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Linearization and control for a perturbed 2D mixing flow dynamical system

Adela Ionescu[1]

[1]Department of Applied Mathematics, University of Craiova, 13, A.I. Cuza Street, 200585, Craiova, Dolj, Romania

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***Corresponding author**: Adela Ionescu, Department of Applied Mathematics, University of Craiova, 13, A.I. Cuza Street, 200585, Craiova, Dolj, Romania
Email: adela0404@yahoo.com

Abstract

This paper continues the recent work on feedback linearizing for the 2D mixing flow dynamical system in a slightly perturbed version. The possibility of finding a control u is taken into account, in order to further find the global non-linear controller form of it.

Keywords

mixing flow; feedback linearization; global non-linear controller.

1. Introduction. The mathematical context of feedback linearization.

Feedback is a powerful idea which is used extensively in natural and technological systems. The principle of feedback is simple: base correcting actions on the difference between desired and actual performance. The use of feedback has often resulted in vast improvements in system capability and these improvements have sometimes been revolutionary. The reason for this is that feedback has some truly remarkable properties.

The feedback linearization technique has multiple approaches. Basically it is based on concepts from nonlinear systems theory and contains two fundamental non-linear controller design techniques: input-output linearization and state-space linearization [1,7].

The central idea is to algebraically transform nonlinear systems dynamics into (fully or partly) linear ones, so that linear control techniques can be applied. It is important to notice that this differs entirely from conventional (Jacobian) linearization, because feedback linearization is achieved by *exact state transformation and feedback*, rather than by linear approximations of the dynamics.

An important use of feedback is to change the dynamics of a system. Through feedback, we can alter the behavior of a system to meet the needs of an application: systems that are unstable can be stabilized, systems that are sluggish can be made responsive and systems that have drifting operating points can be held constant.

Control theory provides a rich collection of techniques to analyze the stability and dynamic response of complex systems and to place bounds on the behavior of such systems by analyzing the gains of linear and nonlinear operators that describe their components [7]. The term “control” has many meanings and often varies between scientific communities. Very often, the control is defined as the use of algorithms and feedback in engineered systems. Thus, control includes such examples as feedback loops in electronic amplifiers, setpoint controllers in chemical and materials processing, “fly-by-wire” systems on aircraft and even router protocols that control the traffic flow.

There are few techniques in the feedback linearization theory. The papers that suggest a new approach to control of non-linear systems have been appeared in the years 80s. Sommer [11] proposed a method that transforms a class of non linear time-varying systems into a phase-variable canonical form. Subsequently, Su [12], Hunt [3] et al developed a procedure for global linearization.

In this paper we are focusing on dynamical systems of the form

$$\dot{\mathbf{x}}(\mathrm{t}) = \mathbf{a}(\mathbf{x}(\mathrm{t})) + \mathbf{b}(\mathbf{x}(\mathrm{t}))\mathrm{u}(\mathrm{t}) \tag{1}$$

with **a** and **b** C^{∞} vector fields on an open set U in R^n containing the origin, **a**(0)=0.
Some sufficient conditions on **a** and **b** are required, so that there exists a C^{∞} transformation ***z*** = ***T(x)*** such that the system (1) can be transformed into the *global non-linear controller form:*

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ . \\ . \\ . \\ z_{n-1} \\ f(z) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ \pm\mu(x) \end{bmatrix} u \tag{2}$$

We need in what follows some definitions from algebraic geometry, namely:
Def1. For two vector fields **f** and **g** on R^n, the *Lie bracket* [**f**, **g**] is a vector field defined by

$$[\boldsymbol{f}, \boldsymbol{g}] = \frac{\partial g}{\partial x}\boldsymbol{f} - \frac{\partial f}{\partial x}\boldsymbol{g} \tag{3}$$

where $\partial f/_{\partial x}$, $\partial g/_{\partial x}$ are the jacobians of f and g respectively.
The Lie bracket is also denoted as $[f, g] = (ad^1 f, g)$
Thus we define iteratively

$$(ad^k f, g) = [f, (ad^{k-1} f, g)], \text{ where } (ad^0 f, g) = g \tag{4}$$

Def.2. For a scalar field h and a vector field $\mathbf{f}=(f_1,f_2, \ldots, f_n)^T$ the Lie *derivative of h with respect to **f*** is

$$\langle dh, f\rangle = \frac{\partial h}{\partial x_1} f_1 + \cdots + \frac{\partial h}{\partial x_n} f_n \;\; ; \;\; \langle dh, f\rangle = \nabla h \cdot f \tag{5}$$

The following theorem is basic in the theory. It is proved [7] that if and only if

i) The matrix

$$C = [b \;\; (ad^1 a, b) \;\; (ad^2 a, b) \;\; \cdots \;\; (ad^{n-1} a, b)] \tag{6}$$

has rank n in an x_0 of the open set U, and

ii) Near x0, the distribution

$$D = span\{b, ad_a b, \cdots, ad_a^{n-2} b\} \tag{7}$$

is involutive, then there exists a real valued function α(x) defined in a neighborhood of x_0 such that

$$L_b \alpha(\boldsymbol{x}) = L_{ad_a b}\alpha(\boldsymbol{x}) = \cdots = L_{ad_a^{n-2} b}\alpha(\boldsymbol{x}) = 0, L_{ad_a^{n-1} b}\alpha(\boldsymbol{x}) \neq 0 \tag{8}$$

where $L_X \alpha$ denotes the Lie derivative of the real valued function α(**x**) with regard to the vector field **X.**
In these conditions we can find a control of the form

$$v(\boldsymbol{x}) = d(\boldsymbol{x}) + c(\boldsymbol{x})u \tag{9}$$

where

$$c(\boldsymbol{x}) = L_b L_a^{n-1}\alpha(\boldsymbol{x}), d(\boldsymbol{x}) = L_a^n(\boldsymbol{x}) \;\; , \;\; \boldsymbol{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T \tag{10}$$

such that the initial system is changed into the controllable system

$$\begin{aligned} \dot{z_1} &= z_2 \\ \dot{z_2} &= z_3 \\ &\cdots \\ \dot{z_{n-1}} &= z_n \\ \dot{z_n} &= v \end{aligned} \tag{11}$$

The basic tool involved in this technique is the so-called *controllability matrix* C exposed in (6). It involves the successive Lie brackets of the vectors **a** and **b** and it is mandatory to be non-singular.

2. Recent results in modeling the mixing flow dynamical system

2.1. *Basic framework of mixing flow modeling*

The mixing flow theory appears in an area with far from complete solving problems: the flow kinematics. Its methods and techniques developed the *significant relation between turbulence and chaos*. The turbulence is an important feature of dynamic systems with few freedom degrees, the so-called "far from equilibrium systems". These are widespread between the models of excitable media, and a recent goal is to find a consistent and coherent theory to stand up that a mixing model in excitable media leads to a far from equilibrium model.
Generally, the statistical idea of a flow is represented by a map:

$$x = \Phi_t(X), X = \Phi_{t=0}(X) \tag{12}$$

We say that **X** is *mapped in* ***x*** after a time t. In the continuum mechanics the relation (12) is named *flow*, and it is a diffeomorphism of class C^k. It must satisfy the relation:

$$J = \det(D(\Phi_t(X))) = \det\left(\frac{\partial x_i}{\partial X_j}\right) \tag{13}$$

where D denotes the derivation with respect to the reference configuration, in this case **X**. The relation (13) implies two particles, X_1 and X_2, which occupy the same position **x** at a moment.
The mixing flow concept implies *stretching* and *folding* of material elements mixed in a host fluid. Therefore the basic tools are the study of the deformation measures in length and surface of a material filament d**X** and an area element d**A**.
With respect to **X** there is defined the basic measure of deformation, the *deformation gradient*, **F,** namely:

$$\mathbf{F} = (\nabla_X \Phi_t(\mathbf{X}))^T,\ F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right) \tag{14}$$

where ∇_X denotes differentiation with respect to **X**. According to (14), **F** is non singular. The basic measure for the deformation with respect to **x** is the *velocity gradient*.
The basic deformation measures are the *length deformation* λ and *surface deformation* η, and are defined with the relations [10]:

$$\lambda = (\mathbf{C} : \mathbf{MM})^{1/2},\ \eta = (\det F) \cdot (\mathbf{C}^{-1} : \mathbf{NN})^{1/2} \tag{15}$$

with **C** (=$\mathbf{F}^T \mathbf{F}$) the *Cauchy-Green deformation tensor,* and the vectors **M**, **N** - the orientation versors in length and surface respectively, defined by

$$\mathbf{M} = \frac{d\mathbf{X}}{|d\mathbf{X}|},\ \mathbf{N} = \frac{d\mathbf{A}}{|d\mathbf{A}|} \tag{16}$$

In the above context, we say that the flow $\boldsymbol{x}=\Phi_t(\boldsymbol{X})$ has *a good mixing* if the mean values $D(\ln\lambda)/Dt$ and $D(\ln\eta)/Dt$ are not decreasing to zero, for any initial position P and any initial orientations **M** and **N.**
The deformation tensor **F** and the associated tensors **C**, $\mathbf{C}^{-1}$, form the fundamental quantities for the analysis of deformation of infinitesimal elements [10]. The flows of interest belong generally to two classes: i) flows with a special form of ∇v and ii) flows with a special form of **F**. The second class is of very large interest, as it contains the so-called Constant Stretch History Motion – CSHM flows.

2.2. *Recent results*

The mixing flow mathematical modeling had multiple approaches both in 2d and 3d cases, from the numerical simulations in association with some experimental standpoints in order to study the deformations behavior [6], to computational approaches for studying the behavior of the model in different versions of it [4,5].
The basic starting model is the widespread 2d dynamical system of a mixing flow [10]:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1, \quad -1 \langle K \langle 1, G \rangle 0 \end{cases} \tag{17}$$

In order to collect more statistical data for the turbulent mixing model behavior, few versions of this model were analyzed from computational standpoint. Between them the perturbation with a logistic-type term has to be noticed [5]. The set of parameters values was the same in various simulations, for a better accuracy of the comparative analysis. The phase-portrait analysis offered new features concerning the influence on parameters on the model behaviour.
Changing the standpoint from computational to analytical view, the feedback linearization technique was recently approached for the mixing flow dynamical system. In [4] it was approached a slightly perturbed version of the 2d model, namely the following dynamical system was taken into account:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 + x_1 \\ \dot{x}_2 = K \cdot G \cdot x_1 - x_2, \quad -1 \langle K \langle 1, G \rangle 0 \end{cases} \tag{18}$$

It was found the differential transformation **T**, and further the inverse system associated to (18) as:

$$\dot{z_1} = z_2$$

$$\dot{z_2} = 2G \cdot (Kz_1^2 - z_2^2) \cdot u - 4KG^2 z_1 z_2 \tag{19}$$

The conclusion was that the 2d mixing flow model admits an inverse system in a controllable form. This happens if certain conditions are fulfilled for the vector functions **a**(**x**) and **b**(**x**). In fact the components T_i of the transformation **T** are found from a partial differential equations system [1, 4, 7]. The system has the following general form:

$$\begin{cases} \frac{\partial T_i}{\partial x} \cdot b(\boldsymbol{x}) = 0, i = 1,2,\cdots,n-1 \\ \frac{\partial T_n}{\partial x} \cdot b(\boldsymbol{x}) \neq 0, \qquad \boldsymbol{x} \in R^n \end{cases} \tag{20}$$

Looking at (20) is easy to observe that finding T_i becomes quite complex for n≥3.

3. Finding a control for the 2d mixing flow dynamical system

The purpose of the present paper is finding a control u for the 2d mixing flow dynamical system in the context of the state-space linearization. The slightly perturbed version (18) of the mixing model is taken into account.
Following the technique of the section 2, the vectors **a** and **b** were chosen as follows:

$$\boldsymbol{a}(\boldsymbol{x}) = \begin{bmatrix} Gx_2 \\ KGx_1 \end{bmatrix}, \ \boldsymbol{b}(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}, where\ \boldsymbol{x} = [x_1 \quad x_2]^T \tag{21}$$

Following the calculus of successive Lie brackets (n=2) for the vector fields **a** and **b**, like in (4), we obtain that:

$$[\boldsymbol{a}, \boldsymbol{b}] = (ad^1 \boldsymbol{a}, \boldsymbol{b}) = \begin{bmatrix} 2Gx_2 \\ -2KGx_1 \end{bmatrix} \tag{22}$$

Then the matrix $C = [b \ \ (ad^1 a, b)\,]$ will be given in our case, by the following relation:

$$C = \begin{bmatrix} x_1 & 2Gx_2 \\ -x_2 & -2KGx_1 \end{bmatrix} \tag{23}$$

It is verified that C is non singular. It is found that

$$\det C = 2G(x_2^2 - Kx_1^2) \neq 0 \tag{24}$$

This is true because it is shown [10] that the streamlines of the model (17) satisfy the relation

$$x_2^2 - Kx_1^2 = const, -1 \leq K \leq 1$$

Then we can proceed to find the function α(**x**) like in the system (8).

Solving the partial differential system (8) for the case n=2:

$$L_b \alpha(\boldsymbol{x}) = 0$$

$$L_{ad_a^b} \alpha(\boldsymbol{x}) \neq 0$$

we find α as

$$\alpha(\mathbf{x}) = x_1 x_2, \ \boldsymbol{x} = [x_1 \quad x_2]^T \tag{25}$$

This makes the further calculus not very complicate. Thus, the quantities $L_a\boldsymbol{\alpha}(\boldsymbol{x}), \boldsymbol{L_a^2}\boldsymbol{\alpha}(\boldsymbol{x})$ are calculated and the functions c(**x**) and d(**x**) are found as:

$$c(\boldsymbol{x}) = 2G(Kx_1^2 - x_2^2),\ d(\boldsymbol{x}) = 4KG^2x_1x_2,\ \boldsymbol{x} = [x_1 \quad x_2]^T \tag{26}$$

We introduce further these functions in the formula (9) of the control, and, taking into account that we can usually take a linear form of the control v as:

$$v = a_1x_1 + a_2x_2,\ a_1, a_2 \in R$$

We find the control u as follows:

$$u(\boldsymbol{x}) = \frac{v(x) - d(x)}{c(x)} = \frac{a_1x_1 + a_2x_2 - 4KG^2x_1x_2}{2G(Kx_1^2 - x_2^2)},\ -1 \le K \le 1, G > 0 \quad \boldsymbol{x} = [x_1 \quad x_2]^T \tag{27}$$

4. Conclusions

In the present paper a control was found for the 2d mixing flow dynamical system in a slightly perturbed version. The basic conclusion is that the mixing model in this form admits a control. In this choice of the vector fields **a** and **b,** the conditions are fulfilled for finding a non-singular controllability matrix associated to the system and further proceed for all calculus. Note that the vector fields **a** and **b** have not a unique form, their choice can be so that to be accomplished the specified conditions.

It is important to mention that in this theory of feedback linearization, we are dealing in fact with a switch between the controls u and v. For obtaining u we need another control v which we suppose to be linear. Isidori specifies also this feature [7]. Similar results are found in [4]. This confirms the basic advantage of the feedback linearization:" base correcting actions on the difference between desired and actual performance".

It must be noticed an important issue from previous analysis [6]: although the basic mixing flow is a linear model, when adding similar terms to the model, in 2d case, the model turns its behaviour into a *far from equilibrium one*. The control u has an important role in controlling the chaotic behavior motion of the dynamical system and further obtaining the so-called "time-history" of the trajectory in the non-controlled and controlled form. Such an interesting exhibition is realized in [9].

Therefore an important target is analyzing the equivalent controllable form of the mixing flow dynamical system, the so-called "global non-linear controller" in the form (2). For a number of reasons, such as easy of finding a control law, it is often advantageous to work with such an equivalent system rather than with the original one. It is expected that introducing a control in a chaotic dynamical system, its behavior changes and makes it easier controllable. Studying the initial model via its equivalent non-controller form could help to a better use of the chaotic approaches in the sense of Smale [2].

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