Omega\alpha) \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha)$: from 8 and 3 by modus ponens.
10) $(\Box\alpha \leftrightarrow \Box\Omega\alpha)$: from 9 and 6 by modus ponens.
11) $A\alpha \vdash (\Box\alpha \leftrightarrow \Box\Omega\alpha)$: by 1—10.
12) $\vdash (A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$: from 11 by the rule of introduction of $\rightarrow$.

Here you are.

The theorem-scheme $(A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$ may be instantiated by the following nontrivial philosophical principles.

a) $A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box G\alpha)$: the natural-law principle of *equivalence of necessary being and necessary posi-*

tive-moral-value (necessary goodness), represented in works by Aristotle, Ulpian, and Aquinas.

b) **$A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box O\alpha)$**: the natural-law principle of *equivalence of necessary being and necessary norm (duty)*, represented in works by Cicero, I. Kant, and H. Kelsen.

c) **$A\alpha \rightarrow (\Box O\alpha \leftrightarrow \Box G\alpha)$**: the principle of equivalence of the *normative* (deontic) and the *evaluative* options of formulating the natural-law doctrine. This principle follows logically from a) and b).

10. The logic square and hexagon of opposition of three different kinds of alethic modality "necessary"

Even the basic modal logic [Bull, Segerberg 1984] deals with plenty of somewhat different axiomatizations and intuitions. The syntax differences represent corresponding ones in semantics. The variety of intuitions mirror substantial content differences of modal notions in general and of the alethic necessity ones in particular. In this paper I would like to consider logic interconnections among three different kinds of necessity.

In addition to the classical alethic modality $\Box$ (necessary) let us introduce two different kinds of nonclassical alethic modality "necessary", namely, $\Box^1\alpha$ and $\Box^2\alpha$ by the below definitions.

Definition DF-1: $\Box^1\alpha \leftrightarrow (\Box\alpha \ \& \ \Box\Box\alpha)$.

Definition DF-2: $\Box^2\alpha \leftrightarrow (\Box\alpha \ \& \ \neg\Box\Box\alpha)$.

Corollary: $\Box\alpha \leftrightarrow (\Box^1\alpha \vee \Box^2\alpha)$.

If the equivalences are accepted then the logic interconnections among the alethic modalities $\Box\alpha$, $\Box^1\alpha$, $\Box^2\alpha$ are modeled graphically by the logic square and hexagon of opposition represented below by fig. 3.

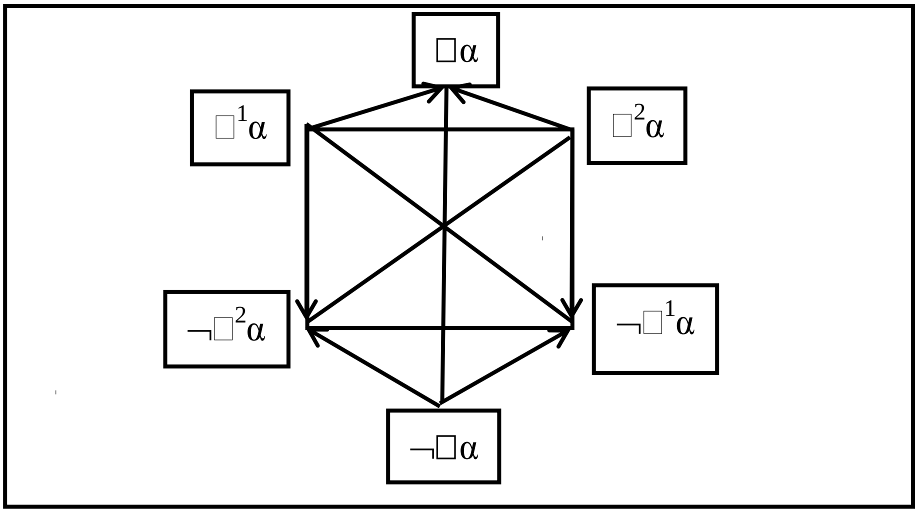

Fig. 3. Square and hexagon of opposition of three different kinds of alethic modality "necessary"

Taking into an account the difference among $\Box\alpha$, $\Box^1\alpha$, $\Box^2\alpha$, it is possible to demonstrate that $\vdash (A\alpha \rightarrow \Box^1\alpha)$ by the following succession of formula-schemes.

1) $(A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$: the above-proved theorem-scheme.
2) $(A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Box\alpha))$: from 1 by substituting $\Box$ for $\Omega\alpha$.
3) $A\alpha$: assumption.
4) $(\Box\alpha \leftrightarrow \Box\Box\alpha)$: from 2 and 3 by modus ponens.
5) $A\alpha \leftrightarrow (K\alpha \ \& \ (\Box\alpha \ \& \ \Box\neg S\alpha \ \& \ \Box(\beta \leftrightarrow \Omega\beta))$: axiom scheme AX-3.
6) $A\alpha \rightarrow (K\alpha \ \& \ (\Box\alpha \ \& \ \Box\neg S\alpha \ \& \ \Box(\beta \leftrightarrow \Omega\beta))$: from 5 by the rule of elimination of $\leftrightarrow$.

7) $K\alpha \,\&\, \Box\alpha \,\&\, \neg\Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)$: from 3 and 6 by modus ponens.
8) $\Box\alpha$: from 7 by the rule of elimination of &.
9) $(\Box\alpha \rightarrow \Box\Box\alpha)$: from 4 by the rule of elimination of $\leftrightarrow$.
10) $\Box\Box\alpha$: from 9 and 8 by modus ponens.
11) $(\Box\alpha \,\&\, \Box\Box\alpha)$: from 8 and 10 by the rule of introduction of &.
12) $\Box^{1}\alpha$: from 11 by the above-given definition DF-1.
13) $A\alpha \vdash \Box^{1}\alpha$: by 1—12.
14) $\vdash (A\alpha \rightarrow \Box^{1}\alpha)$: from 13 by the rule of introduction of $\rightarrow$.

11. Theorem-scheme $(A\alpha \rightarrow \Box\Omega\alpha)$

The following succession of formula-schemes is a proof of the theorem.

1) **$(A\alpha \rightarrow (\Box\alpha \leftrightarrow \Box\Omega\alpha))$:** the above-proved theorem-scheme
2) **$A\alpha$:** assumption
3) **$(\Box\alpha \leftrightarrow \Box\Omega\alpha))$:** from 1 and 2 by modus ponens
4) **$A\alpha \leftrightarrow (K\alpha \,\&\, (\Box\alpha \,\&\, \Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta))$:** axiom scheme AX-3.
5) **$A\alpha \rightarrow (K\alpha \,\&\, (\Box\alpha \,\&\, \Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta))$:** from 4 by the rule of elimination of $\leftrightarrow$.
6) **$(K\alpha \,\&\, (\Box\alpha \,\&\, \Box\neg S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta))$:** from 5 and 2 by modus ponens
7) **$\Box\alpha$:** from 6 by the rule of elimination of **&**.
8) **$(\Box\alpha \rightarrow \Box\Omega\alpha))$:** from 3 by the rule of elimination of $\leftrightarrow$.
9) **$\Box\Omega\alpha$:** from 7 and 8 by modus ponens
10) **$A\alpha \vdash \Box\Omega\alpha$:** by 1—9.
11) **$\vdash (A\alpha \rightarrow \Box\Omega\alpha)$:** from 10 by the rule of introduction of $\rightarrow$.

Here you are.

Evidently, the formal proofs submitted in this paper are not interesting from the viewpoint of pure mathematics proper, but the theorems are nontrivial from the content-epistemology-viewpoint, and their formal proofs are important for the field of application of mathematics, namely, for the general theory of knowledge, which is not reduced to pure mathematics proper.

For instance, the theorem-scheme $(A\alpha \rightarrow \Box\Omega\alpha)$ is exemplified by $(A\alpha \rightarrow \Box G\alpha)$ and $(A\alpha \rightarrow \Box O\alpha)$. The formula-schemes $\Box G\alpha$ and $\Box O\alpha$ represent (respectively) the metaphysical grounds of *axiological* (evaluative) and *deontic* (normative) options of formulating the *immutable and universal natural-law* in ethics and jurisprudence. The formula-schemes $A\alpha$, $\Box G\alpha$, $\Box O\alpha$ (and the ones composed of them) are meaningless for the *empiricist*-minded moralists and lawyers resolutely rejecting existence of a-prior knowledge. Legal and moral positivists resolutely denying existence of the immutable and universal natural-law evaluate it as a metaphysical chimera. However, in spite of the extreme positivist tendency there is a possibility to overcome the old contradiction between sensualism-empiricism and rationalism-a-prior-ism. The present paper submits an attempt of realizing the possibility by consistent synthesizing the two opposites deprived of their extreme formulations. The relation between the two is *not the contradictoriness but the contrariety* one.

12. Conclusion

The above-considered logically formalized axiomatic theory Ξ is consistent. The formula $(Aq \rightarrow q)$ is a theorem in Ξ, but $(Eq \rightarrow q)$ and $(Kq \rightarrow q)$ are not. This means that the logical contradiction between the classical epistemic logic [in which $(Kq \rightarrow q)$ is a theorem] and the *evolutionary* epistemology is eliminated. Owing to Ξ, the problem formulated in the introduction is solved. Moreover, in Ξ the formula $(Aq \rightarrow \Box^{1}\alpha)$ is a theorem but $(\Box q \rightarrow q)$ is not.

References

[1] Adler, M.J. *Six Great Ideas*. New York: Simon & Schuster Inc., 1997, 256 p.

[2] Béziau J-Y. The New Rising of the Square of Opposition, J.-Y. Beziau, D. Jacqette, eds. *Around and Beyond the Square of Opposition*, Basel, Birkhäuser, 2012a, pp. 3-19.

[3] Béziau J-Y. "The Power of the Haxagon", *Logica Universalis*, 2012b, vol. 6, no. 1-2, pp. 1-43.

[4] Blanché, R. *Structures intellectuelles. Essai sur l'organisation systématique des concepts*, Paris, Vrin, 1966, 151 p.

[5] Bull, R., Segerberg, K. "Basic Modal Logic," in D. Gabbay, F. Guenthner (eds.). *Handbook of Philosophical Logic, vol. II: Extensions of Classical Logic*. Dordrecht: D. Reidel Publishing Company, 1984, pp. 1-88.

[6] Campbell, D.T. "Natural selection as an epistemological model", in Raoul Naroll and Ronald Cohen (Eds.), *A handbook of method in cultural anthropology*, New York: National History Press, 1970, pp. 51-85.

[7] Campbell, D.T. "Evolutionary Epistemology", in: P. A. Schilpp (ed.) *The philosophy of Karl R. Popper*, LaSalle, IL: Open Court, 1974, pp. 412—463.

[8] Campbell, D.T. "Evolutionary epistemology", in: *Evolutionary epistemology, rationality, and the sociology of knowledge*, 1987, pp. 47-89.

[9] Campbell, D.T. "Epistemological roles for selection theory", in: *Evolution, cognition, and realism: Studies in evolutionary epistemology*, 1990, pp. 1-19.

[10] Curley, E., Walski, G. "Spinoza's Necessitarianism Reconsidered // R. Gennaro and C. Huenemann (eds.). *New Essays on the Rationalists*. Oxford: Oxford University Press, 1999, pp. 241–62.

[11] Feyerabend P. *Against method: outline of an anarchist theory of knowledge*, London: NLB; Atlantic Highlands: Humanities Press, 1975, 339 p.

[12] Hilpinen, R. (ed.). *Deontic Logic: Introductory and Systematic Readings*. Dordrecht: D. Reidel, 1971. 184 p.

[13] Hintikka, J. *Knowledge and belief. An introduction to the logic of the two notions*. Ithaca: Cornell university press, 1962. 179 p.

[14] Hintikka, J. *Knowledge and the known. Historical perspectives in epistemology*. Dordrecht-Boston: D. Reidel, 1974. 243 p.

[15] Hintikka, J., Hintikka M.B. *The logic of epistemology and the epistemology of logic. Selected essays*. Dordrecht, etc.: Kluwer, 1989. 245 p.

[16] Hume, D. "A Treatise of Human Nature being an Attempt to Introduce the Experimental Method of Reasoning into Moral Subjects." Ed. T.H. Green and T.H. Grose. *The Philosophical Works of David Hume in Four Volumes. Vol. II*. London: Longmans, Green, and Co., 1874, pp. 1-374.

[17] Hume, D. "An Enquiry Concerning Human Understanding," in: Mortimer J. Adler (Ed.). *Great Books of the Western World. V. 33*. Chicago; London: Encyclopedia Britannica, Inc. 1994, pp. 451- 509.

[18] Kalinowski, G. *La Logique des norms*, Paris: Presses Universitaires de France, 1972. 218 p.

[19] Koistinen, O. "Spinoza's Proof of Necessitarianism," *Philosophy and Phenomenological Research*. 2003. V. 66. N. 2, pp. 283–310.

[20] Kostyuk, V.N. *Ehlementy modal'noj logiki [Elements of modal logic]*. Kiev: Naukova dumka, 1978, 180 p. (in Russian)

[21] Kripke, S.A. "Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculi," *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 1963. Volume 9, Issue 5-6, pp. 67–96.

[22] Kripke, S.A. "Semantical Analysis of Modal Logic II: Non-Normal Modal Propositional Calculi," in: *The Theory of Models (Proceedings of the 1963 International Symposium at Berkley)*. Amsterdam: North-Holland, 1965, pp. 206–220.

[23] Kuhn T.S. *The Structure of Scientific Revolutions*. Chicago; London: The University of Chicago Press, 2012, 217 p.

[24] Leibniz, G.W. Generales Inquisitiones de Analysi Notionum et Veritatum, In: Couturat, L. *Opuscules et Fragments Inédits de Leibniz: extraits de la Bibliothèque royale de Hanovre*, Paris: Alcan, 1903. pp. 356-399.

[25] Leibniz, G.W. *Theodicy: essays on the goodness of God, the freedom of man, and the origin of evil*. London: Routledge & K. Paul, 1952. 448 p.

[26] Leibniz, G.W. "Elementa Juris Naturalis", in: *G.W. Leibniz. Philosophishe Schriften. Erster Band (1663-1672)*. Berlin: Akademie-Verlag, 1971. S. 431-485.

[27] Leibniz, G.W. *New essays on human understanding*. Cambridge [Eng.]; New York: Cambridge University Press, 1981. 402 p.

[28] Lobovikov, V.O. "An Axiomatic Definition of the Domain of Adequateness of the Rationalistic Optimism of G.W. Leibniz, D. Gilbert, and K. Gödel," The Siberian Journal of Philosophy. 2016a. Vol. 14. No 4. pp. 69-81.

[29] Lobovikov, V.O. "An Axiomatization of Philosophical Epistemology," *Journal of Tomsk State University. Series: Philosophy, Sociology, Political Studies*, 2016b. Vol. 36. No 4, pp. 69-78. DOI: 10.17223/1998863X/36/7

[30] Lobovikov, V.O. "Square and Hexagon of Opposition of 'A-Priori Knowledge' and 'Empirical One' (Eliminating an Impression of Logic Contradiction between Leibniz' and Gödel's Statements)," in: Jean-Yves Beziau, Arthur Buchsbaum, and Manuel Correia (Eds.). *Handbook of Abstracts of 5th World Congress on the Square of Opposition (November 11-15, 2016, Easter Island – Rapa Nui, Chile)*. Santiago, Chile: Pontifical Catholic University, Chile, 2016c, pp. 33-34.
[31] Lobovikov, V.O. "A Formal Deductive Proof of Equivalence of Evaluative Modalities of Moral Goodness, Utility and Pleasure within an Axiomatic System of Epistemology from the Assumption of A-priori-ness of Knowledge (An Axiomatic Definition of the Scope of Relevant Applying G. Moore's Doctrine of the Naturalistic Fallacies in Ethics)," *Journal of Tomsk State University. Series: Philosophy, Sociology, Political Studies*. 2017a. Vol. 39. No 3, pp. 30-39. DOI: 10.17223/1998863X/39/4.
[32] Lobovikov, V.O. "Uniting normal and non-normal modal logics by one axiomatic system of philosophical epistemology," *Scientific Journal "Discourse-P"*, 2017b, Vol. 14, No 2, pp. 163-169.
[33] Lobovikov, V.O. "Evolutionary epistemology and non-normal modal logic of knowledge," *Journal of Tomsk State University. Series: Philosophy, Sociology, Political Studies*, 2018a. Vol. 41. No. 1, pp. 5-14. DOI: 10.17223/1998863X/41/1
[34] Lobovikov, V.O. "Moving from the Opposition of Normal and Not-Normal Modal Logics to Universal Logic: Synthesizing T, S4, Tr, Verum and Falsum systems by the Square and Hexagon," In: Jean-Yves Beziau, Arthur Buchsbaum, and Christophe Rey (eds). *Handbook of the 6th World Congress and School on Universal Logic, June 16-26, 2018* Vichy, France: Université Clermont Auvergne, 2018b, pp. 449-450. UNILOG'2018 https: / / www.uni-log.org/Vichy2018
[35] Locke, J. "An Essay Concerning Human Understanding," in: Mortimer J. Adler (Ed.). *Great Books of the Western World. V. 33*. Chicago; London: Encyclopedia Britannica, Inc. 1994, pp. 85-395.
[36] Mach, E. *The analysis of sensations, and the relation of the physical to the psychical*. Chicago; London: Open Court, 1914. 408 p.
[37] Miller, J.A. "Spinoza's Possibilities," *Review of Metaphysics*. 2001. V. 54. N. 4, pp. 779–815.
[38] Moore, G.E. *Principia Ethica*. Cambridge: At the University press, 1903, 232 p.
[39] Newlands, S. "The Harmony of Spinoza and Leibniz," *Philosophy and Phenomenological Research*. 2010. V. 81. N.1, pp. 64–104.
[40] Newlands, S. "Leibniz and the Ground of Possibility," *The Philosophical Review*. 2013. V. 122. N. 2, pp. 155–187.
[41] Newlands, S. "Spinoza's Modal Metaphysics," *The Stanford Encyclopedia of Philosophy* (Winter 2013 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2013/entries/spinoza-modal/>.
[42] Popper, K.R. *Conjectures and refutations: the growth of scientific knowledge*. London; New York: Routledge, 2002. 582 p.
[43] Popper, K.R. *The logic of scientific discovery*. London; New York: Routledge, 1992. 479 p.
[44] Popper, K.R. *Objective Knowledge: An Evolutionary Approach*. Oxford [Eng.]: Clarendon Press; New York: Oxford University Press, 1979. 395 p.
[45] Priest, G. "What is a Non-Normal World?" *Logique et Analyse*. 1992. Vol. 139-140, pp. 291-302.
[46] Priest, G. *An Introduction to Non-Classical Logic: From If to Is, 2nd Edition*, Cambridge, New York: Cambridge University Press, 2008. 614 p.
[47] Ruse, M. "Evolutionary Epistemology: Can Sociobiology Help?" *Sociobiology and Epistemology. Synthese Library. Volume 180*. Dordrecht: D. Reidel, 1985, pp. 249-265.
[48] Spinoza, B. "Ethics," in: Mortimer J. Adler (Ed.). *Great Books of the Western World. V. 28*. Chicago; London: Encyclopedia Britannica, Inc. 1994, pp. 589-697.
[49] Tulmin, S. "The Evolutionary Development of Natural Science," *American Scientist*. 1967. Vol 55, pp. 456-471.
[50] Wittgenstein, L. *Tractatus logico-philosophicus*. London; New York: Routledge & K. Paul, 1992. 207 p.
[51] Wright, G.H. von. *Philosophical Papers. Vol. I: Practical Reason*. Oxford: Basil Blackwell, 1983, 214 p.
[52] Wuketits, F.M. *Evolutionary Epistemology and Its Implications for Humankind*. Albany, N.Y.: State University of New York Press, 1990. 262 p.