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Well-Posedness and Long-Time Dynamics of the Rotating Boussinesq and Quasigeostrophic Equations

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Abstract

Most differential equations occurring in multiscale modelling of physical and biological systems cannot be solved analytically. Numerical integrations do not lead to a desired result without qualitative analysis of the behavior of the equation's solutions. The authors study the quasigeostrophic and rotating Boussinesq equations describing the motion of a viscous incompressible rotating stratified fluid flow, which refers to PDE that are singular problems for which the equation has a parabolic structure (rotating Boussinesq equations) and the singular limit is hyperbolic (quasigeostrophic equations) in the asymptotic limit of small Rossby number. In particular, this approach gives as a corollary a constructive proof of the well-posedness of the problem of quasigeostrophic equations governing modons or Rossby solitons. The rotating Boussinesq equations consist of the Navier-Stokes equations with buoyancy-term and Coriolis-term in beta-plane approximation, the divergence-constraint, and a diffusion-type equation for the density variation. Thus the foundation for the study of the quasigeostrophic and rotating Boussinesq equations is the Navier-Stokes equations modified to accommodate the effects of rotation and stratification. They are considered in a plane layer with periodic boundary conditions in the horizontal directions and stress-free conditions at the bottom and the top of the layer. Additionally, the authors consider this model with Reynolds stress, which adds hyper-diffusivity terms of order 6 to the equations. This course focuses primarily on deriving the quasigeostrophic and rotating Boussinesq equations for geophysical fluid dynamics, showing existence and uniqueness of solutions, and outlining how Lyapunov functions can be used to assess energy stability. The main emphasis of the course is on Faedo-Galerkin approximations, the LaSalle invariance principle, the Wazewski principle and the contraction mapping principle of Banach-Cacciopoli. New understanding of quasigeostrophic turbulence called mesoscale eddies and vortex rings of the Gulf Stream and the Agulhas Current Retroflection could be helpful in creating better ocean and climate models.

Keywords

Wave-Current Interactions, Atmospheric and Oceanic Dynamics, Dynamical Systems and Turbulence, Lagrangian and Eulerian Analysis, Rotating Stratified Fluid Flows.

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CHAPTER 1

Introduction

1.1 Observations of mesoscale eddies and vortex rings

Mesoscale eddies in marine science are of particular interest to meteorologists and oceanographers because satellite-tracked surface drifters and analysis of observed motion of floats suggest the structures are an important mechanism for transport of salinity, kinetic energy, available potential energy and enstrophy, the latter being the integrated squared vorticity. The meticulously detailed observations of Lutjeharms *et al* [74, 73] reveal that heat and salinity exchange around the Agulhas Current Retroflection takes place through mesoscale ring detachment with an associated volume transport of approximately $0.5-1.5Sv$ ($1Sv=10^6 m^3 s^{-1}$). The consequent transport within these currents is undisputedly complex as validated by observed motion of convoluted spaghetti floats plots called *SOFAR* and *RAFOS*. These include such mesoscale vortices as the Gulf Stream rings.

In particular, it is evident from the investigations [68, 40, 18, 19, 55] that heat and salinity exchange around the Middle Atlantic Bight occurs through the mesoscale ring detachment of the Gulf Stream. The investigation of the flow patterns of such upwelling fronts are therefore of concern for biological studies of the highly productive ecosystems where nutrient budgets play a significant role. Furthermore, understanding the dynamics of exchange processes by mesoscale eddies is essential so that their effects in shelf-water transport can be accounted for, as observed by Joyce *et al* [45, 46, 47] in their utilization of hydrographic data and acoustic doppler current profiles to estimate total volume transport for a streamer of the Middle Atlantic Bight shelf water. Similarly, satellite-tracked dipole structure images of Hooker *et al* [40, 41, 42] provide additional details in the observation of mesoscale eddies. Specifically, advanced mesoscale observation techniques of Hooker *et al* [40, 41, 42] have shown long-period fluctuations in the ring eccentricity as well as vorticity and rotation rate of WCR82B, the Gulf Stream warm core ring that detached in February 1982. This important discovery serves as a paradigm that the ocean is full of mesoscale eddies that should be considered as part of a dynamically linked ring system. Nevertheless, the meager information available on the observations of mesoscale eddies using moored instruments and remote sensors made it possible to reveal various discrepancies between mathematical modelling of mesoscale eddies utilizing the system of nonlinear differential equations of geophysical fluid dynamics.

1.2 Modelling of mesoscale eddies and vortex rings

Since its inception in the 19th century through the efforts of Poincare and Lyapunov, the theory of dynamical systems addresses the qualitative behaviour of dynamical systems as understood from models. From this perspective, the modelling of dynamical processes in applications requires a detailed understanding of the processes to be analyzed. This deep understanding leads to a model, which is an approximation of the observed reality and is often expressed by a system of ordinary/partial, underdetermined (control), deterministic/stochastic differential or difference equations. While models are very precise for many processes, for some of the most challenging applications of dynamical systems (such as climate dynamics), the development of such models is notably difficult. In the derivation of fluid models one tries to make models as well-posed as possible, building into them behavior designed to achieve manifestation of order-chaotic processes and Lorenz-like butterfly effects. We proceed to note that the modelling of the dynamics of the ring systems or mesoscale and synoptic scale eddies is imperative due to limitations of current oceanographic observations. Explorations using satellite-tracking of vorticity structures is usually masked by heating of the surface layer and may not always be detected by satellite measurements. In the description of strategies utilized in observing geophysical phenomena, unsatisfactory results from oceanographic and meteorological interest, is the fact that deployment of floats such as

SOFAR and RAFOS is at the cutting edge of remote sensing technology and very expensive especially in the forecasting of mesoscale and synotic scale eddies. In this regard, the design and utilization of geophysical fluid dynamics models is imperative in supplementing costly observations and experiments described above in an effort of understanding the dynamics of mesoscale eddying coherent structures.

Over the past decades, there have emerged several elegant complementary approaches driving current research in vortex dynamics work, using models from geophysical fluid dynamics. The first approach involved construction of closed form solutions which delivered a dynamically consistent theory of geophysical vortical coherent structures. For example, there are two distinct and robust developments of exact close-form vortex solutions called modons. The building blocks of modons are sinusoidal and Bessel functions. Researchers developing to these solutions proliferated in the past decades and include the translating modons group directed by Flierl *et al* [20, 17, 18, 79] and the rotating modons group driven by Mied *et al* [62, 54, 53, 68]. These exact close-form vortex solutions have been used to simulate oceanic ring systems. In Lipphardt *et al* [68, 18, 55, 19, 61], rotating modon solutions are utilized to investigate WCR82B. Although closed form results compare well with oceanographic observations, this approach has fallen short of elucidating the problem due to challenges associated with nonlinearities of the system of partial differential equations. Moreover, most feature models for mesoscale eddies are quasigeostrophic which we derive later and from observational evidence cannot acount for ageostrophic motions which we derive later are significant in the calculation of ring systems quantities not directly measured, such as ring energies, enstrophy, and Lagrangian transport. The baroclinic dissipative quasigeostrophic equations govern the evolution of the streamfunction, denoted by ψ, whenever the Rossby number is asymptotically small and are given by

$$\begin{aligned}
&\frac{\partial q}{\partial t} + J(\psi, q) = \frac{Ek}{Ro}\triangle q \\
&\frac{\partial \rho}{\partial t} + J(\psi, \rho) = \frac{1}{Ed}\triangle \rho \\
&q = \triangle\psi - \frac{\partial^2 \psi}{\partial z^2} + \beta y \\
&\rho = -\frac{\partial \psi}{\partial z} \\
&u = -\frac{\partial \psi}{\partial y} \\
&v = \frac{\partial \psi}{\partial x}
\end{aligned} \tag{1.1}$$

where the nondimensional fields $\underline{u}$, q, ρ, and ψ are fluid velocity, potential vorticity, density and streamfunction, respectively. The geophysically relevant parameter Ro is the Rossby number, Ek is the Ekman number, β is the reference reciprocal Coriolis parameter. Here $J(.,.)$ is the Jacobian operator and the operator $\frac{\partial}{\partial t} + J(.,.)$ represents advection along fluid particle trajectories. Indeed, the translating modons of Flierl *et al* [20, 17, 18, 79] and the rotating modons of Mied *et al* [62, 54, 53, 68] represent closed form vortex solutions of the above quasigeostrophic potential vorticity equation when for example dissipation terms are neglected. Other useful simplifying assumptions include the replacement of vertical coordinate by density so that the quasigeostrophic equations may lend themselves to discretization in the vertical resulting in layered models. We remark here that the introduction of two-layer

quasigeostrophic models leads to the indepedence of horizontal velocity with respect to height and the phenomena is called *barotropic* which complement *baroclinic*, the change of horizontal velocity with height as in the above equations (1.1). In order to elucidate chaotic mixing of barotropic flows such as eddies and jets, Lagrangian tranport of the quasigeostrophic system is investigated utilizing the system

$$\frac{dx}{dt} = -\frac{\partial}{\partial y}\psi(x, y, t),$$

$$\frac{dy}{dt} = \frac{\partial}{\partial x}\psi(x, y, t),$$

where $\psi(x, y, t)$ is the streamfunction. Lagrangian transport, which is the lodestar in the analysis of transient and aperiodic geophysical phenomena, is defined with respect to the different regimes that arise in a flow. Fluid exchange between these regimes is examined by analyzing the unstable and stable invariant manifolds of certain distinguished parcel trajectories in the flow. By virtue of this technique, Pierrehumbert [80] analyzed solutions of the barotropic quasigeostophic equations with flow fields giving a large-amplitude meandering configuration and vortex motions.
Numerically generated velocity vector fields of the barotropic quasigeostrophic equations using a pseudospectral scheme were developed by Flierl *et al* [21, 81]. Similarly, the flow fields of their numerical results yield a meandering jet with vortical coherent structures. In Miller *et al* [63, 64], a perturbed current evolves nonlinearly into a large-amplitude meandering configuration that propagates zonally. Although there are two dominant time scales, the flow is aperiodic and dissipative, characteristics that challenge standard techniques in the theoretical investigation of geophysical phenomena. Nevertheless, the meager information available on the system of nonlinear differential equations of geophysical fluid dynamics made it possible to reveal various discrepancies between mathematical modelling of jets and mesoscale vortices and observations of jets and mesoscale vortices using moored instruments and remote sensors.

1.3 Prototype models for rotating Boussinesq and quasigeostrophic flows

In what follows, we briefly review the equations governing the dynamics of the ocean with emphasis on the rotating Boussinesq equations. The equations for the conservation of mass, energy, salt and momentum budget in a rotating framework of reference are simplified by the Boussinesq assumption which states that the effect of compressiblity is negligible in the balance equations, with the exception of the buoyancy term and equation of state. The resulting system of partial differential equations governs the flow of a viscous incompressible stratified fluid with the Coriolis force. Some preliminary remarks are offered on other geophysical fluid dynamics models that are derived from the equations of a viscous incompressible stratified fluid with the Coriolis force where use is made of a perturbation analysis of the flow fields with respect to a geophysically relevant parameter called the Rossby number, which compares advection to the Coriolis force, when we give a nondimensional formulation of the novel modelling system. At the zero order we obtain the geostrophic equations and in this limit the equations reduce to a balance between the Coriolis force and the pressure gradient. First-order terms in the Rossby number yield the quasigeostrophic equations that govern the evolution of geostrophic pressure. These remarks serve as a motivational proposal to name the flow of a viscous incompressible fluid with ambient rotation and density heterogeneity or stratification, the rotating Boussinesq equations. For reasons that will become clear in the sequel, mesoscale eddies and gyres describe geophysical eddying currents on the scale of at least the Rossby deformation radius. In the present context, Rossby deformation radius is defined as the distance covered by a wave travelling at a given speed during one inertial period. Prototype problems include dynamics of oceanic vortices, atmospheric vortex blocking, rings of the Agulhas Current, and the Gulf Stream [75, 74, 73, 68, 40, 18, 61] ring systems.

As a first effort, it is instructive for mesoscale eddies and gyres to consider the beta-plane approximation. The validity of the β-plane approximation is based on scale analysis and the plausible notion of mapping the curved earth's surface

onto a plane. Another consequence of the approximation is restriction to geophysical phenomena with length scales substantially smaller than
the radius of the earth [8, 13, 77]. Thus, each approximation or assumption is suitable for only a certain range of geophysical situations and the treatment that follows will not exhaustively cover all the possible situations. The beta-plane approximation to the Coriolis f and the reciprocal Coriolis parameter f_* is specified by the two-term Taylor series

$$f = f_0 + \beta_0 y$$
$$f_* = r_0\beta_0 - \frac{f_0}{r_0} y$$

where $f_0 = 2\varpi \sin\varphi_0$ is the Coriolis parameter at reference altitude φ_0; $y = (\varphi - \varphi_0)r_0$ is the coordinate oriented southward, ϖ is the angular rate for a rotating framework of reference, r_0 is the earth's radius, and $\beta_0 = 2\varpi r_0^{-1}\cos\varphi_0$ is the beta parameter. The inclusion of the distinguishing geophysical rotary term in the balance of momentum asserts that the inertial acceleration of geophysical fluids is the decomposition of the relative acceleration and and the Coriolis acceleration due to the rotating framework of reference. The significance of the Coriolis rotary term is in the generation of planetary or Rossby waves that support geophysical jets and mesoscale vortices. Thus we study the rotating Boussinesq equations describing the motion of a viscous incompressible stratified fluid in a rotating system which is relevant, e.g., for Lagrangian coherent structures. These equations consist of the Navier-Stokes equations with buoyancy-term and Coriolis-term in beta-plane approximation, the divergence-constraint, and a diffusion-type equation for the density variation. They are considered in a plane layer with periodic boundary conditions in the horizontal directions and stress-free conditions at the bottom and the top of the layer. Additionally, we consider this model with Reynolds stress, which adds hyper-diffusivity terms of order 6 to the equations.

1.4 Limitations of standard techniques

In the description of strategies utilized in modelling and observing geophysical phenomena, results are unsatisfactory due to challenges associated with nonlinearities of the systems from the mathematical analysis viewpoint, whereas from the oceanographic and meteorological interest, deployment of floats such as SOFAR and RAFOS is at the cutting edge of remote sensing technology and very expensive especially in the forecasting of mesoscale and synotic scale eddies. For example, since the quasigeostrophic equation is quadratic in the streamfunction, there is no general solution for this system of nonlinear partial differential equations. Additionally, the translating modons of Flierl *et al* [20, 17, 18, 79] and the rotating modons of Mied *et al* [62, 54, 53, 68] represent closed form vortex solutions of the quasigeostrophic model without dissipation, and as such the models have not accounted for the spin-down of mesoscale eddies.
These limitations lead suggest the need for alternative perspectives for studying solutions of the models. One such perspective is that of Lagrangian transport in oceanic flows, which has benefited from progress in dynamical systems theory. By courtesy of the dynamical systems approach, the trajectories of particles of fluid corresponding to quasigeostrophic flows is determined by the kinematic problem

$$\frac{dX(t)}{dt} = U(X(t),t), \quad X(0) = X_0,$$

where U is the velocity field satisfying the rotating Boussinesq equations and quasigeostrophic system of partial differential equations.
Other techniques anchored on the dynamical systems approach entail the well-posedness and existence of Lipschitz invariant manifolds for the resulting system of partial differential equations from geophysical fluid dynamics. With this

approach many properties of solutions to the feature models can be deduced without resorting to the cumbersome project of solving the rotating Boussinesq equations equations numerically. Moreover, well-posedness results anticipate that numerical approximations such as finite-difference schemes to the derivatives in the equations are convergent in the sense of the Lax equivalence theorem, i.e., a given finite-difference scheme to a well-posed initial-boundary value problem converges to the solution of of a partial differential equation with the rate of convergence specified by the order of accuracy of the finite-difference scheme.

Existence and uniqueness of solutions for the inviscid rotating Boussinesq equations is tackled by Bourgeois and Beale [9], who also demonstrate that when the Rossby number is asymptotically small, the inviscid quasigeostrophic solutions are accurate approximations of solutions of the inviscid rotating Boussinesq equations for small initial data. The essential part of their analysis and results is in establishing necessary and sufficient criteria for the suppression of fast-scale motions in inviscid rotating Boussinesq equations

Most feature models currently rely on quasigeostrophic dynamics which fortunately is amenable to slowly-modulated Hamiltonian systems theory. Thus, the splendid work of Bourgeois and Beale [9] on the inviscid ageostrophic dynamics serves as an improvement in those feature models that emphasize quasigeostrophic dynamics. For example, as by observed Joyce *et al* [45, 46, 47] in their utilization of moored instruments as well as Hooker *et al* [40, 41, 42] in the remote sensing of warm-core rings, the ring system has bimodal distribution of spin-down with short-lived mean of 54 days and a long-lived mean of 229 days. It is not clear, however, if such results qualitatively and quantitatively match those generated by dynamically consistent translating modons of Flierl *et al* [20, 17, 18, 79] and dynamically consistent rotating modons of Mied *et al* [62, 54, 53, 68]. In the splendid report of Lipphardt *et al* [68, 18, 55, 19, 61], it is argued that there is a need for dissipative eddy-permitting models for the examination of mesoscale eddies that spin-down under the action of viscosity.

1.5 The purpose of this research and innovation

Concrete analysis of the equations for ocean dynamics can assist in the long range to build a synthesis of techniques in the framework of the theory of dynamical systems. In order to elucidate ageostrophic dynamics, it is desirable for the theory of dynamical systems to recognize and be be able to supplement, or in some cases even supplant, the expensive observational and experimental framework of oceanography developed. In this section we give the basic setup for our problems and state the main results. It should be noted at the outset, that the questions of well-posedness, stability, attractors and invariant manifolds for non-homogeneous Navier-Stoke type equations already commands a large body of work that will be utilized in the nonlinear analysis of the rotating Boussinesq equations.

We address in a concrete way dynamical challenges for the rotating Boussinesq equations. The problem of rotating Boussinesq equations continue to be of interest for various purposes and the thesis aims are therefore as follows:

- A sysnthesis of wellposedness, stability, attractors and invariant manifolds anticipated in ageostrophy;
- The problem of ageostrophic flows is among noteworthy and accessible mathematical problems and continue to have oceanographic and meteorogical interest especially for the modelling and forecasting of mesoscale and synoptic scale eddies;
- The improvement beyond quasigeostrophy is imperative since the Rossby number need not be asymptotically small especially in situations such as the Gulf Stream ring systems and the Agulhas Current Retroflection;
- The rationale of the Boussinesq and the hydrostatic approximations as well as the assumption of β-plane approximation serve as paradigmatic models that validate and guide theoretical, computational and experimental research on dynamical systems.

Further new accomplishments of this investigation entail the proof that the general three-dimensional space variables initial-boundary value problem for the rotating Boussinesq equations with Reynolds stress has a unique solution in the large that enjoys useful properties such as differentiability with respect to initial conditions. Furthermore, we illustrate

that uniqueness and continuity with respect to initial conditions of the solution $U(t)$ of the initial-value problem generate a dynamical system provided by continuous solution operators $S(t)$, $S(t)$, $t \in \mathbb{R}_+$. We further prove that the solution operators $S(t)$, $t \in \mathbb{R}_+$ are injective and as a result we obtain the solution operators $S(t)$ defined for all time $t \in \mathbb{R}$. The crucial properties of the solution operators which we establish are as follows:

$$S(t)U_0 = U(t) \equiv S(U_0, t)$$

satisfying the *group property*

$$S(t)U(s) = S(s)U(t)$$
$$S(0)U = U \quad \forall \ t, s \in \mathbb{R}.$$

The basic ingredients for the proofs are the Faedo-Galerkin method, the Lebesgue dominated convergence theorem and the mini-max principle. The estimates are presented in terms of geophysically relevant parameters. As regards to the nonlinear stability of the solutions of the initial-boundary value problem for the rotating Boussinesq equations, according to more refined a priori estimates that we develop, one of the criterion necessary and sufficient for nonlinear stability of the rest state is achieved with the flow energy and entropy production provided with

$$E(t) = \frac{1}{2}\{\| U\|^2 + \frac{Ek}{Ro}\| \bigtriangledown \underline{u}\|^2 + \frac{1}{Ed}\| \bigtriangledown \rho\|^2 + 2R\int_\Omega \rho w dx\}$$

which is specification of energy and the entropy production in the Sobolev H^1-norm whenever $Ga \le R < (\frac{1}{Ro} + \frac{Ro}{Fr^2})$ is valid. For instance, the stability criterion number Ga is given by $Ga = 80$ as in the rigorous investigations of Galdi *et al* [28, 27].

It is crucial to note that the techniques employed in developing the nonlinear stability criteria have been completed without carrying out closed form wave solutions for mesoscale and synoptic eddies. The presence of stratification and Reynolds stress introduces some new considerations that need special attention since in this more general setting, exact close-form solutions are not easily accessible. Thus, some cases of oceanographic or meteorological interest such as vortex spin-down, the decay of energy in the the nonlinear stability criteria need not account for the dissipation of energy and enstrophy for mesoscale and synoptic wave motions. However, we consider the energetic stability criteria as a first effort towards elucidating such geophysical phenomena. Furthermore, we observe that nonlinear stability criteria may give the possibility of a more rigorous investigation of the numerical approximation of the solution in the sense of the Lax equivalence theorem. One of the advantages of the results presented in this thesis is that they may be considered as more refined generalization for the dissipation of the flow energy and the entropy production provided by

$$E(t) = \frac{1}{2}\| U(t)\|^2 = \frac{1}{2}\int_\Omega (\underline{u}^2 + \rho^2) dx,$$

which is specification of energy and the entropy production in the L^2-norm.

In the proof of results for the attractor of the general three-dimensional space variables initial-boundary value problem for the rotating Boussinesq equations with Reynolds stress we infer from more refined a priori estimates that the attractor is given by the ω-limit set of $Q = B_{2\rho_2}$,

$$A = \omega(Q) = \cap_{s \ge 0} Cl(\cup_{t \ge s} S(t)Q).$$

where $B_{2\rho_2}$ denotes an open ball in phase space of radius $2\rho_2$, which depends on geophysically relevant parameters. Utilitiy of the group property and continuity of of the solution operators $S(t)$ defined for all time $t\in \R$, gives the following invariance property of the above established attractor:

$$S(t)A = A, \ \forall t \in \mathbb{R}.$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the general three-dimensional space variables initial-boundary value problem for the rotating Boussinesq equations with Reynolds stress is both positively and negatively invariant and consists of orbits or trajectories that are defined for all $t \in \mathbb{R}$.

CHAPTER 2

Models for rotating Boussinesq equations

2.1 Models for rotating Boussinesq equations

2.1.1 Description of rotating Boussinesq equations

Rotating Boussinesq equations

The objective of this investigation is to develop prototype geophysical fluid dynamics models as the first effort towards understanding the impact of rotating Boussinesq equations in the ocean. First, we review the fundamental assumptions and techniques involved in the derivation of a system of partial differential equations governing geophysical fluid flows [8, 13, 77]. The evolution equations for the quantities of interest, which may be scalar or vector or tensor valued are derived with the assistance of Reynolds' transport theorem, conservation laws and constitutive assumptions [5, 8, 32]. The basic fields in the description of the motion and states of geophysical phenomena are the velocity, the pressure, the density, the temperature and the salinity. The equations governing these fields consists of the mass balance or continuity equation, the momentum equation, the energy equation and the equation for salt budget.

Consider a geophysical fluid occupying a region $B \in \mathbb{R}^3$ as depicted in figure 2.1 for the benchmark Lagrangian coherent structures utilizing the DsTool package program of Worfolk, Guckenheimer *et al*, [109] which greatly helped codify the discipline.

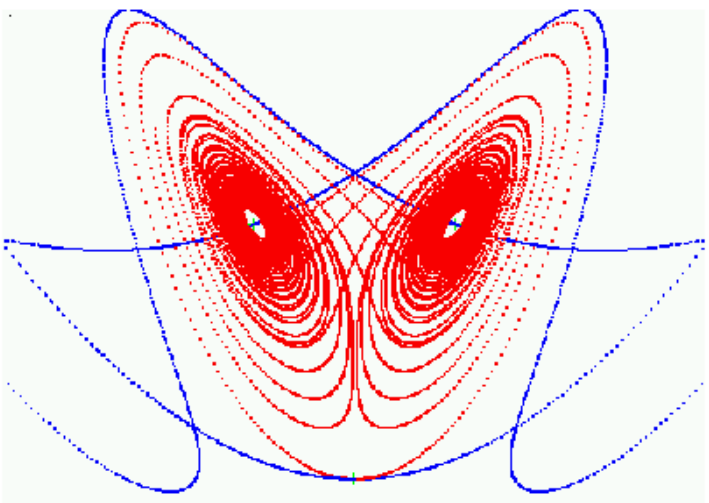

Figure 2.1 Lorenz chaotic attractor

For clarity, conventional vector, tensor and indicial notation are utilized. Also, for convenience in all that follows, we will adopt a Cartesian coordinate system on a spherical rotating earth. The x-axis is directed westward, the y-axis is southward and the z-axis is oriented upward.

Heuristically, Reynolds' transport theorem asserts that the rate at which the integral of density for a geophysical fluid parcel over $V(t)$ is changing is equal to the sum of the rate performed with $V(t)$ fixed in its current position and the

rate at which density is transported out of the fluid parcel $V(t)$ across its boundary. Consequently, the use of the divergence theorem to transform the surface integral into a volume integral gives the equation

$$\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho + \rho \nabla.\underline{u} = 0, \tag{2.1}$$

for mass balance where $\underline{u} = (u, v, w)$ is Eulerian velocity of the fluid, ρ is density, t is time, and $\underline{x} = (x, y, z)$ is the coordinate system. Equation 2.1 is the continuity equation for compressible fluids. Similarly, in order to obtain the *equation of motion* we utilize Reynolds' transport theorem, and the conservation hypothesis that the rate of change of momentum transported in a geophysical fluid parcel of volume $V(t)$ is balanced by the resultant of both body and surface forces. Concerning the contact or surface forces, as in the analysis of the relative motion of a fluid near a point and Cauchy's constitutive hypothesis, we represent surfaces forces by the stress tensor σ. Further, it is instructive to decompose the velocity gradient $\nabla \underline{u}$ into the superposition of symmetric rate-of-deformation tensor D and the skew-symmetric vorticity tensor Ω defined by

$$\begin{aligned} D &= \frac{1}{2}(\nabla \underline{u} + \nabla^T \underline{u}) \\ \Omega &= \frac{1}{2}(\nabla \underline{u} - \nabla^T \underline{u}). \end{aligned} \tag{2.2}$$

Next, we consider the Cauchy's stress tensor σ using the constitutive assumption

$$\sigma = -p + 2\mu(D - \frac{1}{3}\nabla.\underline{u}) \tag{2.3}$$

for a Newtonian viscous compressible fluid. Here μ is dynamic viscosity and p is pressure. In addition, consideration of the important ambient rotation of geophysical fluids and application of the simplifying effect of the continuity equation yield the following equation for the conservation of linear momentum customarily called the equation of motion

$$\rho(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u} + g\underline{k}) = -\nabla p + \mu \triangle \underline{u} + \frac{\mu}{3}(\nabla.\underline{u}), \tag{2.4}$$

where g is gravitation, $\underline{f} = (0, f, f_*)$ is the earth's rotation, f is Coriolis parameter and f_* is reciprocal Coriolis parameter.

Scale analysis ensures consideration of the meriodinal change in the Coriolis parameter and the reciprocal Coriolis parameter. According to the investigations [13, 77], nonlinear Rossby waves and vortex coherent structures such as alternating cyclones and anticyclones of the Gulf stream and Agulhas current span numerous degrees of latitude; and for them, it is imperative to consider the meridional change in the Coriolis parameter f and the reciprocal Coriolis parameter f_*. The β-plane approximation to the Coriolis f and the reciprocal Coriolis f_* is obtained by considering $\frac{y}{r_0}$ as sufficiently small and invoking the 2-term Taylor series

$$\begin{aligned} f &= f_0 + \beta_0 y, \\ f_* &= r_0\beta_0 - \frac{f_0}{r_0} y, \end{aligned} \tag{2.5}$$

where $f_0 = 2\varpi \sin\varphi_0$ is the Coriolis parameter at reference altitude φ_0; $y = (\varphi - \varphi_0)r_0$ is the coordinate oriented southward, ϖ is the angular rate for rotating framework of reference, r_0 is the earth's radius and $\beta_0 = 2\varpi r_0^{-1}\cos\varphi_0$ is the beta parameter. We note that the β-plane approximation is valid whenever the term $\beta_0 y$ is small compared to the leading term f_0. The Cartesian coordinate system where the beta term $\beta_0 y$ is not retained is called the f-plane approximation, that is,

$$\begin{aligned} f &= 2\varpi \sin\varphi_0, \\ f_* &= 2\varpi \cos\varphi_0. \end{aligned} \tag{2.6}$$

The inclusion of the distinguishing geophysical rotary term in the balance of momentum asserts that the inertial acceleration of geophysical fluids is the decomposition of the relative acceleration and and the Coriolis acceleration due to the rotating framework of reference. The significance of the Coriolis rotary term is in the generation of planetary or Rossby waves that support geophysical jets and mesoscale vortices. And the validity of the approximation (2.5) is a consequence of restriction to geophysical phenomena with length scales substantially smaller than the radius of the earth [8, 13, 77]. Consequently, no appeal will be made in this investigation to the spherical geometry of the earth which in curvilinear coordinate system contain challenging extraneous curvature terms. Formal passage from the β-plane approximation to retain the extraneous curvature terms corresponding to the full geometry of the spherical rotating earth is treated in [8, 77]. The mathematical analysis of the resulting geophysical fluid equations with extraneous curvature terms is examined in Lions *et al* [105, 104].

Equations (2.1)-(2.4) are supplemented by equation of state, energy and salt budgets. In order to derive these results, utility of the first law of thermodynamics and Reynolds's transport theorem assert that the rate of change of the internal energy supplied to a geophysical fluid parcel is balanced by the heat out of the fluid parcel and the power or work done by the system against external forces. Thus, by consideration of the power produced by the surfaces forces and Fourier's consitutive hypothesis for the rate of heat and subsequent application of the continuity equation give the following temperature form of the energy equation

$$\rho C_v\left(\frac{\partial T}{\partial t} + \underline{u}.\nabla T\right) + p\nabla.\underline{u} \;=\; \kappa_T \triangle T + \mu\nabla.(\underline{u}.\nabla\underline{u}) - \frac{2}{3}\mu(\nabla.\underline{u})^2 \tag{2.7}$$

An alternative derivation of the energy equation may be achieved by exploiting the enthalpy relation [5].
The equations must be completed with the addition of the state equation

$$\rho \;=\; \rho_0[1 - \alpha_T(T - T_0) + \alpha_S(S - S_0)] \tag{2.8}$$

and the following salt budget

$$\frac{\partial S}{\partial t} + \underline{u}.\nabla S = \kappa_S \triangle S \tag{2.9}$$

which asserts that ocean water conserve their salt content except in the presence of diffusion. In the evolution equations (2.7)-(2.9), \$T\$ is absolute temperature, C_v is heat capacity, κ_T is thermal conductivity, S is salinity, T_0, S_0, ρ_0 are reference values of temperature, salinity and density, α_T is coefficient of thermal expansion, α_S is coefficient of saline contraction and κ_S is coefficient of saline diffusion. The coupled equations (2.1)-(2.9) yield the following system of partial differential equations governing general heterogenous compressible geophysical fluid flows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho + \rho \nabla.\underline{u} = 0, \\
&\rho(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u} + g\underline{k}) = -\nabla p + \mu \Delta \underline{u} + \frac{\mu}{3}(\nabla.\underline{u}), \\
&\rho C_v(\frac{\partial T}{\partial t} + \underline{u}.\nabla T) + p\nabla.\underline{u} = \kappa_T \Delta T + \mu \nabla.(\underline{u}.\nabla \underline{u}) - \frac{2}{3}\mu(\nabla.\underline{u})^2, \\
&\rho = \rho_0[1 - \alpha_T(T - T_0) + \alpha_S(S - S_0)], \\
&\frac{\partial S}{\partial t} + \underline{u}.\nabla S = \kappa_S \Delta S.
\end{aligned} \tag{2.10}$$

Next, the equations for the conservation of mass, momentum, energy and salt budget are simplified by the Boussinesq assumption which states that the effect of compressiblity is negligible in the balance equation except in the buoyancy term and the equation of state. Employing the additive decomposition

$$\rho = \rho_0 + \acute{\rho}(x, y, z, t), \quad \acute{\rho} \ll \rho_0 \tag{2.11}$$

in (2.1) and retaining terms multiplied by ρ_0, we obtain

$$\nabla.\underline{u} = 0 \tag{2.12}$$

which is the continuity equation for an incompressible fluid. Physically, this statement means that conservation of mass has become conservation of volume. Another implication is the elimination of sound waves. Substitution in (2.10) of the additive decompositions (2.11) and

$$p = p_0(z) + \acute{p}(x, y, z, t)$$

with $p_0(z) = P_0 - \rho_0 gz$ being the hydrostatic pressure, taking into account the continuity equation (2.12) and retaining dominating terms, we obtain the set of equations for a general heterogeneous incompressible geophysical fluid flows:

$$\begin{aligned}
&\nabla.\underline{u} = 0, \\
&\rho_0(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u}) + g\rho\underline{k} = -\nabla p + \mu \Delta \underline{u}, \\
&\rho_0 C_v(\frac{\partial T}{\partial t} + \underline{u}.\nabla T) = \kappa_T \Delta T, \\
&\rho = \rho_0[1 - \alpha_T(T - T_0) + \alpha_S(S - S_0)], \\
&\frac{\partial S}{\partial t} + \underline{u}.\nabla S = \kappa_S \Delta S.
\end{aligned} \tag{2.13}$$

We set $\kappa_{\bar{T}} = \frac{\kappa_T}{\rho_0 C_v}$, and $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity.

In order to circumvent the challenges of directly accounting for molecular diffusion and oceanic salt finger pattern formations that result from the competitive effects of the diffusivities of heat and salt budgets, we examine the case where the salt and heat diffusivities are assumed to be equal to the eddy diffusivity, that is, $\kappa = \kappa_T = \kappa_S$. The choice of the eddy diffusivity [13, 77], which forms the basis of the model analyzed here, incorporates the ubiquitous geophysical phenomena that temperature, density and salinity structures of the ocean are influenced primarily by chaotic transport and mixing within jet streams and geophysical eddying currents on the scale of the Rossby deformation radius. The

Rossby deformation radius is defined as the distance covered by a wave travelling at a given speed during one inertial period. The existence and persistence of jet streams and other geophysical coherent structures in the ocean make plausible that molecular diffusion is weak to be directly significant in the evolution equations governing the observed organized motions of a viscous incompressible stratified fluid with the Coriolis force.

Application of (2.11) into the equation of state (2.8) and appying the operators $\triangle$ and $\frac{\partial}{\partial t}+\underline{u}.\triangledown$ and taking (2.13) in consideration yield

$$\frac{\partial \rho}{\partial t}+\underline{u}.\triangledown \rho = \kappa \triangle \rho. \tag{2.14}$$

Thus, dropping the primes from ρ and p, the above Boussinesq approximation gives the following system of primitive partial differential equations

$$\begin{aligned} &\rho_0(\frac{\partial \underline{u}}{\partial t}+\underline{u}.\triangledown \underline{u}+\underline{f}\times\underline{u})+g\rho\underline{k}=-\triangledown p+\mu\triangle\underline{u},\\ &\triangledown.\underline{u}=0,\\ &\frac{\partial \rho}{\partial t}+\underline{u}.\triangledown\rho=\kappa\triangle\rho. \end{aligned} \tag{2.15}$$

These equations govern the flow of a viscous incompressible stratified fluid with the Coriolis force. Some preliminary remarks are offered on other geophysical fluid dynamics models that are derived from the equations (2.15) where use is made of a perturbation analysis of the flow fields at the Rossby number on the order of unity or less. In particular, at the zero-order we obtain the geostrophic equations and in this limit the rotating Boussinesq equations reduces to a balance between the Coriolis force and the pressure gradient. Subsequently, first-order terms in the Rossby number yield the quasigeostrophic equations that govern the evolution of geostrophic pressure.

Next, concerning the the boundary conditions we assume the fields to be periodic along the horizontal coordinates x and y with periods L_1 and L_2, respectively. The flow domain is assigned on an open rectangular periodic region $B=\{(x,y,z)\in(0,L_1)\times(0,L_1)\times(0,h)\}$ with boundary $\Gamma=\{z=0,h\}$. The domain \$B\$ represents the flow region and the surfaces $z=0$ and $z=h$ represent the lower and upper boundaries of the ocean, respectively. To system (2.15) we prescribe boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=0 \ \ at \ \ \Gamma,\\ &\rho(z=0)=\rho_l\\ &\rho(z=h)=\rho_u \end{aligned} \tag{2.16}$$

where ρ_l and ρ_u are constants. In order to eliminate rigid motions we consider the case

$$\int_B u dx=\int_B v dx=0.$$

We introduce the following conducting solutions

$$\rho=\bar{\rho}-\rho_l+\frac{z}{2}(\rho_u-\rho_l)$$

$$p=\bar{p}+z\rho_l g-\frac{g}{4}z^2(\rho_u-\rho_l)$$

for density and pressure, respectively. Substitution in (2.15) lead to the system of partial differential equations

$$\begin{aligned} &\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u} + \frac{g}{\rho_0}\rho \underline{k} = -\frac{1}{\rho_0}\nabla p + \nu \triangle \underline{u}, \\ &\nabla . \underline{u} = 0, \\ &\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho + (\rho_u - \rho_l)\underline{u}.\underline{k} = \kappa \triangle \rho. \end{aligned} \tag{2.17}$$

Rotating Boussinesq equations with Reynolds stress

Since its inception in the 19th century through the efforts of Poincare and Lyapunov, the theory of dynamical systems addresses the qualitative behaviour of dynamical systems as understood from models. From this perspective, the modelling of dynamical processes in applications requires a detailed understanding of the processes to be analyzed. This deep understanding leads to a model, which is an approximation of the observed reality and is often expressed by a system of ordinary/partial, underdetermined (control), deterministic/stochastic differential or difference equations. While models are very precise for many processes, for some of the most challenging applications of dynamical systems (such as climate dynamics), the development of such models is notably difficult. We embark on the the following additive decomposition of flow fields for the rotating Boussinesq equations (2.15):

$$\begin{aligned} \underline{u} &= < \underline{u} > + \underline{u}', \\ p &= < p > + p', \\ \rho &= < \rho > + \rho'. \end{aligned} \tag{2.18}$$

Here the time-averaging operator $<.>$ is defined by

$$< \underline{u} >= \frac{1}{\tau}\int_{\tau} \underline{u}(\underline{x})\chi(\underline{x})d\underline{x}$$

with τ being a characteristic evolution time-scale and $\chi(\underline{x})$ a probability density function. The terms in the splitting (2.18) represent coherent and incoherent flow fields, respectively. Furthermore, we assume that the incoherent velocity field satisfies

$$< \underline{u}' >= 0.$$

The formal additive decomposition of the flow fields of the rotating Boussinesq equations into coherent and incoherent terms is justified partly by the realization that solutions to the primitive equations of geophysical fluid dynamics contain both fast and time scales with frequencies proportional to the Rossby number.

Employing the decomposition (2.18) in the rotating Boussinesq equations (2.15) and invoking the time-averaging operator, the system

$$\begin{aligned} &\rho_0(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u}) + g\rho \underline{k} = -\nabla p + \mu \triangle \underline{u} + \nabla . A \\ &\nabla . \underline{u} = 0 \\ &\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho = \kappa \triangle \rho + \nabla . W. \end{aligned} \tag{2.19}$$

for coherent flow fields in which A and W are Reynolds stress [77] fields. The presence of the effective dissipative terms $\nabla . A$ and $\nabla . W$ implies that although the incoherent flow fields have zero average, the momentum and density

fluxes of the fluctions, which are quadratic in the incoherent fluctions, need not vanish when the time-averaging operator is employed. As far as the contributions of the incoherent flow fields are concerned, we observe that a prototypical fluid parcel with fluctuation velocity u' in the x-direction will transport z-momentum $\rho_0 w'$ with a nonzero flux across the surface given by the quadratic relation $-\rho_0 < u' w' >= A_{xz}$. It is similarly convenient to introduce the following quadratic protocols for other Reynolds stress fields

$$
\begin{aligned}
A_{xx} &= -\rho_0 < u' u' >, \\
A_{yy} &= -\rho_0 < v' v' >, \\
A_{zz} &= -\rho_0 < w' w' >, \\
A_{yx} &= -\rho_0 < u' v' >= A_{xy}, \\
A_{zx} &= -\rho_0 < u' w' >= A_{xz}, \\
A_{zy} &= -\rho_0 < v' w' >= A_{yz}, \\
W_x &= -\rho_0 < u' \rho' >, \\
W_y &= -\rho_0 < v' \rho' >, \\
W_z &= -\rho_0 < w' \rho' >.
\end{aligned}
\tag{2.20}
$$

It is useful from the oceanographic and meteorological perspectives to idealize the Reynolds stress fields as a representation of wind stress quantities. To be more specific, we consider the wind stress comprising primarily the trades that is communicated to ocean water through absorption of a shear stress and is responsible for driving oceanic circulations in the form of jets and eddying currents that impact the earth's weather [13, 77]. It is important to realize that the additive decomposition of the ageostrophic flow quantities into coherent and incoherent terms and consideration of the averaging operator to obtain the Reynolds stress fields or wind stress fields have resulted in a set of evolution equations that are not closed. We adopt the following Pedlosky closure protocols [77] for the Reynolds stress fields (2.20):

$$
\begin{aligned}
A_{xx} &= -2\varepsilon\rho_0 \triangle^5 \frac{\partial u}{\partial x}, \\
A_{yy} &= -2\varepsilon\rho_0 \triangle^5 \frac{\partial v}{\partial y}, \\
A_{zz} &= -2\varepsilon\rho_0 \triangle^5 \frac{\partial w}{\partial z}, \\
A_{yx} &= -\varepsilon\rho_0 \triangle^5 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = A_{xy}, \\
A_{zx} &= -\varepsilon\rho_0 \triangle^5 (\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}) = A_{xz}, \\
A_{zy} &= -\varepsilon\rho_0 \triangle^5 (\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y}) = A_{yz},
\end{aligned}
\tag{2.21}
$$

and

$$
\begin{aligned}
W_x &= -\delta(\frac{\partial^2 u}{\partial x^2} + \triangle^5 \frac{\partial \rho}{\partial x}), \\
W_y &= -\delta(\frac{\partial^2 v}{\partial y^2} + \triangle^5 \frac{\partial \rho}{\partial y}), \\
W_z &= -\delta(\frac{\partial^2 w}{\partial z^2} + \triangle^5 \frac{\partial \rho}{\partial z}),
\end{aligned}
\tag{2.22}
$$

where ε and δ are parameters. The geophysical motivation of the choice of the closure protocols (2.21-2.22) will become clear in the sequel when we give a mathematical treatment of the resulting evolution equations. As a validation from a dynamical systems approach, the closure protocols lead to well-posedness in the sense of Hadamard and the existence of attractors for the solution of the system of partial differential equations. The closure protocols (2.21-2.22) are the simplest which allows existence of stability and attraction. Another validation desirable from the craft [35, 31] of stability and attraction follows from the fact that a, siginificant quantity in analyzing whether the solution of an evolution equation is bounded in some suitable norm is that of Lyapunov functional of the system with interpretations such as fictitious energy, enstrophy and entropy production which are decreasing along solutions. The equations governing the flow of ageostrophic equations with Reynolds stress (2.21-2.22) induce damping mechanisms in the nature of viscosity, diffusion, stratification and rotation effects that manifest themselves in the evolution equation with the existence of Lyapunov functionals which are equivalent to some norm induced by the inner product of solution.

We now proceed with the derivation of our geophysical fluid dynamics model. Employing the Pedlosky closure hypothesis into the equations (2.19) for coherent flow fields gives the following primitive partial differential equations governing rotating Boussinesq equations with Reynolds stress:

$$
\begin{aligned}
&\rho_0(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u} + \varepsilon \triangle^6 \underline{u}) + g\rho\underline{k} = -\nabla p + \mu \triangle \underline{u}, \\
&\nabla.\underline{u} = 0, \\
&\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho + \delta \triangle^6 \rho = \kappa \triangle \rho.
\end{aligned}
\tag{2.23}
$$

The presence of effective dissipative terms $\triangle^6 \underline{u}$ and $\triangle^6 \rho$ is equivalent to stress due to the impact of the incoherent terms on the coherent flow. Alternatively, the dissipative terms account for the wind stress along the ocean surface. Next, concerning the boundary conditions, we assume the fields to be periodic along the coordinates x, y and z with periods L_1, L_2 and L_2, respectively. The flow domain is assigned on an open rectangular periodic region

$$B = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, L_3)\}.$$

To system (2.23) we prescribe the following space-periodic boundary conditions for the flow fields

$$\begin{aligned} \underline{u}(\underline{x} + Le_i, t) &= \underline{u}(\underline{x}, t) \\ \rho(\underline{x} + Le_i, t) &= \rho(\underline{x}, t) \\ p(\underline{x} + Le_i, t) &= p(\underline{x}, t) \end{aligned} \tag{2.24}$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$ and with the simplifying assumption that the period $L = L_1 = L_2 = L_3$. We further assume the derivatives of $\underline{u}$, ρ and p are also space-periodic. In order to eliminate rigid motions we consider the case where the average velocity and pressure vanish

$$\int_B \underline{u}(\underline{x}, t) dx = 0,$$
$$\int_B p(\underline{x}, t) dx = 0.$$

Substitution in (2.23) reduce to the following:

$$\begin{aligned} &\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} + \underline{f} \times \underline{u} + \frac{g}{\rho_0} \rho \underline{k} + \varepsilon \triangle^6 \underline{u} = -\frac{1}{\rho_0} \nabla p + \nu \triangle \underline{u}, \\ &\nabla . \underline{u} = 0, \\ &\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho + (\rho_u - \rho_l) \underline{u}.\underline{k} + \delta \triangle^6 \rho = \kappa \triangle \rho. \end{aligned} \tag{2.25}$$

2.1.2 The initial-boundary value problems

Rotating Boussinesq equations

In this section we introduce the nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force without Reynolds stress. In order to achieve this objective, the system of partial differential equations and boundary conditions are made nondimensional with the following length and velocity scales of motions in which rotation and stratification effects are essential:

$$x = L\bar{x}, y = L\bar{y}, z = H\bar{z}, t = \frac{L}{U}\bar{t}$$

$$u = U\bar{u}, v = U\bar{v}, w = \frac{UH}{L}\bar{w}, L_1 = L\bar{L_1}$$

$$L_2 = L\bar{L_2}, \rho = \frac{1}{gH} f_0 \rho_0 UL\bar{\rho}, p = f_0 \rho_0 UL\bar{p}$$

$$\rho_u - \rho_l = \frac{1}{g} N^2 \rho_0, \bar{\beta} = \frac{L \cos \varphi_0}{r_0 Ro \sin \varphi_0}$$

where $T = \frac{L}{U}$ represent the time scale, U, represent the horizontal velocity scale, W represent the vertical velocity scale, L represent the length scale, H represent the vertical length scale, etc. Distinguishing attributes of

geophysical fluid dynamics are due to the effects of rotation, density heterogeneity or stratification and scales of motion. To ensure the measure of the effect of variations of density defined by stratification frequency

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz}$$

is real(which implies static stability of the stratified fluid), we assume $\frac{d\rho}{dz} < 0$ for $0 \le z \le h$. It is known that an unstable density stratification leads to rapid convective motions. Employing the beta-plane approximation and substituting the above scaling into the system (2.17) and omitting bars from nondimensional quantities, we obtain the following nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid with the Coriolis force:

$$Ro(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u}) - (1 + yRo\beta_0)\underline{u} \times \underline{k} + \rho \underline{k} = -\nabla p + Ek \triangle \underline{u},$$
$$\nabla . \underline{u} = 0,$$
$$\frac{Fr^2}{Ro}(\frac{\partial \rho}{\partial t} + \underline{u}.\nabla \rho) + \underline{u}.\underline{k} = \frac{Fr^2}{Ro(Ed)} \triangle \rho,$$

where we set the aspect ratio $\frac{H}{L}$ to unity. The evolution equations have been scaled so that the relative order of each term is measured by the dimensionless parameter multiplying it. Here the parameter $Ro = \frac{U}{Lf_0}$ is the Rossby number which compares the inertial term to the Coriolis force; $Fr = \frac{U}{NH}$ is the Froude number which measures the importance of stratification; $Ek = \frac{\nu}{H^2 f_0}$ measures the relative importance of frictional dissipation; $Ed = \frac{\nu}{f_0 N^2}$ is the nondimensional eddy diffusion coefficient. The ratio $\frac{Fr^2}{Ro}$ measures the significant influence of both rotation and stratification to the dynamics of the flow. The corresponding initial-boundary conditions are specified as follows: We make the vertical density boundary conditions homogeneous using

$$\rho = \bar{\rho} + \rho_l + \frac{1}{g} z \rho_0 N^2$$

$$p = \bar{p} - z\rho_l - \frac{1}{2g} z^2 \rho_0 N^2$$

Omitting bars the conditions (2.16) become

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \rho = 0 \ \ at \ \ \Gamma,$$
$$\int_B u dx = \int_B v dx = 0.$$

And the flow domain reduces to $B = \{(x, y, z) \in (0, L) \times (0, L) \times (0,1)\}$ with boundary $\Gamma = \{z = 0,1\}$. Furthermore,velocity and density are given at initial time $t = 0$

$$\underline{u}(\underline{x},0)=u_0(\underline{x})$$
$$\rho(\underline{x},0)=\rho_0(\underline{x}).$$

We conclude this section with recapitulation of the set of equations that will be analyzed in this investigation. The evolutionary equation for the β-plane rotating Boussinesq equations governing geophysical fluid flows is the following initial-boundary value problem:

$$Ro(\frac{\partial \underline{u}}{\partial t}+\underline{u}.\bigtriangledown \underline{u})-(1+yRo\beta_0)\underline{u}\times\underline{k}+\rho\underline{k}=-\bigtriangledown p+Ek\bigtriangleup\underline{u},$$
$$\bigtriangledown.\underline{u}=0, \tag{2.26}$$
$$\frac{Fr^2}{Ro}(\frac{\partial \rho}{\partial t}+\underline{u}.\bigtriangledown\rho)+\underline{u}.\underline{k}=\frac{Fr^2}{Ro(Ed)}\bigtriangleup\rho,$$

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=\rho=0 \ \ at \ \ \Gamma, \tag{2.27}$$
$$\int_B udx=\int_B vdx=0.$$
$$\underline{u}(\underline{x},0)=u_0(\underline{x})$$
$$\rho(\underline{x},0)=\rho_0(\underline{x}).$$

Rotating Boussinesq equations with Reynolds stress

In this section we introduce the nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force with Reynolds stress. In order to achieve this objective, the system of partial differential equations and boundary conditions are made nondimensional with the following length and velocity scales of motions in which rotation and stratification effects are essential:

$$x=L\bar{x}, y=L\bar{y}, z=H\bar{z}, t=\frac{L}{U}\bar{t}$$
$$u=U\bar{u}, v=U\bar{v}, w=\frac{UH}{L}\bar{w}, L_1=L\bar{L}_1$$
$$L_2=L\bar{L}_2, \rho=\frac{1}{gH}f_0\rho_0UL\bar{\rho}, p=f_0\rho_0UL\bar{p}$$
$$\rho_u-\rho_l=\frac{1}{g}N^2\rho_0, \bar{\beta}=\frac{L\cos\varphi_0}{r_0Ro\sin\varphi_0}.$$

Employing the beta-plane approximation and substituting the above scaling into the system (2.17) and omitting bars from nondimensional quantities, we obtain the following nondimensional form of the equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force with Reynolds stress:

$$Ro(\frac{\partial \underline{u}}{\partial t}+\underline{u}.\bigtriangledown \underline{u})-(1+yRo\beta_0)\underline{u}\times\underline{k}+\rho\underline{k}+Ek\bigtriangleup^6\underline{u}=-\bigtriangledown p+Ek\bigtriangleup\underline{u},$$
$$\bigtriangledown.\underline{u}=0,$$
$$\frac{Fr^2}{Ro}(\frac{\partial \rho}{\partial t}+\underline{u}.\bigtriangledown\rho)+\underline{u}.\underline{k}+\frac{Fr^2}{Ed(Ro)}\bigtriangleup^6\rho=\frac{Fr^2}{Ro(Ed)}\bigtriangleup\rho,$$

where we set the aspect ratio $\frac{H}{L}$ to unity. The evolution equations have been scaled so that the relative order of each term is measured by the dimensionless parameter multiplying it.

The corresponding initial-boundary conditions are specified as follows: The space-periodicity of the fields and the vanishing of the average velocity and pressure are represented by

$$\begin{aligned}
\underline{u}(\underline{x}+Le_i,t) &= \underline{u}(\underline{x},t),\\
\rho(\underline{x}+Le_i,t) &= \rho(\underline{x},t),\\
p(\underline{x}+Le_i,t) &= p(\underline{x},t),\\
\int_B \underline{u}(\underline{x},t)dx &= 0,\\
\int_B p(\underline{x},t)dx &= 0,
\end{aligned}$$

where the flow domain $B=\{(x,y,z)\in(0,L)\times(0,L)\times(0,L)\}$ is the periodic region. We conclude this chapter with recapitulation of the set of equations that will be analyzed in this investigation. The evolutionary equation for the β -plane rotating Boussinesq equations with Reynolds stress is the following initial-boundary value problem:

$$\begin{aligned}
&Ro(\frac{\partial \underline{u}}{\partial t}+\underline{u}.\nabla\underline{u})-(1+yRo\beta_0)\underline{u}\times\underline{k}+\rho\underline{k}+Ek\triangle^6\underline{u}=-\nabla p+Ek\triangle\underline{u},\\
&\nabla.\underline{u}=0,\\
&\frac{Fr^2}{Ro}(\frac{\partial\rho}{\partial t}+\underline{u}.\nabla\rho)+\underline{u}.\underline{k}+\frac{Fr^2}{Ed(Ro)}\triangle^6\rho=\frac{Fr^2}{Ro(Ed)}\triangle\rho,
\end{aligned}\tag{2.29}$$

$$\begin{aligned}
\underline{u}(\underline{x}+Le_i,t) &= \underline{u}(\underline{x},t),\\
\rho(\underline{x}+Le_i,t) &= \rho(\underline{x},t),\\
p(\underline{x}+Le_i,t) &= p(\underline{x},t),\\
\int_B \underline{u}(\underline{x},t)dx &= 0,\\
\int_B p(\underline{x},t)dx &= 0,
\end{aligned}\tag{2.30}$$

$$\begin{aligned}
\underline{u}(\underline{x},0) &= u_0(\underline{x})\\
\rho(\underline{x},0) &= \rho_0(\underline{x}).
\end{aligned}\tag{2.31}$$

CHAPTER 3
Nonlinear analysis preliminaries

3.1 Elements of Hilbert spaces and Sobolev spaces

This is an introductory chapter on the analysis of nonlinear partial differential equations using techniques from dynamical systems theory. The main goal is to present techniques and results for nonlinear PDE on unbounded domains in a self-contained format. This is achieved by developing the theory with an introduction to finite-dimensional dynamics defined by ordinary differential equations; next section is focused on dynamics in countably many dimensions, typically defined by PDE on bounded domains. Topics in ODE typically covered in a graduate introduction to the subject, including existence and uniqueness, linear systems, ω-limit sets and attractors, chaos, stable/unstable/center manifold theory, and bifurcations. The author has used this as a nice way to introduce a certain perspective on nonlinear dynamics and begin to build a toolkit that will be utilized and further developed in later chapters on PDE. This is quite useful because often a proof technique appears for the first time in this finite-dimensional setting, and this base case can then be referred to when studying similar ideas in more complicated settings in the later chapters. This setting is used to introduce a variety of key ideas in PDE theory, including semigroup theory, Sobolev spaces, and nonlinear analysis, as well as to discuss important issues like smoothing and compactness. It is very helpful to see these ideas presented first in this countable context, because one can really see where the finite-dimensional theory starts to breaks down. This chapter is focused on the Navier-Stokes equation on a bounded domain, and includes basic results on local existence and uniqueness and a discussion of the important open millennium problem and of why standard methods fail to address it. Although this chapter seemed somewhat distinct from the overall theme of the article, we still appreciated its inclusion, since the Navier-Stokes equation is such a well known system and a natural place to apply many of the techniques developed in the article up to this point.

The purpose of this introductory section is to develop the definitions of relevant function spaces and briefly examine propositions appropriate for analysis of the evolution equations established in the previous chapter. In the investigation of the initial-value problems, it should be proved that the solution of the equations in a variety of function spaces such as Hilbert spaces is well-posed using the auxiliary results available from this chapter. The proof of existence and uniqueness of solution

to initial-value problems utilizing Hilbert spaces and Sobolev spaces techniques will be of central importance in this thesis. After the functional formulation of the problems is given, we proceed to

investigate existence, uniqueness and nonlinear stability of solutions. We represents the flow region by a bounded domain denoted by the symbol $B = \{(x, y, z) \in \mathbb{R}^3\}$ of the three-dimensional Euclidean space. We assume B has a locally Lipschitz boundary $\Gamma = \partial B$, that is, Γ is locally the graph of a Lipschitz function which holds when B is of class at least C^1. The scalar-, vector-, and tensor-valued functions are assumed to be real and locally summable in the sense of Lebesgue, whereas their derivatives will be interpreted in the generalized sense of the theory of distributions of Schwartz. Thus, throughout this work we will use the standard Lebesgue spaces $L^p(B)$, $1 \le p \le \infty$, which consist of p-integrable functions on B with corresponding norms [84, 85]

$$\| \underline{u}\|_{L^p}^p = \int_B |\underline{u}(x)|^p \, dx,$$

$$\| \underline{u}\|_{L^\infty} = sup_{x \in B} ess \, | \underline{u}(x)|$$

where dx is a notation for the Lebesque measure on $\mathbb{R}^3$ and the assumption of identification through the *a.e.* equivalence relation. The standard Lebesgue spaces $L^p(B)$ are complete normed linear spaces in the sense that every Cauchy sequence in the space converges. We denote the inner product of the Hilbert space $L^2(B)$ by

$$(\underline{u},\underline{v}) = \int_B \underline{u}(x).\underline{v}(x)dx$$

and the associated norm is defined with

$$(\underline{u},\underline{u}) = \| \underline{u}\|^2_{L^2}.$$

The idea of an inner product can be more abstractly formalized, with $(.,.)$ as a specific example. For a general set of elements $u, v, w, \cdots \in L^2(B)$ and scalars $c_1, c_2, \cdots \in C$, an inner product by definition is a bilinear form that satisfies the following conditions:

$$(u,v) = \overline{(v,u)}$$
$$(u, v+w) = (u,v) + (u,w)$$
$$(u, c_1 v) = c_1 (u,v)$$
$$(u,u) \geq 0 \;\; with \;\; (u,u) = 0 \Leftrightarrow u = 0.$$

The norm of $u \in L^2(B)$,

$$\| u\| = (u,u)^{1/2},$$

in effect yields the distance of the element u from the origin of the Hilbert space. The main point to note is that the Cauchy-Schwarz inequality(H$\ddot{o}$lder inequality for $L^p(B)$)

$$|(u,v)| \leq \| u\| \;\; \| v\|$$

and the triangle inequality(Minkowski inequality for $L^p(B)$)

$$\| u \pm v\| \leq \| u\| + \| v\|$$

are a consequence of Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

$\frac{1}{p} + \frac{1}{q} = 1$, and the convexity of the function $\phi(t) = t^p$ on the interval $[0,\infty)$ for $1 \leq p < \infty$ where a and b are nonnegative scalars. The function spaces of real continous functions on B are represented by the notation $C(B)$. The spaces of infinitely continuous differentiable functions on B with a compact support in B is denoted by $C_0^\infty(B)$. We define a function to be of compact support in the domain B if it is nonzero only on a bounded subdomain B_s of the domain B with the subdomain lying at a positive distance from Γ. We also utilize the Sobolev spaces $H^k(B)$, $k = -1, 0, 1, \cdots$, with generalized derivatives up to order k belonging to $L^2(B)$ and norms $\|.\|_k$ which are Hilbert spaces endowed with the inner product

$$\big((\underline{u},\underline{v})\big)_k = \sum_{[\alpha] \leq m} (D^\alpha \underline{u}, D^\alpha \underline{v}) = \sum_{[\alpha] \leq m} \int_B D^\alpha \underline{u}(x).D^\alpha \underline{v}(x)dx$$

where $\alpha = (\alpha_1 \ldots \alpha_n)$ is a multi-index with each α_i a nonnegative integer; $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha u$ denotes the partial derivative $\partial^{|\alpha|} u / \partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The Sobolev \$k\$-norm associated with the above inner product is defined by

$$\| \underline{u}\|_k^2 = \big((\underline{u}, \underline{u})\big)_k$$

whereas the H^k -seminorm is given by

$$\| \underline{u}\|_k^2 = \sum_{|\alpha|=k} \| D^\alpha \underline{u}\|_{L^2}^2.$$

For example, in the case of scalar-valued functions then the tedious and involved expression for the Sobolev k -norm of a function u simplifies to the following:

$$\| u\|_k^2 = \int_{-\infty}^{\infty} |u(x)|^2 \, dx + \sum_{\alpha=1}^{k} \int_{-\infty}^{\infty} |\frac{\partial^\alpha u}{\partial x^\alpha}(x)|^2 \, dx.$$

It follows that the concept of Sobolev spaces underlies the construction of a priori energy type estimates for suitable functions and their derivatives up to order k. Thus, in order to prove well-posedness and other properties of solution to initial-value problems we resort to a priori energy type estimates which are derived from the bound of the L^2 -norm of the solution in terms of the Sobolev k -norm of the initial condition. In the sequel we will require the Sobolev space $H^{k,per}(B_p)$ of space-periodic functions in the periodic domain

$$B_p = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0, L_3)\}.$$

Let $\underline{u}(x)$ represent a space-periodic function or, equivalently, the space-periodic extension of a function. Define the Fourier series expansion

$$\underline{u}(x) = \sum_{k \in Z^3} u_k exp(2i\pi k. \frac{x}{L})$$

with $\overline{u_k} = u_{-k}$,

$$\sum_{k \in Z^3} (1 + |k|^2) |u_k|^2 < \infty.$$

Furthermore, in the space-periodic case we assume that the flow average vanishes

$$\frac{1}{|B_p|} \int_{B_p} \underline{u}(x) dx = 0.$$

The rigorous characterization of the Sobolev spaces $H_{even}^{k,per}(B_p)$ and $H_{odd}^{k,per}(B_p)$ of functions belonging to $H^{k,per}(B_p)$ that are even and odd, respectively, in z may be found in the work of Bourgeois and Beale [9]. We will have need of the following useful result from [9] that characterizes the spaces $H_{even}^{k,per}(B_p)$ and $H_{odd}^{k,per}(B_p)$:

Lemma 3.1.1 Suppose here the flow domain is given by $B = \{(x, y, z) \in (0, L_1) \times (0, L_2) \times (0,1)\}$ with the boundary $\Gamma = \{z = 0,1\}$. Consider a function $u \in H^k(B)$ for a given integer k. Then u can be extended to $H_{even}^{k,per}(B_p) \Leftrightarrow$ all odd z derivatives of u with index less than k are zero on the boundary Γ of B. Also, u may be extended to $H_{odd}^{k,per}(B_p) \Leftrightarrow$ all even z derivatives of u with index less than k including u itself are zero on the boundary Γ of B.

Numerous a priori estimates and important properties have been developed for Sobolev spaces and are customarily referred to as the Sobolev's embedding theorem and the Rellich's compact injection theorem. These results are established in the expos$e$ [66, 92, 91, 90]. The Sobolev space $H_0^1(B)$ of functions in $H^1(B)$ which vanish on Γ in the sense of traces and its dual $H^{-1}(B)$ will be employed. Duality pairing of two elements $l \in H^{-1}(B)$ and $u \in H_0^1(B)$ is represented by the notation $< l, u >$. Often the interval $I = (a,b)$ will be the interval $(0,T)$, $T > 0$ fixed. The following proposition from Temam [92, 91, 90] considers the sense in which integration by parts is valid in functional formulation of evolution equations:

Lemma 3.1.2 Suppose Y is a given complete normed linear space with dual denoted by Y' and assume $u, f \in L^1(I;Y)$. Then the following results are equivalent:

(1) u is *a.e.* equal to the Lebesgue integral of f, that is, there is $\xi \in Y$ such that for $t \in I$ *a.e.*

$$u(t) = \xi + \int_0^t f(s)ds.$$

(2) Given a test function $\phi \in C_0^\infty(a,b)$,

$$\int_a^b \frac{d\phi}{dt} u(t)dt = -\int_a^b f(t)\phi(t)dt.$$

(3) Given $\eta \in Y'$, then the following is valid in the sense of distributions on (a,b)

$$\frac{d}{dt} < u, \eta > = < f, \eta > . \tag{3.1}$$

Whenever conditions $(1)-(3)$ are satisfied, $f = \frac{du}{dt}$ is considered the Y-valued distribution derivative of u and in this case u is *a.e.* equal to an element of $C(I;Y)$.

Next, we consider the following useful result from Temam [92, 91, 90] that will be utilized in the sequel:

Lemma 3.1.3 Consider $V \subset H \subset V'$, where the injections are continuous and each Hilbert space is dense in the following one. If a function $u \in L^2((a,b);V)$ and its derivative $\frac{du}{dt}$ is an element of $L^2((a,b);V')$, then \$u\$ is *a.e.* equal to an element of $C(I;H)$ as in the above lemma and the following

$$\frac{d}{dt} |u|^2 = 2 < \frac{du}{dt}, u > \tag{3.2}$$

is valid in the scalar distribution sense on (a,b) since the functions $t \to |u(t)|^2$ and $t \to < \frac{du}{dt}(t), u(t) >$ are both elements of $L^1(a,b)$.

Now we proceed with other significant results from Sobolev spaces. By the Poincar$e$-Friedrichs inequality, the H^1-seminorm $|\underline{u}|_1$ is equivalent to the Dirichlet norm

$$\| \nabla \underline{u} \|_{L^2} = \| \underline{u} \|_1$$

which is a norm on $H_0^1(B)$. For brevity of notation, the function spaces of vector-, or tensor-valued functions which have components in one of the spaces defined above, will be denoted by the same symbol. In the investigation of the differentiability properties of generalized solutions to the initial-boundary value equations reformulated in variational form, the following Banach spaces are employed: given that $I=(a,b)\subset\mathbb{R}$ is an open interval and Y is a complete normed space, then $L^p(I;Y(B))\$,\$1\le p\le\infty$, are spaces of functions from I into Y which are Banach spaces with corresponding norms

$$\| f\|^p_{L^p(I;Y(B))}=\int_a^b\| f(t)\|^p_Y dt$$

$$\| f\|_{L^\infty(I;Y(B))}=sup_{t\in I}ess\| f(t)\|_Y.$$

Thus, $L^p(I;H^k(B))$, $1\le p\le\infty$, are function spaces which consist of p-integrable functions with values in $H^k(B)$. Similar function spaces $C(\bar{I};H^k(B))$ are the spaces of vector functions $\underline{u}(x,t)$ such that $\underline{u}(.,t)$ is an element of $H^k(B)$ for all $t\in\bar{I}$ and the function $t\to\underline{u}(.,t)$ with values in $H^k(B)$ is continuous on $\bar{I}$. Similarly, $C_b(\bar{I};H^k(B))$ denotes the space of vector functions $\underline{u}(x,t)$ such that $\underline{u}(.,t)$ is an element of $H^k(B)$ for all $t\in\bar{I}$ and the function $t\to\underline{u}(.,t)$ with values in $H^k(B)$ is a bounded continuous function on $\bar{I}$.

3.2 Properties of some functionals and operators

In the functional formulation of the initial-boundary value problems, the following function spaces of divergence-free or solenoidal vector functions in the sense of the theory of distributions will be
utilized:

$$Q=\{\underline{u}\in L_2(B):\nabla.\underline{u}=0,\ \underline{u}\ \ satisfies\ \ Space-periodicity\},$$

$$X=\{\tau:\tau_{ij}\in L_2(B),\tau_{ij}=\tau_{ji},\tau_{ij}=0,\ a.e.\in B\}$$

$$H=\begin{cases}H_1\times Closure\ of\ \ Q\ in\ L_2(B) & \text{Space-periodicity}\\ H_1\times H_0 & \text{Condition(2.27)}\end{cases}$$

$$V=\begin{cases}V_1\times Closure\ of\ \ Q\ in\ H^1(B) & \text{Space-periodicity}\\ V_1\times V_0 & \text{Condition(2.27)}\end{cases}$$

which are Hilbert spaces equipped, respectively, with norms $\|.\|_H\equiv\|.\|_{L^2}$, $\|.\|_V\equiv\|.\|_1$ and $\|.\|_X\equiv\|.\|_{L^2}$ corresponding to the product Hilbert structure. The rigorous construction and useful features of Hilbert spaces of divergence-free or solenoidal vector functions can be found in the noteworthy and accesible work of Ladyzhenskaya [66]. Here $H_1=L_2(B)$ and V_1 is the space of functions in $H^1(B)$ satisfying condition (2.27) which are Hilbert spaces endowed, respectively, with norms $\|.\|_{H_1}\equiv\|.\|_{L^2}$ and $\|.\|_{V_1}\equiv\|.\|_1$. And the function spaces H_0 and V_0 are given by

$$H_0=\{\underline{u}\in L_2(B):\nabla.\underline{u}=0\ \ in\ \ B,\ \underline{u}.\mathbf{n}=0\ \ at\ \Gamma\},$$

with $\mathbf{n}$ the unit outward normal on the boundary Γ and

$$V_0=\{\underline{u}\in V_1:\nabla.\underline{u}=0\}$$

which are Hilbert spaces with norm denoted by $\|.\|_1$. Additionally, we need the following function space of tensor-valued functions

$$\mathbf{H}^k = H^k(B) \cap X.$$

Let $\alpha = \frac{Ek}{Ro}$ or $\frac{1}{Ek}$. It is useful to set

$$A_s \underline{u} = -\alpha \Pi \triangle \underline{u},$$

where $\underline{u} \in D(A_s)$, the domain of the Stokes operator A_s, and Π is the orthogonal projection of $L^2(B)$ onto H_0. Due to the presence of the boundary condition (2.27) and the flow region $B = \{(x, y, z) \in (0, L) \times (0, L) \times (0,1)\}$ with boundary $\Gamma = \{z = 0,1\}$, we have

$$\Pi \triangle \underline{u} = \triangle \underline{u} - \nabla \Phi$$

with Φ satisfying the Neumann problem corresponding to the Laplace operator:

$$\triangle \Phi = 0 \ \ in \ \ B,$$

$$\frac{\partial \Phi}{\partial z} = \triangle w \ \ at \ \ \Gamma.$$

The domain of the Stokes operator A_s is defined by

$$D(A_s) = V_0 \cap H^2(B),$$

and is endowed with the norm $\| \underline{u} \|_{D(A_s)} = \| A_s \underline{u} \|_{L^2}$ which is equivalent to the natural norm $\|.\|_2$ of the Sobolev space $H^2(B)$. Moreover, the Laplace's operator defined by

$$A_l \rho = -\alpha \triangle \rho$$

has domain given by

$$D(A_l) = V_1 \cap H^2(B).$$

In order to ensure the solution of an evolution equation is bounded in some suitable norm, we need to consider the following Hilbert space

$$D(L) = \begin{cases} \{U \in \mathrm{Y} : \int_B (|\triangle \underline{u}|^2 + |\triangle \rho|^2) dx < \infty, \ \rho, \underline{u} \ \ satisfy \ \ (2.27)\} \\ \{U \in \mathrm{Y} : \int_B (|\triangle^6 \underline{u}|^2 + |\triangle^6 \rho|^2) dx < \infty, \ \underline{u} \ \ satisfy \ \ (2.27)\}. \end{cases}$$

We recall that the Stokes operator A_s and the Laplace's operator A_l may be regarded as ubounded self-adjoint positive linear operators, respectively, from $D(A_s)$ into H_0 and from $D(A_l)$ into H_1 defined by

$$(A_s \underline{u}, \underline{v}) = ((\underline{u}, \underline{v})) = \big((\underline{u}, \underline{v})\big)_1,$$

$$(A_l \underline{u}, \underline{v}) = ((\underline{u}, \underline{v})) = \big((\underline{u}, \underline{u})\big)_1$$

for all $\underline{u}, \underline{v} \in D(A_l)$ and $D(A_s)$. Furthermore, A_l^{-1} and A_s^{-1} are compact self-adjoint linear operators in H_1 and H_0. The basic function space $D(L^n)$ is Hilbert space with norm $\| U \|_{D(L^n)} = \| L^n U \|$. Further, the dual of $D(L^{\frac{1}{2}})$ is denoted by $D(L^{\frac{-1}{2}})$. Under appropriate assumptions [92, 91, 90, 22], it is possible to establish existence of the following injections

$$D(L^{\frac{1}{2}}) = V \subset D(L) \subset \mathrm{Y} = H \subset D(L^{\frac{-1}{2}}). \tag{3.3}$$

The density and compactness of the injections will find applications in the sequel when we prove well-posedness of solutions to the initial-boundary value problems (2.29-2.31) and (2.26-2.28). In addition to the Stokes operator and Laplace operator, with the assistance of the orthogonal projection Π we will need the bilinear mapping $\mathbf{b}(.,.)$ defined by

$$\mathbf{b}(\underline{u},\underline{v}) = \begin{cases} \Pi(\underline{u}.\nabla)\underline{v} = \mathbf{b}_s(\underline{u},\underline{v}) & \text{Condition(2.27)} \\ \\ (\underline{u}.\nabla)\underline{v} = \mathbf{b}_l(\underline{u},\underline{v}) & \text{Space-periodicitycondition} \end{cases}$$

where $\underline{u},\underline{v} \in D(A_s)$ for the bilinear operator $\mathbf{b}_s(\underline{u},\underline{v})$ and $\underline{u} \in D(A_s)$, $\underline{v} \in D(A_l)$ for the bilinear operator $\mathbf{b}_l(\underline{u},\underline{v})$.

The quantity of central importance in analyzing whether the solution of an evolution equation is bounded in a suitable norm is that of Lyapunov functional of the system. We will illustrate in the sequel a technique for constructing Lyapunov functionals with interpretations such as energy, enstrophy and entropy production which are decreasing along solutions. The equations governing the flow of an incompressible stratified fluid under the Coriolis force induce damping mechanisms in the nature of viscosity, diffusion, stratification and rotation effects that manifest themselves in the evolution equation with the existence of Lyapunov functionals which are equivalent to some norm induced by the inner product of solution. Specifically, the techniques used are spectral properties of the linear operator and a priori estimates. Now we recall ideas which are required in the formulation of the *nonlinear stability criteria* [27, 28] to be customized for use in rotating Boussinesq equations. A self-adjoint operator L is considered essentially dissipative if the following hold:

$$(LU,U) \le 0 \quad \forall \quad U \in D(L)$$
$$(LU,U) = 0 \Rightarrow U = 0.$$

The complement of essentially dissipative is called essentially non-dissipative. By the spectral theorem [27, 28, 50], essential dissipativity is equivalent to the spectrum of L being nonnegative and zero not an eigenvalue. For every ssentially dissipative operator L, the bilinear form

$$(U,W)_L = -(LU,W) \ \forall \ U,W \in D(L)$$

defines a scalar product in $D(L)$. We denote by H_L the completion of $D(L)$ in the norm $\|.\|_L$ given by

$$\| U\|_L^2 = -(LU,U) \ \forall \ U \in D(L).$$

In fact the case of essential dissipativity of linear operators in Hilbert spaces developed here may be considered as an extension of advanced matrix theory in $\mathbb{R}^n$, the finite-dimensional space. The most crucial utility of essential dissipativity is to obtain exponential decay estimates for solutions of the initial-boundary value problems (2.29-2.31) and (2.26-2.28) that yield asymptotic stability of solutions. The ideas can be less abstractly formalized in the finite dimensional case. First, we observe that each basis of $\mathbb{R}^n$ renders a scalar product denoted by $(.,.)$. It is known that if L is an $n \times n$ matrix for which

$$\alpha_1 < Re(\lambda) < \alpha_2 \tag{3.4}$$

for all λ in the spectrum $\sigma(L)$, of L then there is a basis of $\mathbb{R}^n$ in which

$$\alpha_1(u,u) \le (Lu,u) \le \alpha_2(u,u) \tag{3.5}$$

where α_1 and α_2 are nonzero constants. Thus if we set $u(t)$ to be a solution of a system of linear ordinary differential equations

$$\frac{du}{dt} = Lu,\ u(0) = u_0\ \in H, \tag{3.6}$$

with the matrix L satisfying (3.4-3.5)then we obtain

$$|u_0|\, exp(\alpha_1 t) \leq |u(t)| \leq |u_0|\, exp(\alpha_2 t). \tag{3.7}$$

where $|.|$ is the Euclidean norm. Consequently, if $Re(\lambda) < 0\ \forall \lambda \in \sigma(L)$ we obtain asymptotic stability and attraction of the rest state and whenever $Re(\lambda) > 0\ \forall \lambda \in \sigma(M)$ then the rest state is unstable in the sense of Lyapunov. We emphasize that the results on linear stability (3.7) belong to a finite-dimensional space. Formal passage from the finite dimensional space $\mathbb{R}^n$ to the infinite-dimensional case of Hilbert spaces will be established in the sequel by the virtue of *energetic stability criteria* customized for solutions of initial-value problem. We consider the *orbit* or *trajectory* [91, 90, 14, 15] of motion for an initial-value problem starting at the initial condition $u_0 in H$ to be defined by the set

$$\Xi = \{(u,t) : u \in H,\ t \in \mathbb{R}_+\} = \cup_{t \in \mathbb{R}_+} S(t)u_0.$$

Uniqueness and continuity with respect to initial conditions of the solution $u(t)$ generate a dynamical system prescribed by continuous solution operators $S(t),\ t \in \mathbb{R}_+$ which is a mapping of the phase space H into itself. Whenever the solution operators $S(t),\ t \in \mathbb{R}_+$ are injective then we denote by $S(-t)$ the inverse mapping of $S(t)H$ onto H. As a result we obtain the solution operators $S(t)$ defined for all time $t \in \mathbb{R}$. The crucial properties of the solution operators which will be employed are as follows:

$$S(t)u_0 = u(t) \equiv S(u_0, t)$$

satisfying the *group property*

$$S(t)u(s) = S(s)u(t)$$
$$S(0)u = u\ \ \forall\ t, s \in \mathbb{R}.$$

That the solution operators $S(t)$ are continuous is a consequence of the continuity of $u(t)$ in time and in initial conditions. The group property is a consequence of the injectivity of the solution operators which is equivalent to the backward uniqueness of solution for the initial-value problem. In this work we shall adopt the definition that a set $Q \subset H$ is an invariant set [14, 15, 91, 90, 23, 22] for the dynamical system if the following useful technique for the absorbing or trapping of trajectories in phase space is valid

$$S(t)Q = Q,\ \forall t \in \mathbb{R}.$$

We remark that an invariant set is both positively and negatively invariant and consists of orbits or trajectories that are defined for all $t \in \mathbb{R}$. It is known that of central significance in the analysis of dynamical systems is the asymptotic behavior of trajectories such as homoclinic and heteroclinic orbits[31, 35]. Geophysical fluid dynamical processes are identified with trajectories of a dynamical system in a suitable phase space and the investigation of asymptotic behavior is reduced to the structure of ω-limit sets of these orbits. Invoking these ideas, we define the ω-limit set of $Q \subset H$ with

$$\omega(Q) = \cap_{s \geq 0} Cl(\cup_{t \geq s} S(t)Q),$$

where the closure is taken in the Hilbert space *H*. From the fact that the arbitrary intersection of closed sets is closed, we deduce that ω-limit set of Q given by $\omega(Q)$ is closed. An invariant set $\Omega \subset H$ is defined as an attractor if there is a neighborhood Ξ of Q such that

$$\omega(\Xi) = Q.$$

Beside stability and attraction, it is important to note that other properties for the solution of the initial-boundary value problems are also possible. Differentiability of solution that lies in the fact that trajectories nearby a given trajectory yield a variation on the base trajectory which is approximated, to first order, by a linear nonautonoumous differential equation. Thus, the equation of variations is obtained by linearizing the nonlinear initial-value problem about a solution. The technique of proving injectivity and differentiability of the solution operators is of great importantance as it uses the linearized flow or the variational equation and this is a central geometric construction in dynamical systems. Consider a given dynamical system $u_0 \to S(t)u_0$ generated by the initial-value problem. It is Fr$e$chet differentiable in a Hilbert space Y with differential $L(t,u_0):\xi \to \Psi(t)$ given by the solution of the corresponding variational equation if the following is satisfied

$$\frac{\| S(t)v_0 - S(t)u_0 - L(t,u_0).(v_0 - u_0)\|^2}{\| v_0 - u_0\|^2} = o(\| v_0 - u_0\|)$$

as $v_0 \to u_0$. The Fr$e$chet derivative has properties similar to the derivative in finite dimensional space: the chain rule holds and the mean value theorem is valid [35]. The assertion of the following Gronwall's inequality [36, 92, 91, 90] will obtain use in establishing a priori energy type estimates, for instance, on the difference between two solutions of the nonlinear initial-value problems which as always is required in proving backward uniqueness and Fr$e$chet differentiability of solution.

Lemma 3.2.1 Suppose f, g and u are nonnegative locally integrable functions on an interval $I = (t_0, \infty)$ with the derivative of u, $\frac{du}{dt}$, locally integrable on $I = (t_0, \infty)$ and satisfying the differential inequality

$$\frac{du}{dt} \le gu + f \quad for \;\; t \ge t_0. \tag{3.8}$$

If in addition the estimates,

$$\int_t^{t+\alpha} g(s)ds \le \alpha_1,$$

$$\int_t^{t+\alpha} f(s)ds \le \alpha_2,$$

$$\int_t^{t+\alpha} u(s)ds \le \alpha_3,$$

hold for $t \ge t_0$ where $\alpha, \alpha_1, \alpha_2$ and α_3 are nonnegative constants then the following Gronwall's inequality is valid:

$$u(t+\alpha) \le (\frac{\alpha_3}{\alpha} + \alpha_2)exp(\alpha_1) \;\; \forall\, t \ge t_0.$$

CHAPTER 4

Well-posedness of solutions

4.1 Existence and uniqueness of solutions

In the functional formulation of the initial-boundary value problems, the following function spaces of divergence-free or solenoidal vector functions in the sense of the theory of distributions will be
utilized:

$$Q=\{\underline{u}\in L_2(B):\nabla.\underline{u}=0,\ \underline{u}\ \ satisfies\ \ Space-periodicity\},$$

$$X=\{\tau:\tau_{ij}\in L_2(B),\tau_{ij}=\tau_{ji},\tau_{ij}=0,\ a.e.\in B\}$$

$$H=\begin{cases}H_1\times Closure\ of\ \ Q\ in\ L_2(B) & \text{Space-periodicity}\\ H_1\times H_0 & \text{Condition(2.27)}\end{cases}$$

$$V=\begin{cases}V_1\times Closure\ of\ \ Q\ in\ H^1(B) & \text{Space-periodicity}\\ V_1\times V_0 & \text{Condition(2.27)}\end{cases}$$

which are Hilbert spaces equipped, respectively, with norms $\|.\|_H\equiv\|.\|_{L^2}$, $\|.\|_V\equiv\|.\|_1$ and $\|.\|_X\equiv\|.\|_{L^2}$ corresponding to the product Hilbert structure. The rigorous construction and useful features of Hilbert spaces of divergence-free or solenoidal vector functions can be found in the noteworthy and accesible work of Ladyzhenskaya [66]. Here $H_1=L_2(B)$ and V_1 is the space of functions in $H^1(B)$ satisfying condition (2.27) which are Hilbert spaces endowed, respectively, with norms $\|.\|_{H_1}\equiv\|.\|_{L^2}$ and $\|.\|_{V_1}\equiv\|.\|_1$. And the function spaces H_0 and V_0 are given by

$$H_0=\{\underline{u}\in L_2(B):\nabla.\underline{u}=0\ \ in\ \ B,\ \underline{u}.\mathbf{n}=0\ \ at\ \Gamma\},$$

with $\mathbf{n}$ the unit outward normal on the boundary Γ and

$$V_0=\{\underline{u}\in V_1:\nabla.\underline{u}=0\}$$

which are Hilbert spaces with norm denoted by $\|.\|_1$. Additionally, we need the following function space of tensor-valued functions

$$\mathbf{H}^k=H^k(B)\cap X.$$

Let $\alpha=\dfrac{Ek}{Ro}$ or $\dfrac{1}{Ek}$. It is useful to set

$$A_s\underline{u}=-\alpha\Pi\triangle\underline{u},$$

where $\underline{u}\in D(A_s)$, the domain of the Stokes operator A_s, and Π is the orthogonal projection of $L^2(B)$ onto H_0. Due to the presence of the boundary condition (2.27) and the flow region $B=\{(x,y,z)\in(0,L)\times(0,L)\times(0,1)\}$ with boundary $\Gamma=\{z=0,1\}$, we have

$$\Pi\triangle\underline{u}=\triangle\underline{u}-\nabla\Phi$$

with Φ satisfying the Neumann problem corresponding to the Laplace operator:

$$\triangle\Phi=0\ \ in\ \ B,$$

$$\frac{\partial \Phi}{\partial z} = \triangle w \ \ at \ \ \Gamma.$$

The domain of the Stokes operator A_s is defined by

$$D(A_s) = V_0 \cap H^2(B),$$

and is endowed with the norm $\| \underline{u} \|_{D(A_s)} = \| A_s \underline{u} \|_{L^2}$ which is equivalent to the natural norm $\|.\|_2$ of the Sobolev space $H^2(B)$. Moreover, the Laplace's operator defined by

$$A_l \rho = -\alpha \triangle \rho$$

has domain given by

$$D(A_l) = V_1 \cap H^2(B).$$

In order to ensure the solution of an evolution equation is bounded in some suitable norm, we need to consider the following Hilbert space

$$D(L) = \begin{cases} \{U \in \Upsilon : \int_B (|\triangle \underline{u}|^2 + |\triangle \rho|^2) dx < \infty, \ \rho, \underline{u} \ \ satisfy \ \ (2.27)\} \\ \{U \in \Upsilon : \int_B (|\triangle^6 \underline{u}|^2 + |\triangle^6 \rho|^2) dx < \infty, \ \underline{u} \ \ satisfy \ \ (2.30)\}. \end{cases}$$

We recall that the Stokes operator A_s and the Laplace's operator A_l may be regarded as ubounded self-adjoint positive linear operators, respectively, from $D(A_s)$ into H_0 and from $D(A_l)$ into H_1 defined by

$$(A_s \underline{u}, \underline{v}) = ((\underline{u}, \underline{v})) = \big((\underline{u}, \underline{v})\big)_1,$$
$$(A_l \underline{u}, \underline{v}) = ((\underline{u}, \underline{v})) = \big((\underline{u}, \underline{u})\big)_1$$

for all $\underline{u}, \underline{v} \in D(A_l)$ and $D(A_s)$. Furthermore, A_l^{-1} and A_s^{-1} are compact self-adjoint linear operators in H_1 and H_0. The basic function space $D(L^n)$ is Hilbert space with norm $\| U \|_{D(L^n)} = \| L^n U \|$. Further, the dual of $D(L^{\frac{1}{2}})$ is denoted by $D(L^{\frac{-1}{2}})$. Under appropriate assumptions [92, 91, 90, 22], it is possible to establish existence of the following injections

$$D(L^{\frac{1}{2}}) = V \subset D(L) \subset \Upsilon = H \subset D(L^{\frac{-1}{2}}). \tag{4.1}$$

The density and compactness of the injections will find applications in the sequel when we prove well-posedness of solutions to the initial-boundary value problems (2.29-2.31) and (2.26-2.28). In addition to the Stokes operator and Laplace operator, with the assistance of the orthogonal projection Π we will need the bilinear mapping $\mathbf{b}(.,.)$ defined by

$$\mathbf{b}(\underline{u}, \underline{v}) = \begin{cases} \Pi(\underline{u}.\triangledown)\underline{v} = \mathbf{b}_s(\underline{u}, \underline{v}) & \text{Condition(2.27)} \\ \\ (\underline{u}.\triangledown)\underline{v} = \mathbf{b}_l(\underline{u}, \underline{v}) & \text{Space-periodicitycondition} \end{cases}$$

where $\underline{u}, \underline{v} \in D(A_s)$ for the bilinear operator $\mathbf{b}_s(\underline{u}, \underline{v})$ and $\underline{u} \in D(A_s)$, $\underline{v} \in D(A_l)$ for the bilinear operator $\mathbf{b}_l(\underline{u}, \underline{v})$.

4.1.1 Rotating Boussinesq equations

The purpose of this section is to develop results of existence, uniqueness and differentiability of solution for the initial-boundary value problem of β-plane rotating Boussinesq equations (2.26-2.28). We prove the significant aspects of well-posedness of solution utilizing the machinery developed above and additional accessible techniques constructed in this section. The functional formulation for the evolution equation of β-plane rotating Boussinesq equations (2.26-2.28) now takes the form of the following initial value problem in the Hilbert space Y :

$$\begin{aligned} &\frac{dU}{dt} + LU + N(U) = 0 \\ &U(0) = U_0 \end{aligned} \tag{4.2}$$

where $U = (u, v, w, \rho)$. The linear operator LU and the nonlinear operator $N(U)$ are given by

$$N(U) = B(U,U) - FU$$

$$FU = -(0, 0, \Pi \frac{\rho}{Ro}, \frac{Ro}{Fr^2} w)$$

$$LU = TU + SU$$

$$TU = -\begin{pmatrix} \frac{Ek}{Ro}\Pi \triangle \underline{u} \\ \frac{1}{Ed} \triangle \rho \end{pmatrix} = \begin{pmatrix} A_s \underline{u} \\ A_l \rho \end{pmatrix}$$

$$B(U,U) = B(U) = \begin{pmatrix} \Pi \underline{u} \cdot \bigtriangledown \underline{u} \\ \underline{u} \cdot \bigtriangledown \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_s(\underline{u}, \underline{u}) \\ \mathbf{b}_l(\underline{u}, \rho) \end{pmatrix}$$

$$S = -\Pi(\frac{1 + y\beta Ro}{Ro}) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \forall\, U \in D(L) = D(T).$$

This problem as the Navier-Stokes equations in the presence of stratification but without Coriolis terms, has been examined by Temam *et al* [92, 91, 23, 22] and cited works therein. In order to develop basic results concerning whether there exists a unique solution in the large for the general three-dimensional space variables for initial-value problem (4.2), we derive a priori energy type estimates useful in proving that (4.2) generate a dynamical system, denoted, $U(t) = S(t)U_0$, where $U(t)$ is the unique solution uniformly bounded in finite-time and $S(t)$ is a group of continuous nonlinear solution operators. The principal result concerning the existence of such a unique solution will be proven by the utility of the Faedo-Galerkin technique, a priori energy type and Leray-Schauder principle. As for the linear operator LU of the initial-value problem (4.2), the following is valid

$$(LU,U) = (TU,U) = \frac{Ek}{Ro} \| \bigtriangledown \underline{u}\|^2 + \frac{1}{Ed} \| \bigtriangledown \rho\|^2 \tag{4.3}$$

It follows from [27, 28, 66, 50] that the operator T is self-adjoint and its spectrum $\sigma(T)$ satisfies $\sigma(T) \subseteq [0, \infty)$. The following result on the spectrum of L from [27, 28, 66, 50] will be utilized:

Lemma 4.1.1 $\sigma(L) \subseteq [0,\infty) \cup \coprod$, where $\coprod$ is either empty or an at most denumerable set consisting of isolated, positive eigenvalues $\lambda_n = \lambda_n(\frac{1}{Ro} + \frac{Ro}{Fr^2})$ with finite multiplicity such that

$$\infty > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots,$$

clustering at zero. Furthermore, whenever

$$\frac{1}{Ro} + \frac{Ro}{Fr^2} \leq Ga$$

then $\coprod = \varnothing$ whereas if

$$\frac{1}{Ro} + \frac{Ro}{Fr^2} \geq Ga$$

we have $\coprod \neq \varnothing$. Also, zero is not an eigenvalue whenever the following is satisified:

$$\frac{1}{Ro} + \frac{Ro}{Fr^2} \leq Ga.$$

Before giving the proof of the lemma, we make remarks and investigate the steady linear versions of the nonlinear initial-value problem. The examination demonstrate that the steady linear system has a unique solution and provide useful properties which are employed in the investigation of the nonlinear initial-value problem. We start by recalling that when $\sigma(L) \subseteq (0,\infty)$ which hold if

$$\frac{1}{Ro} + \frac{Ro}{Fr^2} \leq Ga$$

then we obtain

$$(LU,U) = (TU,U) \geq \lambda \| U\|^2 > 0$$

which implies the bilinear form induced by L is coercive and L^{-1} is compact and self-adjoint. This establishes existence and uniqueness of solution to the steady linear case by the Riesz-Fr$e$chet representation theorem or the Lax-Milgram lemma. Through the utility a result from [92, 27, 28, 66, 50], λ is the smallest eigenvalue of Laplace's operator and Stokes operator. Also, the symmetry of the bilinear form $a(U,W)$, together with the coerciveness of $a(U,U)$ imply the existence of a sequence of positive eigenvalues and a corresponding sequence of eigensolutions which yield an orthonormal basis of Y such that

$$\begin{aligned} &LW_i = \lambda_i W_i \\ &0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots, \end{aligned} \tag{4.5}$$

and $\lambda_i \to \infty$ as $i \to \infty$. Moreover,

$$\begin{aligned} &\| L^{\frac{1}{2}} U\| \geq \lambda_1^{\frac{1}{2}} \| U\| \\ &\| LU\| \geq \lambda_1^{\frac{1}{2}} \| L^{\frac{1}{2}} U\| \end{aligned} \tag{4.6}$$

Next, we focus our attention to establishing the above lemma. The flavor of the proof is similar to that given in Kato [50] for the existence of eigensolutions and eigenvalues of perturbed elliptic operators

Proof: According to a perturbation theorem [27, 28, 50, 66], if TU is closed and SU is *relatively compact with respect to* TU then $\sigma(L = T + S)$ and $\sigma(T)$ differ by at most a denumerable number of isolated, positive

eigenvalues of finite multiplicity clustering at zero. Thus it suffices to show that S is relatively compact with respect to T since from [27, 28, 50, 66] we have $\sigma(T) \subseteq [0, \infty)$. By definition, S is relatively compact with respect to T if for any sequence $\{U_n\} \subset \mathrm{Y}$,

$$\| U_n\| + \| TU_n\| \leq \alpha, \ \textit{uniformly in } n, \tag{4.7}$$

then there is a subsequence $U_{n'} \subset \{U_n\}$ such that $SU_{n'}$ is strongly convergent. With strong convergence, we can make strong statements. Indeed, the following inequality is valid

$$\begin{aligned} \| SU_{n'}\|^2 &\leq [5(\frac{1+y\beta Ro}{Ro})^2 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})^2]\| U_{n'}\|^2 \\ &\leq [5(\frac{1+y\beta Ro}{Ro})^2 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})^2]\alpha^2 \end{aligned} \tag{4.8}$$

and by the courtesy of Ascoli-Arzela theorem $\{SU_{n'}\}$ contains a uniformly convergent subsequence which is a Cauchy sequence in a complete normed space Y. Thus $SU_{n'}$ is strongly convergent which illustrates that S is relatively compact with respect to T and the claim of the lemma is achieved. We proceed put together a series of results that will be employed in establishing properties of solution to the initial-value problem (4.2). The following mini-max principle from [92, 23] will be utilized:

$$\lim_{t \to \infty} sup \| \rho(t)\|^2 \leq \| \Omega\| \tag{4.9}$$

where $\| \Omega\|$ is the volume of Ω which represents the flow region $\Omega = \{(x, y, z) \in \mathbb{R}^3\}$. Let $\alpha = max\{\frac{Ek}{Ro}, \frac{1}{Ek}\}$. By the orthogonal property of the nonlinear operator, we have

$$(B(U), U) = \int_\Omega [(\underline{u} \cdot \nabla \underline{u} \cdot \underline{u}) + (\underline{u} \cdot \nabla \rho)\rho] dx = 0 \tag{4.10}$$

Taking the inner product of (4.2) with U, utilizing Cauchy-Schwarz inequality and Young's inequality gives the following:

$$\begin{aligned} &\frac{d}{dt}\| U\|^2 + \alpha\lambda_1\| U\|^2 \\ &\leq \frac{d}{dt}\| U\|^2 + \alpha\| L^{\frac{1}{2}}U\|^2 \\ &\leq \frac{\lambda_1}{\alpha}(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2\| \rho\|^2. \end{aligned} \tag{4.11}$$

The consideration of Gronwall's inequality into (4.11) and invoking the mini-max principle (4.9) provides a priori estimate

$$\| U(t)\|^2 \leq \| U_0\|^2 exp(-\alpha t) + (\frac{Ro}{Fr^2} + \frac{1}{Ro})^4 \frac{\lambda_1^2}{\alpha^2}\| \Omega\|(1 - exp(-\alpha t)) \tag{4.12}$$

which is asymptotic stability with flow energy and entropy production provided by

$$E(t) = \frac{1}{2}\| U(t)\|^2 = \frac{1}{2}\int_\Omega (\underline{u}^2 + \rho^2) dx$$

and exponential dissipation rate $\alpha = max\{\frac{Ek}{Ro}, \frac{1}{Ek}\}$. Furthemore, in the limit as time goes to infinity we obtain

$$\lim_{t\to\infty} sup\| U(t)\|^2 \le (\frac{Ro}{Fr^2} + \frac{1}{Ro})^4 \frac{\lambda_1^2}{\alpha^2} \| \Omega\| = \rho_1^2 \tag{4.13}$$

which shows $U(t)$ is uniformly bounded for all time in Y. Next, by the mini-max principle (4.9) and uniform bound (4.13), it follows that for any $U_0 \in \mathrm{Y}$ and ε there exists $T_1 = T_1(U_0, \varepsilon)$ such that for $t \ge T_1$ then

$$\| \rho(t)\| \le \| \Omega\|^{\frac{1}{2}} + \varepsilon \tag{4.14}$$

Integration of the second inequality in (4.11) and the courtesy (4.14) provide a priori energy type estimate

$$\int_0^{T_1} \| L^{\frac{1}{2}} U(t)\|^2 dt \le \frac{1}{\alpha}\{\| U_0\|^2 + (\frac{Ro}{Fr^2} + \frac{1}{Ro})^2 \frac{\lambda_1}{\alpha}(\| \Omega\|^{\frac{1}{2}} + \varepsilon)^2\} = \rho_{11}^2.$$

Taking the inner product of (4.2) with LU and making use of Cauchy-Schwarz inequality, Young's inequality and Sobolev inequality yield the estimates

$$\begin{aligned} &\frac{d}{dt}\| L^{\frac{1}{2}} U\|^2 + \lambda_1 \alpha \| L^{\frac{1}{2}} U\|^2 \\ &\le \frac{d}{dt}\| L^{\frac{1}{2}} U\|^2 + \alpha \| LU\|^2 \\ &\le 2\alpha(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2 \| \rho\|^2 + \frac{c_0^2}{4\alpha} \| L^{\frac{1}{2}} U\|^6. \end{aligned} \tag{4.15}$$

By setting $c = max\{\frac{c_0^2}{4\alpha}, 2\alpha(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2(\| \Omega\|^{\frac{1}{2}} + \varepsilon)^2\}$ then (4.15) and (4.14) imply that

$$\frac{d}{dt}(1+\| L^{\frac{1}{2}} U\|^2) \le c(1+\| L^{\frac{1}{2}} U\|^2)^3. \tag{4.16}$$

Invoking Gronwall's inequality in the differential inequality (4.16) gives

$$\| L^{\frac{1}{2}} U\|^2 \le 1+\| L^{\frac{1}{2}} U_0\|^2 \tag{4.17}$$

for $t \le T_2(\| L^{\frac{1}{2}} U_0\|) = \dfrac{3}{8c(1+\| L^{\frac{1}{2}} U_0\|^2)}$. Consequently putting the pieces together we obtain

$$Sup_{t\in[0,T_2]} \| L^{\frac{1}{2}} U\|^2 \le 1+\| L^{\frac{1}{2}} U_0\|^2 = \rho_{20}^2 \tag{4.18}$$

which shows that $U(t)$ is bounded for finite-time T_2 in $D(L)$. Integrating the differential inequality (4.16) provide a priori energy type estimate

$$\int_0^{T_2} \| LU(t)\|^2 dt \le \frac{1}{\alpha}\{\| L^{\frac{1}{2}} U_0\|^2 + 2\alpha(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2(\| \Omega\|^{\frac{1}{2}} + \varepsilon)^2 + \frac{c_0^2}{4\alpha}\rho_{20}^6\} = \rho_{21}^2.$$

We now state existence, uniqueness, continuity and differentiability with respect to initial conditions of solution to the initial-value problem (4.2) and utilize the above a priori estimates to prove the results.

Proposition 4.1.2 Under the above hypothesis for $U_0 \in \mathrm{Y}$ given, (4.2) generate a dynamical system $U(t) = S(t)U_0$ satisfying

$$U \in C([0,T];\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}}))$$

for all $T > 0$ and if $U_0 \in D(L^{\frac{1}{2}})$ then

$$U \in C([0,T];D(L^{\frac{1}{2}})) \cap L^2(0,T;D(L))$$

for all $T > 0$ where $T = min\{T_1, T_2\}$ is finite-time.

Proof: For fixed m, the Faedo-Galerkin approximation

$$U_m = \sum_{i=1}^{m} g_{im}(t) W_i$$

of the solution U of (4.2) is defined by the finite-dimensional system of nonlinear ordinary differential equations for $g_{im}(t)$ given by

$$\begin{aligned} &\frac{dU_m}{dt} + LU_m + P_m N(U_m) = 0 \\ &U_m(0) = P_m U_0 \end{aligned} \tag{4.19}$$

where P_m is the orthogonal projection in $\mathrm{Y}, D(L^{\frac{1}{2}}), D(L^{\frac{-1}{2}})$ and $D(L)$ onto the space spanned by the m eigenfunctions of L given in (4.5). Let $\mathbf{X}$ be the finite-dimensional Hilbert space spanned by the m eigenfunctions of L given in (4.5) with inner product $((.,.))$ induced by $D(L)$. Consider a closed ball of radius ρ_{21}^2 contained in $D(L)$. Integration of (4.19) and the above a priori energy type estimates provide a continuous mapping defined by the integral of (4.19) from the closed ball of radius ρ_{21}^2 into itself. Thus, by the Leray-Shauder principle \cite{HALE,LADY} the finite-dimensional system of nonlinear ordinary differential equations (4.19) has at least one solution U_m inside the ball of radius ρ_{21}^2. The passage to the limit

$$m \to \infty \ \ and \ \ T_m \to T$$

follows from the following a priori estimates: Utilizing the differential inequality (4.11) yields

$$\begin{aligned} &\frac{d}{dt} \| U_m \|^2 + \alpha \lambda_1 \| U_m \|^2 \\ &\leq \frac{\lambda_1}{\alpha} \left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \| \rho_m \|^2 \end{aligned} \tag{4.20}$$

which together with a priori estimate ()4.12 and the subsequent one imply U_m remains bounded in

$$L^{\infty}((0,T);\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}})).$$

This useful result combined with the weak compactness implies there is a subsequence also denoted by U_m and

$$U \in L^{\infty}(0,T;\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}}))$$

such that

$$U_m \to \begin{cases} U \in L^2(0,T;D(L^{\frac{1}{2}})) & \text{weakly} \\ U \in L^\infty(0,T;\mathrm{Y}) & \text{weak-star.} \end{cases}$$

Invoking the same inequalities that let to the differential inequality (4.15), give the boundeness of $N(U_m)$ and $P_m N(U_m)$ in $L^2(0,T;D(L^{\frac{-1}{2}}))$. Furthermore, from (4.19), it follows that $\frac{dU_m}{dt}$ is also bounded in $L^2(0,T;D(L^{\frac{-1}{2}}))$. Utility of this result and weak compactness yield

$$\frac{dU_m}{dt} \to \frac{dU}{dt} \in L^2(0,T;D(L^{\frac{-1}{2}}))$$

weakly. The assertion of Lebesgue dominated convergence theorem as well as the above convergence results yield $U_m \to U \in L^2(0,T;\mathrm{Y})$ strongly. With strong convergence we can declare strong assertions. We pass the limit in (4.19) and obtain the required result (4.2).

We obtain

$$U \in C([0,T];\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}})).$$

The result

$$U \in C([0,T];D(L^{\frac{1}{2}})) \cap L^2(0,T;D(L))$$

We conclude that if $U_0 \in D(L^{\frac{1}{2}})$ and by the Faedo-Galerkin technique

$$U \in C([0,T];D(L^{\frac{1}{2}})) \cap L^2(0,T;D(L))$$

for all $T > 0$. According to Gronwall's inequality, uniqueness and continuity with respect to initial conditions of solution $U(t)$ follows from considering the difference between two solutions of the initial-value problem (4.2) and employing a priori estimates (4.12-4.18). Uniqueness and continuity with respect to initial conditions of solution $U(t)$ generate a dynamical system which is described by continuous solution operators $S(t)$, $t \in \mathbb{R}$ defined by

$$S(t)U_0 = U(t) \equiv S(U_0,t)$$

satisfying the group property

$$\begin{aligned} &S(t)U(s) = S(s)U(t) \\ &S(0)U = U \quad \forall \ t,s \in \mathbb{R} \end{aligned} \tag{4.21}$$

such that $t+s \le T$. That the solution operators $S(t)$ are continuous is a consequence of the continuity of $U(t)$ in time and in initial conditions. The group property is a consequence of the injectivity of the solution operators which is equivalent to the backward uniqueness of solution for the initial-value problem (4.2). Next, we turn our attention to establish uniqueness, continuity and Fr$e$chet differentiability with respect to initial conditions of solution $U(t)$ for the initial-value problem (4.2) with the work of Temam [92, 91, 90] as an exploration upon which we build. The technique of the proof uses the linearization of the evolution equation (4.2) about the difference of two given solutions. Let u and

v be solutions of the initial value problem (4.2) corresponding to the initial conditions $u(0)=u_0$ and $v(0)=v_0$, respectively. Then the difference $w=v-u$ satisfies

$$\begin{aligned} &\frac{dw}{dt}+Lw+N(v)-N(u)=0 \\ &w(0)=v_0-u_0 \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} N(v)-N(u) &= B(v,v)-B(u,u) \\ &= B(v,w)+B(w,u) \\ &= B(u,w)+B(w,u)+B(w,w). \end{aligned} \tag{4.23}$$

The linearization of the nonlinear equation (4.22) about a given solution reduces to the variational equation

$$\begin{aligned} &\frac{d\Psi}{dt}+L\Psi+l_0(t)\Psi=0 \\ &\Psi(0)=v_0-u_0=\xi \end{aligned} \tag{4.24}$$

where $l_0(t)w=B(u(t),w)+B(w,u(t))$ and $l_0(t)$ is a linear bounded operator from $D(L^{\frac{1}{2}})$ to $D(L^{\frac{-1}{2}})$. Consequently, the equation of variation is a linear nonautonomous ordinary differential equation. The difference $\varphi=w-\Psi$ satisfies

$$\begin{aligned} &\frac{d\varphi}{dt}+L\varphi+l_0(t)\varphi+l_1(t;w(t))=0 \\ &\varphi(0)=0 \end{aligned} \tag{4.25}$$

where $l_1(t;w(t))=B(w,w)$. From the last equality of (4.23), we have $N(v)-N(u)=l_0(t)w+l_1(t;w)$.

Proposition 4.1.3

If $u_0\in \mathrm{Y}$ and u is the unique solution of the nonlinear initial-value problem (4.2) then the corresponding variational equation (4.24) has a unique solution Ψ satisfying

$$\Psi\in C([0,T];\mathrm{Y})\cap L^2(0,T;D(L^{\frac{1}{2}}))$$

for all $T>0$ where $T=min\{T_1,T_2\}$ is finite-time. Moreover, the dynamical system $u_0\to S(t)u_0$ is Fr$e$chet differentiable in Y with differential $L(t,u_0):\xi\to\Psi(t)$ given by the solution of the variational equation.

Proof: Existence and uniqueness of solution to the variational equation (4.24) is proved by the courtesy of the splendid Faedo-Galerkin technique and the above a priori energy type estimates. Taking the inner product of the equation (4.22) with w and application of Cauchy-Schwarz inequality, Young inequality and Sobolev inequalities yield

$$\frac{d}{dt}\| w\|^2+\| L^{\frac{1}{2}}w\|^2\le k^2\| w\|^2$$

where $k=c_1\rho_{21}$ and ρ_{21} is given in the estimate (4.18). By the virtue of the Gronwall's inequality we obtain

$$\| v(t)-u(t)\|^2\le exp(k^2T)\| v_0-u_0\|^2 \tag{4.27}$$

$\forall t \in (0,T)$. The sharp a priori estimate (4.27) illustrates forward uniqueness and continuity with respect to initial conditions of solution. Furthermore, invoking the estimates (4.26) yield

$$\int_0^t \| L^{\frac{1}{2}} w(s)\|^2 ds \leq exp(k^2 T) \| v_0 - u_0\|^2.$$

Additionally, taking the inner product of equation (4.25) with φ we obtain

$$\frac{d}{dt}\| \varphi\|^2 + \frac{1}{2}\| L^{\frac{1}{2}}\varphi\|^2 \leq k^2 \| \varphi\|^2 + \frac{c_1^2}{\lambda_1}\| L^{\frac{1}{2}} w(t)\|^3 \tag{4.28}$$

By the utility of Gronwall's inequality we obtain the estimate

$$\| \varphi(t)\|^2 \leq \frac{c_1^2}{\lambda_1}\int_0^T \| L^{\frac{1}{2}} w(s)\|^3 ds$$

for all $t \in [0,T]$. Consequently it follows that

$$\| \varphi(t)\|^2 \leq \frac{c_1^2}{\lambda_1} exp(3k^2 T/2) \| v_0 - u_0\| \tag{4.29}$$

which gives the required result

$$\frac{\| S(t)v_0 - S(t)u_0 - L(t,u_0).(v_0 - u_0)\|^2}{\| v_0 - u_0\|^2} = o(\| v_0 - u_0\|)$$

as $v_0 \to u_0$. Thus, the dynamical system $S(t): u_0 \to u(t)$ is Fr$e$chet differentiable at u_0 in the Hilbert space Y.

Next, we investigate the problem of proving backward uniqueness of solution for the initial-value equation (4.2) that establish the injectivity properties and group properties of the solution operators $S(t)$. The backward uniqueness of solution is proved in approximately the same way as Fr$e$chet differentiability result using the standard log-convexity method that has been employed in [3, 91]. In order to motivate this objective of the injectivity of the solution operators, consider two solutions u and v of the initial-value problem elaborated above such that at time $t = t_*$ both u and v satisfy the equation (4.2) for $t \in (t_* - \epsilon, t_*)$ and $u(t_*) = v(t_*)$. Given that this hypothesis is valid, backward uniqueness of the solution operators is accomplished if $u(t) = v(t)$ for all time $t < t_*$ whenever the solutions are well-defined. Alternatively, suppose the solution operators satisfy

$$S(t) = u(t+s) \forall \ t > 0 \ \ s \in \mathbb{R}$$

then for $\tau \in (0, \epsilon)$, the problem of backward uniqueness is
whether the following hold:

$$S(\tau)u(t_* - \tau) = S(\tau)v(t_* - \tau) \ \Rightarrow u(t_* - \tau) = v(t_* - \tau) \tag{4.30}$$

which is the injectivity of the solution operators $S(\tau)$.

Proposition 4.1.4 Suppose the following hold

$$u, v \in L^{\infty}(0,T; V = D(L^{\frac{1}{2}})) \cap L^2(0,T; H = D(L))$$

then the difference $w = v - u$ satisfies the differential equation (4.22) for $t \in (0,T)$ and the solution operators $S(t)$ are injective.

We give a sketch of the proof due to Temam [91].

Proof: We will use the notation $\varphi=\varphi(t)$ for the quotient

$$\varphi = \frac{\| \varphi(t)\|_V^2}{\| \varphi(t)\|_H^2}$$
$$= \frac{(L\varphi(t), \varphi(t))}{(\varphi(t), \varphi(t))}.$$

Invoking Cauchy-Schwarz inequality, Young's inequality, Sobolev inequalities, continuity properties of the nonlinear operator $B(w,w)$ established above and the fact that $(Lw - \varphi w, w) = 0$ we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\varphi &= \frac{((\frac{dw}{dt}, w))}{\| w\|^2} - \frac{\| w\|_V^2}{\| w\|^4}(\frac{dw}{dt}, w) \\
&= \frac{1}{\| w\|^2}(\frac{dw}{dt}, Lw - \varphi w) \\
&= \frac{1}{\| w\|_V^2}(N(v) - N(u) - Lw, Lw - \varphi w) \\
&= -\frac{\| Lw - \varphi w\|^2}{\| w\|^2} + \frac{1}{\| w\|^2}(Lw, N(v) - N(u)) \\
&\leq -\frac{\| Lw - \varphi w\|^2}{2\| w\|^2} + \frac{\| N(v) - N(u)\|^2}{2\| w\|^2} \\
&\leq -\frac{\| Lw - \varphi w\|^2}{2\| w\|^2} + k^2\varphi
\end{aligned} \tag{4.31}$$

where $N(v) - N(u)$ is given in (4.23) and the function $k(t) = k_1(t) + B(w,w)$ is defined by the mapping

$$k_1(t) : t \to c(\| u(t)\|_V^{\frac{1}{2}} \| Lu(t)\|^{\frac{1}{2}} + \| v(t)\|_V^{\frac{1}{2}} \| Lv(t)\|^{\frac{1}{2}}) \in L^2(0,T).$$

Utility of Gronwall's inequality into (4.31) gives

$$\varphi(t) \leq \varphi(0) exp(2\int_0^t k^2(s)ds) \;\; t \in (0,T).$$

Thus, if the difference $w = v - u$ satisfies the above conditions and

$$w(\tau) = 0 \;\; \Rightarrow \;\; w(t) = 0, \;\; t \in [0,T].$$

We proceed by contradiction. Hence, suppose that $\| w(t_0) \neq 0\|$ for $t_0 \in [0,T)$. As a result by continuity we have $\| w(t) \neq 0\|$ on some open interval $(t_0, t_0 + \epsilon)$ and we employ the notation $t_* \leq T$ for the largest time for which

$$w(t) \neq 0 \;\; \forall t \in [t_0, t_*).$$

It follows that $w(t_*) = 0$. In the interval $[t_0, t_*)$ the mapping given by $t \to log\| w(t)\|$ is well-defined and the fact that $w(t)$ satisfies the differential equation (4.22) we obtain

$$\frac{d}{dt} log \frac{1}{\| w\|} \leq 2\varphi + k^2.$$

Consequently for the time $t \in [t_0, t_*)$ the following is valid:

$$log \frac{1}{\| w\|} \leq log \frac{1}{\| w(t_0)\|} + \int_{t_0}^{T} 2\varphi(s) + k^2(s) ds$$

which illustrates the boundedness of $\frac{1}{w(t)}$ from above as the time $t \to t_*$ from below and this is a contradiction.

4.1.2 Rotating Boussinesq equations with Reynolds stress

The goal of this section is to establish results of existence, uniqueness and differentiability of the solutions for the initial-boundary value problem of β-plane rotating Boussinesq equations with Reynolds stress (2.29-2.31). We prove the significant aspects of well-posedness of solution utilizing the machinery developed above and additional accessible techniques constructed in this section. The functional formulation for the evolution equation of β-plane rotating Boussinesq equations with Reynolds stress (2.29-2.31) takes the following form of an initial value problem in the Hilbert space Y :

$$\begin{aligned} &\frac{dU}{dt} + LU + N(U) = 0 \\ &U(0) = U_0 \end{aligned} \tag{4.32}$$

where $U = (u, v, w, \rho)$. The boundary conditions have been taken to be space-periodic in order to simplify the analysis. Concerning the linear operator LU and the nonlinear operator $N(U)$ we have

$$N(U) = B(U,U) + CU - FU$$

$$FU = -(0, 0, \frac{\rho}{Ro}, \frac{Ro}{Fr^2} w)$$

$$CU = -\begin{pmatrix} \frac{Ek}{Ro} \triangle \underline{u} \\ \frac{1}{Ed} \triangle \rho \end{pmatrix} = \begin{pmatrix} A_l \underline{u} \\ A_l \rho \end{pmatrix}$$

$$B(U,U) = B(U) = \begin{pmatrix} \underline{u} \cdot \nabla \underline{u} \\ \underline{u} \cdot \nabla \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_l(\underline{u}, \underline{u}) \\ \mathbf{b}_l(\underline{u}, \rho) \end{pmatrix}$$

$$LU = TU + SU$$

$$TU = -\begin{pmatrix} \frac{Ek}{Ro} \triangle^6 \underline{u} \\ \frac{1}{Ed} \triangle^6 \rho \end{pmatrix} = \begin{pmatrix} A_l^6 \underline{u} \\ A_l^6 \rho \end{pmatrix}$$

$$S = -(\frac{1 + y\beta Ro}{Ro}) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \forall\, U \in D(L) = D(T).$$

In order to develop basic results concerning whether there exists a unique solution in the large for the general three-dimensional space variables for initial-value problem (4.32), we derive a priori energy type estimates useful in proving that (4.32) generate a dynamical system, denoted, $U(t) = S(t)U_0$, where $U(t)$ is the unique solution uniformly bounded in time and $S(t)$ is a group of continuous nonlinear solution operators. The principal result concerning the existence of such a unique solution will be proven by the utility of the Faedo-Galerkin technique, a priori energy type estimates and Leray-Schauder principle. The linear operator LU of the initial-value problem (4.32) satisfies

$$(LU,U) = (TU,U) \equiv a(U,U) \tag{4.33}$$

where $a(.,.)$ is symmetric bilinear form. It follows from [92, 27, 66, 50] that this implies L is self-adjoint. Additionally,

$$(LU,U) = (TU,U) \geq \lambda \| U\|^2 > 0 \tag{4.34}$$

which implies the bilinear form induced by L is coercive and L^{-1} is compact and self-adjoint. By the courtesy of an excellent result from [92, 27, 66, 50] , λ is the smallest eigenvalue of Laplace's operator and Stokes operator. Also, the symmetry of the bilinear form $a(U,W)$, together with the coerciveness of $a(U,U)$ imply the existence of a sequence of positive eigenvalues and a corresponding sequence of eigensolutions which yield an orthonormal basis of Y such that

$$\begin{aligned} &LW_i = \lambda_i W_i \\ &0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots, \end{aligned} \tag{4.35}$$

and $\lambda_i \to \infty$ as $i \to \infty$. Moreover,

$$\begin{aligned} &\| L^{\frac{1}{2}}U\| \geq \lambda_1^{\frac{1}{2}} \| U\| \\ &\| LU\| \geq \lambda_1^{\frac{1}{2}} \| L^{\frac{1}{2}}U\| \end{aligned} \tag{4.36}$$

The following useful inequalities are utilized in the derivation of a priori energy type estimates:

$$(B(U),U) = \int_\Omega [(\underline{u} \cdot \nabla \underline{u} \cdot \underline{u}) + (\underline{u} \cdot \nabla \rho)\rho] dx = 0 \tag{4.37}$$

$$\| CU\|^2 \leq \lambda_1 \| L^{\frac{1}{2}}U\|^2 \leq \lambda_1^{\frac{1}{2}} \| LU\| \ \ \| L^{\frac{1}{2}}U\| \tag{4.38}$$

Since $LC = CL$, the inequality

$$\| L^{\frac{1}{2}}CU\|^2 \leq \lambda_1 \| LU\|^2 \tag{4.39}$$

holds. By the courtesy of the Sobolev inequality and continuity properties of the nonlinearity \$B(U)\$ the following bounds are satisfied

$$\| B(U)\|^2 \leq c_0^2 (CU,U)[\| CU\|^2 + \| U\|^2] \leq c_0^2 (\lambda_1 + 1/\lambda_1) \| U\| \ \ \| L^{\frac{1}{2}}U\|^2 \| LU\| \tag{4.40}$$

By the virtue of Cauchy-Schwarz inequality, Young's inequality and (4.40) the following is satisfied

$$|(B(U) + CU, LU)| \leq 54(c_1^4 \| U\|^2 \| L^{\frac{1}{2}}U\|^2 + \lambda_1) \| L^{\frac{1}{2}}U\|^2 + \frac{1}{4} \| LU\|^2 \tag{4.41}$$

where $c_1 = c_0^2(\lambda_1 + 1/\lambda_1)$. Also, from [22] the following a priori estimates is valid

$$\| L^{\frac{1}{2}} B(U)\|^2 \leq c\| LU\|^2. \tag{4.42}$$

Concerning the solution to the steady linear case we have

$$\begin{aligned}(LU+CU,U) &= (TU,U)+(CU,U)\\ &\geq (\lambda+\lambda_1^{\frac{1}{6}})\| U\|^2 > 0,\end{aligned} \tag{4.43}$$

which implies $L+C$ is positive and hence the bilinear form it induces is coercive. This establishes existence and uniqueness of solution to the steady linear case by the Riesz-Fr$e$chet representation theorem or the Lax-Milgram lemma. The following mini-max principle from [92, 23] will be utilized:

$$\lim_{t\to\infty} sup\| F\|^2 \leq (\frac{Ro}{Fr^2}+\frac{1}{Ro})^2\| \Omega\|^2 \tag{4.44}$$

where $\| \Omega\|$ is the measure of Ω. Taking the inner product of (4.32) with \$U\$, and invoking the inequalites (4.43), (4.36) and Young's inequality yield the following differential inequality:

$$\begin{aligned}&\frac{d}{dt}\| U\|^2 + \alpha\lambda_1\| U\|^2\\ &\leq \frac{d}{dt}\| U\|^2 + \alpha\| L^{\frac{1}{2}}U\|^2\\ &\leq \frac{1}{\alpha}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2\| \Omega\|^2\end{aligned} \tag{4.45}$$

where $\alpha = \lambda + \lambda_1^{\frac{1}{6}}$. The consideration of Gronwall's inequality in (4.45) provides a priori estimate

$$\| U(t)\|^2 \leq \| U_0\|^2 exp(-\alpha t) + \frac{1}{\alpha^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2\| \Omega\|^2(1-exp(-\alpha t)) \tag{4.46}$$

which is asymptotic stability with flow energy and entropy production provided by

$$E(t) = \frac{1}{2}\| U(t)\|^2 = \frac{1}{2}\int_\Omega (\underline{u}^2+\rho^2)dx$$

and exponential dissipation rate $\alpha = \lambda + \lambda_1^{\frac{1}{6}}$. Thus, we are able to assert that the following asymptotic estimate is valid

$$\lim_{t\to\infty} sup\| U(t)\|^2 \leq \frac{1}{\alpha^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2\| \Omega\|^2 \tag{4.47}$$

which shows $U(t)$ is uniformly bounded in time in Υ. Taking the inner product of (4.2) with LU and using inequalities (4.36) and (4.41), we find that

$$\begin{aligned}&\frac{d}{dt}\| L^{\frac{1}{2}}U\|^2 + \lambda_1\| L^{\frac{1}{2}}U\|^2\\ &\leq \frac{d}{dt}\| L^{\frac{1}{2}}U\|^2 + \| LU\|^2\\ &\leq 2(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2\| \Omega\|^2 + 108(c_1^4\| U\|^2\| L^{\frac{1}{2}}U\|^4 + \lambda_1\| L^{\frac{1}{2}}U\|^2)\end{aligned} \tag{4.48}$$

We estimates (4.45) using the inequality

$$\lim_{t\to\infty} sup\int_t^{t+1} \| \bar{L^2}U(s)\|^2 ds \le (1/\alpha^2 + 1/\alpha^3)(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2 \| \Omega\|^2$$

and by the virtue of Gronwall's inequality in (4.48) we obtain a priori estimate

$$\| \bar{L^2}U(t)\|^2 \le \rho_1^2 \tag{4.49}$$

where $\rho_1^2 = \nu_1\nu_2$, and the parameters ν_1 and ν_2 are given by the following involved and tedious expressions, respectively,

$$\nu_1 = (2 + 107\lambda_1(1/\alpha^2 + 1/\alpha^3) + 1/\alpha^2 + 1/\alpha^3)(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2 \| \Omega\|^2$$

and

$$\nu_2 = exp(108c_0^2(1/\alpha^2 + 1/\alpha^3)(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2 \| \Omega\|^2).$$

Consequently from the last inequality the following is valid

$$\lim_{t\to\infty} sup\| \bar{L^2}U(t)\|^2 \le \rho_1^2 \tag{4.50}$$

which shows $U(t)$ is uniformly bounded in time in $D(\bar{L^2})$. Similarly, taking the inner product of (4.32) with L^2U and making use of the inequality (4.36) and (4.42) yield

$$\begin{aligned} &\frac{d}{dt}\| LU\|^2 + \lambda_1\| LU\|^2 \\ &\le \frac{d}{dt}\| LU\|^2 + \| \bar{L^2}U\|^2 \\ &\le 3c^2\| LU\|^4 + 3\lambda_1\| LU\|^2 + 3(\frac{1}{Ro^2} + \frac{Ro^2}{Fr^4})\rho_1^2 \end{aligned} \tag{4.51}$$

Using the differential inequality (4.48) and invoking H$\ddot{o}$lder inequality we obtain a priori energy type estimate

$$\int_t^{t+1} \| LU(s)\|^2 ds \le \nu_3^2$$

where $\nu_3^2 = a_1 + a_2$ denotes the sum of the involved expressions

$$a_1 = \lambda_1(1/\alpha^2 + 1/\alpha^3)(\frac{Ro}{Fr^2} + \frac{1}{Ro})^2 \| \Omega\|^2 + \rho_1^2$$

and

$$a_2 = 108c_0^2(1/\alpha^2 + 1/\alpha^3)^2(\frac{Ro}{Fr^2} + \frac{1}{Ro})^4 \| \Omega\|^4.$$

Substitution of this a priori energy type estimate into (4.51) yields

$$\| LU(t)\|^2 \le \rho_2^2 \tag{4.52}$$

with ρ_2^2 denoting

$$\rho_2^2 = 3c^2\nu_3^4 + 2\lambda_1\nu_3^2 + 3(\frac{1}{Ro^2} + \frac{Ro^2}{Fr^4})\rho_1^2.$$

Consequently, putting the estimates together we obtain

$$\lim_{t\to\infty} sup\| LU(t)\|^2 \le \rho_2^2 \tag{4.53}$$

which shows $U(t)$ is uniformly bounded in time in $D(L)$. We now state existence, uniqueness, continuity and differentiability with respect to initial conditions of solution to the initial-value problem (4.32) and invoke the above a priori estimates to prove the result.

Proposition 4.1.5 Under the above hypothesis for $F \in D(L^{\frac{1}{2}})$ and $U_0 \in \mathrm{Y}$ given, (4.32) generate a dynamical system $U(t) = S(t)U_0$ satisfying

$$U \in C([0,T];\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}}))$$

for all $T > 0$. Moreover, if $U_0 \in D(L^{\frac{1}{2}})$ then

$$U \in C([0,T];D(L^{\frac{1}{2}})) \cap L^2(0,T;D(L))$$

for all $T > 0$.

Proof: For fixed m, the Faedo-Galerkin approximation

$$U_m = \sum_{i=1}^{m} g_{im}(t)W_i$$

of the solution U of (4.32) is defined by the finite-dimensional system of nonlinear ordinary differential equations for $g_{im}(t)$ given by

$$\begin{aligned} &\frac{dU_m}{dt} + LU_m + P_m B(U_m) + CU_m = P_m F \\ &U_m(0) =_m U_0 \end{aligned} \tag{4.54}$$

where P_m is the orthogonal projection in $\mathrm{Y}, D(L^{\frac{1}{2}}), D(L^{\frac{-1}{2}})$ and $D(L)$ onto the space spanned by the m eigenfunctions of L given in (4.35). Let $\mathbf{X}$ be the finite-dimensional Hilbert space spanned by the m eigenfunctions of L given in (4.35) with inner product $((.,.))$ induced by $D(L)$. Consider a closed ball of radius ρ_2^2 given by (4.53) contained in $D(L)$. Integration of (4.54) and the above a priori energy type estimates provide a continuous mapping defined by the integral of (4.54) from the closed ball of radius ρ_2^2 given by (4.53) into itself. Thus, by the Leray-Shauder principle [36, 66] the finite-dimensional system of nonlinear ordinary differential equations (4.54) has at least one solution U_m inside the ball of radius ρ_2^2. The passage to the limit

$$m \to \infty \ \ and \ \ T_m \to T = \infty$$

follows from the following a priori estimates: Utilizing the differential inequality (4.54) provides

$$\frac{d}{dt}\| U_m\|^2 + \alpha\lambda_1\| U_m\|^2 \le \frac{1}{\alpha}\left(\frac{Ro}{Fr^2} + \frac{1}{Ro}\right)^2 \tag{4.55}$$

which implies U_m remains bounded in

$$L^{\infty}((0,T);\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}})).$$

This useful result combined with the weak compactness implies there is a subsequence also denoted by U_m and

$$U \in L^{\infty}(0,T;\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}}))$$

such that

$$U_m \to \begin{cases} U \in L^2(0,T;D(L^{\frac{1}{2}})) & \text{weakly} \\ U \in L^{\infty}(0,T;\mathrm{Y}) & \text{weak-star.} \end{cases}$$

Using (4.37) and (4.40), we obtain the boundeness of $B(U_m)$ and $P_m B(U_m)$ in $L^2(0,T;D(L^{\frac{-1}{2}}))$. Furthermore, from (4.54), $\frac{dU_m}{dt}$ is also bounded in $L^2(0,T;D(L^{\frac{-1}{2}}))$. Utility of this result and weak compactness yield

$$\frac{dU_m}{dt} \to \frac{dU}{dt} \in L^2(0,T;D(L^{\frac{-1}{2}}))$$

weakly. The assertion of Lebesgue dominated convergence theorem as well as the above convergence results yield $U_m \to U \in L^2(0,T;\mathrm{Y})$ strongly. We pass the limit in (4.54) and obtain the required result (4.32). We obtain

$$U \in C([0,T];\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}})).$$

Similarly, from (4.48) and (4.51) we conclude that if $U_0 \in D(L^{\frac{1}{2}})$ then by the Faedo-Galerkin technique

$$U \in C([0,T];D(L^{\frac{1}{2}})) \cap L^2(0,T;D(L))$$

for all $T > 0$. According to Gronwall's inequality, uniqueness and continuity with respect to initial conditions of solution $U(t)$ follows from considering the difference between two solutions of the initial-value problem (4.32) and employing a priori estimates (4.47-4.53). Uniqueness and continuity with respect to initial conditions of solution $U(t)$ generate a dynamical system which is prescribed by continuous solution operators $S(t)$, $t \in \mathbb{R}$ defined by

$$S(t)U_0 = U(t) \equiv S(U_0, t)$$

satisfying the group property

$$\begin{aligned} &S(t)U(s) = S(s)U(t) \\ &S(0)U = U \quad \forall \ t,s \in \mathbb{R}. \end{aligned} \tag{4.56}$$

That the solution operators $S(t)$ are continuous is a consequence of the continuity of \$U(t)\$ in time and in initial conditions. The group property is a consequence of the injectivity of the solution
operators which is equivalent to the backward uniqueness of solution for the initial-value problem (4.32). Next, we turn our attention to establish uniqueness, continuity and Fr\$\acute{e}\$chet differentiability with respect to initial conditions of solution $U(t)$ for the initial-value problem (4.32). As in the previous section, the technique of the proof uses the linearization of the evolution equation (4.32) about the difference of two given solutions. For the sake of completeness, we repeat the analysis here. Consider u and v satisfying the evolution equation (4.32) corresponding to the initial conditions $u(0) = u_0$ and $v(0) = v_0$, respectively. The precise analysis of the solution difference $w = v - w$ yields

$$\begin{aligned} &\frac{dw}{dt}+Lw+N(v)-N(u)=0 \\ &w(0)=v_0-u_0 \end{aligned} \tag{4.57}$$

where

$$\begin{aligned} N(v)-N(u) &= B(v,v)-B(u,u)+Cw \\ &= B(v,w)+B(w,u)+Cw \\ &= B(u,w)+B(w,u)+B(w,w)+Cw. \end{aligned} \tag{4.58}$$

The linearization of (4.57) along the dynamical system trajectory satisfies the variational equation

$$\begin{aligned} &\frac{d\Psi}{dt}+L\Psi+l_0(t)\Psi=0 \\ &\Psi(0)=v_0-u_0=\xi \end{aligned} \tag{4.59}$$

where $l_0(t)w=B(u(t),w)+B(w,u(t))+Cw$ and $l_0(t)$ is linear bounded operator from $D(L^{\frac{1}{2}})$ to $D(L^{\frac{-1}{2}})$. Consequently, the equation of variation is a linear nonautonomous ordinary differential system. We shall be concerned with the difference $\varphi=w-\Psi$ and the following is valid

$$\begin{aligned} &\frac{d\varphi}{dt}+L\varphi+l_0(t)\varphi+l_1(t;w(t))=0 \\ &\varphi(0)=0 \end{aligned} \tag{4.60}$$

where $l_1(t;w(t))=B(w,w)$. The remarkable feature of the last equality is that from (4.58), we have $N(v)-N(u)=l_0(t)w+l_1(t;w)$. We are now in a position to state the keystone result:

Proposition 4.1.6 If $u_0\in \mathrm{Y}$ and u is the unique solution of (4.32) then the variational equation (4.59) has a unique solution Ψ satisfying

$$\Psi\in C([0,T];\mathrm{Y})\cap L^2(0,T;D(L^{\frac{1}{2}}))$$

for all $T>0$. Moreover, the dynamical system $u_0\to S(t)u_0$ is Fr$e$chet differentiable in Y with differential $L(t,u_0):\xi\to\Psi(t)$ given by the solution of the variational equation.

Proof: Existence and uniqueness of solution to (4.59) is a consequence the Faedo-Galerkin technique and the use of a priori estimates together with the Gronwall's inequality. Next, we exhibit a series of a priori estimates which will be required in establishing the above proposition. Taking the inner product of (4.57) with w and application of Cauchy-Schwarz inequality, Young's inequality and Sobolev inequalities yield

$$\frac{d}{dt}\| w\|^2+\| L^{\frac{1}{2}}w\|^2\le k^2\| w\|^2, \tag{4.61}$$

where $k=c_1\rho_2$ and ρ_2 is given in the estimate (4.52). According to the Gronwall's inequality the following is valid

$$\| v(t)-u(t)\|^2\le exp(k^2T)\| v_0-u_0\|^2 \tag{4.62}$$

$\forall t\in(0,T)$. The sharp a priori estimate (4.62) yield forward uniqueness and continuity with respect to initial conditions of solution. Furthermore, from (4.61) we obtain a priori energy type estimate

$$\int_0^t \| L^{\frac{1}{2}} w(s)\|^2 ds \leq exp(k^2 T) \| v_0 - u_0\|^2.$$

By the utility of the inner product of (4.60) with φ we obtain the differential inequality

$$\frac{d}{dt}\| \varphi\|^2 + \frac{1}{2}\| L^{\frac{1}{2}}\varphi\|^2 \leq k^2 \| \varphi\|^2 + \frac{c_1^2}{\lambda_1}\| L^{\frac{1}{2}} w(t)\|^3 \tag{4.63}$$

We repeat the construction using Gronwall's inequality and find a priori estimate

$$\| \varphi(t)\|^2 \leq \int_0^T \| L^{\frac{1}{2}} w(s)\|^3 ds$$

$\forall t \in (0,\infty)$. Consequently we obtain the bound

$$\| \varphi(t)\|^2 \leq \frac{c_1^2}{\lambda_1} exp(3k^2 T/2) \| v_0 - u_0\| \tag{4.64}$$

which implies the needed result

$$\frac{\| S(t)v_0 - S(t)u_0 - L(t,u_0).(v_0 - u_0)\|^2}{\| v_0 - u_0\|^2} = o(\| v_0 - u_0\|)$$

as $v_0 \to u_0$. Hence, the dynamical system $S(t): u_0 \to u(t)$ is Fr$e$chet differentiable at $u_0 \in \mathrm{Y}$.

Next, we examine the problem of proving backward uniqueness of solution for the initial-value equation (4.32) that establish the injectivity properties and group properties of the solution operators $S(t)$. The backward uniqueness of solution is proved in approximately the same way as Fr$e$chet differentiability result using the standard log-convexity method as in the previous section. For the sake of completeness of presentation we cover the proof of the injectivity of solution operators here. In order to motivate this objective, consider two solutions u and v of the initial-value problem such that at time $t = t_*$ both u and v satisfy (4.32) for $t \in (t_* - \epsilon, t_*)$ and $u(t_*) = v(t_*)$. Given that this hypothesis is valid, backward uniqueness of the solution operators is accomplished if $u(t) = v(t)$ for all time $t < t_*$ whenever the solutions are well-defined. Alternatively, suppose the solution operators satisfy

$$S(t) = u(t+s) \forall \ \ t > 0 \ \ s \in \mathbb{R}$$

then for $\tau \in (0,\epsilon)$, the problem of backward uniqueness is whether the following hold:

$$S(\tau)u(t_* - \tau) = S(\tau)v(t_* - \tau) \ \Rightarrow u(t_* - \tau) = v(t_* - \tau) \tag{4.65}$$

which is the injectivity of the solution operators $S(\tau)$. The following significant result on the injectivity of the solution operators is valid:

Proposition 4.1.7 Suppose the following hold

$$u, v \in L^{\infty}(0,T; V = D(L^{\frac{1}{2}})) \cap L^2(0,T; H = D(L))$$

then the difference \$w=v-u\$ satisfies the differential equation (4.57) for $t \in (0,T)$ and the solution operators $S(t)$ are injective.

We give a sketch of an elegant proof due to Temam [91.]

Proof: We will use the notation $\varphi = \varphi(t)$ for the quotient

$$\varphi = \frac{\| \varphi(t)\|_V^2}{\| \varphi(t)\|_H^2}$$
$$= \frac{(L\varphi(t), \varphi(t))}{(\varphi(t), \varphi(t))}.$$

Invoking Cauchy-Schwarz inequality, Young's inequality, Sobolev inequalities, continuity properties of the nonlinear operator $B(w,w)+Cw$ developed above and the fact that $(Lw-\varphi w, w)=0$ we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\varphi &= \frac{((\frac{dw}{dt}, w))}{\| w\|^2} - \frac{\| w\|_V^2}{\| w\|^4}(\frac{dw}{dt}, w) \\
&= \frac{1}{\| w\|^2}(\frac{dw}{dt}, Lw-\varphi w) \\
&= \frac{1}{\| w\|_V^2}(N(v)-N(u)-Lw, Lw-\varphi w) \\
&= -\frac{\| Lw-\varphi w\|^2}{\| w\|^2} + \frac{1}{\| w\|^2}(Lw, N(v)-N(u)) \\
&\le -\frac{\| Lw-\varphi w\|^2}{2\| w\|^2} + \frac{\| N(v)-N(u)\|^2}{2\| w\|^2} \\
&\le -\frac{\| Lw-\varphi w\|^2}{2\| w\|^2} + k^2\varphi
\end{aligned} \tag{4.66}$$

where $N(v)-N(u)$ is given in (4.58) and the function $k(t)=k_1(t)+B(w,w)+Cw$ is defined by the mapping

$$k_1(t): t \to c(\| u(t)\|_V^{\frac{1}{2}} \| Lu(t)\|^{\frac{1}{2}} + \| v(t)\|_V^{\frac{1}{2}} \| Lv(t)\|^{\frac{1}{2}}) \in L^2(0,T).$$

Utility of Gronwall's inequality into (4.66) gives

$$\varphi(t) \le \varphi(0) exp(2\int_0^t k^2(s)ds) \;\; t \in (0,T).$$

Thus, if the difference $w=v-u$ satisfies the above conditions and

$$w(\tau)=0 \;\; \Rightarrow \;\; w(t)=0, \;\; t\in[0,T].$$

We proceed by contradiction. Hence, suppose that $\| w(t_0)\neq 0\|$ for $t_0 \in [0,T)$. As a result by continuity we have $\| w(t)\neq 0\|$ on some open interval $(t_0, t_0+\epsilon)$ and we employ the notation $t_* \le T$ for the largest time for which

$$w(t) \neq 0 \;\; \forall t \in [t_0, t_*).$$

It follows that $w(t_*)=0$. In the interval $[t_0, t_*)$ the mapping given by $t \to log\| w(t)\|$ is well-defined and the fact that $w(t)$ satisfies the differential equation (4.22) we obtain

$$\frac{d}{dt} log \frac{1}{\| w\|} \le 2\varphi + k^2.$$

Consequently for the time $t \in [t_0, t_*)$ the following is valid:

$$log\frac{1}{\| w\|} \leq log\frac{1}{\| w(t_0)\|} + \int_{t_0}^{T} 2\varphi(s) + k^2(s)ds$$

which illustrates the boundedness of $\frac{1}{w(t)}$ from above as the time $t \to t_*$ from below and this is a contradiction.

CHAPTER 5

Nonlinear stability of solutions

5.1 Aspects of stability and attraction

In this section we consider nonlinear stability and attractors of solutions of rotating Boussinesq equations by adapting a priori estimates employed in establishing well-posedness and differentiability of solutions. One of the advantages of the results presented in this chapter is that they may be considered as more refined generalization of results such as (4.12) and (4.46) for the dissipation of flow energy and the entropy production provided by

$$E(t) = \frac{1}{2}\| U(t)\|^2 = \frac{1}{2}\int_\Omega (\underline{u}^2 + \rho^2)dx,$$

which is specification of energy and the entropy production in the L^2-norm. Hence, the results of this section on nonlinear stability differ in form rather than in essence from their counterparts utilized in establishing well-posedness and differentiability of solutions. The problem of showing that the rest state of system (4.2) or (4.32) is nonlinearly stable is investigated using ideas developed in Galdi *et al* [27, 28]. An important quantity in analyzing whether the solution of an evolution equation is bounded in some suitable norm is that of Lyapunov functional or a priori energy type estimates of the system as we have already noted in establishing well-posedness. We will illustrate in the sequel a technique for constructing Lyapunov functionals with interpretations such as energy, enstrophy and entropy which are decreasing along solutions. The equations governing the flow of an incompressible stratified fluid under the Coriolis force induce damping mechanisms in the nature of viscosity, diffusion, stratification and rotation effects that manifest themselves in the evolution equation with the existence of Lyapunov functionals which are equivalent to some norm induced by the inner product of solution. Here we state a technique for constructing generalized Lyapunov functionals that furnish necessary and sufficient conditions for nonlinear stability. Specifically, the tools used are spectral properties of the linear operator and a priori estimates for the nonlinear operator. Now we recall definitions which are required in the formulation of the *nonlinear stability criteria* to be customized for use in the solutions of the rotating Boussinesq equations. Consider the following prototype initial-value problem

$$\frac{dU}{dt} = LU + N(U),\ U(0) = U_0. \tag{5.1}$$

where LU is linear operator and $N(U)$ is nonlinear operator in the sense that its Fr$e$chet derivative at zero vanishes. A self-adjoint linear operator L for the initial-value problem (5.1) is called *essentially dissipative* if

$$\begin{aligned} &(LU,U) \le 0 \quad \forall \ \ U \in D(L) \\ &(LU,U) = 0 \ \Rightarrow \ \ U = 0 \end{aligned} \tag{5.2}$$

The complement of essentially dissipative is called essentially non-dissipative. By the spectral theorem [27, 50], essential dissipativity is equivalent to the spectrum of L being nonnegative and zero not an eigenvalue. For every essentially dissipative operator L, the bilinear form

$$(U,W)_L = -(LU,W) \ \forall \ U,W \in D(L) \tag{5.3}$$

defines a scalar product in $D(L)$. We denote by H_L the completion of $D(L)$ in the norm $\|.\|_L$ given by

$$\| U\|_L^2 = -(LU,U).$$

According to the condition (5.2) on essential dissipativity of linear operators, we are in a position to state necessary and sufficient criteria for nonlinear stability using the Lyapunov functional

$$J(U) = -(LU, U) \tag{5.4}$$

and generalized energy flow energy and entropy production

$$E(t) = \frac{1}{2}\{\| U\|^2 + \| U\|_L^2\} \tag{5.5}$$

for the paradigmatic example of initial-value problem (5.1).

Proposition5.1.1 Suppose L is essentially dissipative and the nonlinear operator satisfies

$$\| NU\|^2 \le c\| U\|_L^2 (\| LU\|^2 + \| U\|_L^2) \tag{5.6}$$

where $c > 0$. Then the rest state of system (5.1) is asymptotically stable, that is,

$$J(U) \le \frac{1}{\eta}\{2E(0)(1 + \frac{1}{\gamma\mu_0})exp[\frac{4(1-\gamma)}{\mu_0\gamma}]\}/(t+1), \tag{5.7}$$

whenever

$$\gamma = (1 - cE(0)) > 0, \tag{5.8}$$

where $\eta > 0$ and $\mu_0 = \frac{1}{2}$. Next assume *L* is essentially non-dissipative and the nonlinear operator satisfies

$$\| NU\|^2 \le cJ(U)(\| LU\|^2 + \| U\|^2) \tag{5.9}$$

$c > 0$. Then the rest state is unstable in the sense of Lyapunov.

This proposition is proved in approximately the same way as the analogous fact for the finite-dimensional linear problem with the assistance of a priori energy type estimates that dominates the nonlinearity by linear elliptic operators.

Proof: Taking the inner product of the equation (5.1) with U yields

$$\frac{1}{2}\frac{d}{dt}(U, U) = (LU, U) + (NU, U).$$

Similarly, taking the inner product of the equation (5.1) with LU gives

$$\frac{1}{2}\frac{d}{dt}(LU, U) = (LU, LU) + (LU, NU).$$

Putting the estimates together provide the following equation for the evolution the generalized energy functional (5.5)

$$\frac{dE}{dt} = (LU, U) + (NU, U) - (LU, LU) - (LU, NU). \tag{5.10}$$

Applying Cauchy-Schwarz inequality and Young's inequality in (5.10) yields

$$\begin{aligned} \frac{dE}{dt} &\le (LU, U) + \frac{3}{2}\| LU\|^2 + \| U\| \ \ \| NU\| + \frac{1}{2}\| NU\|^2 \\ &\le -\mu_0(\| LU\|^2 + |U|_L^2) + \| U\| \ \ \| NU\| + \frac{1}{2}\| NU\|^2. \end{aligned} \tag{5.11}$$

Substitution of a priori estimate (5.6) in (5.11) and the courtesy of the generalized energy functional (5.5) provides the differential inequality

$$\frac{dE}{dt} \leq -\mu_0(\| LU\|^2 + \| U\|_L^2)[1 - c^{\frac{1}{2}} \| U\|_L \| U\|(\| LU\|^2 + \| U\|_L^2)^{\frac{-1}{2}} - \frac{c}{2}\| U\|_L^2]$$

$$\leq -\mu_0(\| LU\|^2 + \| U\|_L^2)[1 - \frac{c}{2}\| U\|_L^2] \tag{5.12}$$

$$\leq -\mu_0(\| LU\|^2 + \| U\|_L^2)[1 - cE].$$

According to the inequality (5.12) and assumption (5.8) we obtain a priori energy estimate

$$\mu_0(\| LU\|^2 + \| U\|_L^2)(1 - cE(0)) \leq -\frac{dE}{dt}$$

which implies

$$\int_0^\infty \mu_0(\| LU\|^2 + \| U\|_L^2)(1 - \frac{c}{\mu_1}E(0))dt \leq 2E(0).$$

In this respect it should be noticed that the following a priori energy type estimate holds

$$\int_0^\infty (\|LU\|^2 + \| U\|_L^2)dt \leq \frac{2E(0)}{\mu_0\gamma}.$$

We proceed by recalling the useful property

$$-\frac{1}{2}\frac{d}{dt}(LU, U) = -(LU, LU) + (LU, NU)$$

and invoking the Cauchy-Schwarz inequality, Young's inequality and a priori estimate (5.6) which yield the following differential inequality

$$\frac{d}{dt}\| U\|_L^2 \leq 2c\| U\|_L^2(\| LU\|^2 + \| U\|_L^2). \tag{5.13}$$

In order to establish the hypotheses of Gronwall's inequality we note that (5.13) may be restated equivalently as follows:

$$\frac{d\Phi}{dt} \leq h_1(t) + \frac{2c}{\mu_1}h_2(t)\Phi \tag{5.14}$$

where

$$\Phi(t) = \| U\|_L^2(t+1),$$

$$h_1(t) = \| U\|_L^2,$$

and

$$h_2(t) = \| LU\|^2 + \| U\|_L^2.$$

Thus, the following holds

$$\Phi(t) \leq exp[2c\int_0^t h_2(s)ds]\{\Phi(0) + \int_0^t exp[-2c\int_0^s h_2(\alpha)d\alpha]h_1(s)ds\}.$$

which is the assertion of Gronwall's inequality. Moreover, through the use of estimates from above we deduce the following a priori estimate that guarantee the hypotheses of the Gronwall's lemma are valid

$$\int_0^\infty h_1(t)dt \leq \int_0^\infty h_2(t)dt \leq \frac{2E(0)}{\mu_0\gamma}.$$

Another application of relation (5.8) yields a priori estimate

$$2c\int_0^{\infty} h_2(t)dt \le \frac{4cE(0)}{\mu_0\gamma} = \frac{4(1-\gamma)}{\mu_0\gamma}.$$

Consequently, we have

$$exp[\frac{2c}{\mu_1}\int_0^{\infty} h_2(t)dt] \le exp[\frac{4(1-\gamma)}{\mu_0\gamma}].$$

Similarly, putting the above estimates together we obtain the following finite energy solution

$$\Phi(t) \le 2E(0)(1+\frac{1}{\gamma\mu_0})exp[\frac{4(1-\gamma)}{\mu_0\gamma}]. \tag{5.15}$$

Substitution of $\Phi(t) = \| U\|_1^2(t+1)$ into (5.15) gives the desired nonlinear stability result (5.7) since $J(U) \le \frac{1}{\eta}\| U\|_1^2$ whenever L is essentially dissipative. Hence, as a bonus we obtain the proof of Gronwall's lemma. The instability result holds by following the proof given in (27, 28).

For the rest states, asymptotic stability and attraction coincide. However, for more complicated invariant sets than rest states, an attractor may be considered as the stronger notion of stability than the one given above. The following proposition [93, 92, 91, 90, 23, 22] on the characterization of attractors and determining of the compactness of an ω -limit set is required:

Proposition 5.1.2 Suppose that for $Q \subset H$, $Q \neq \varnothing$, and that for some $s>0$ the set $\cup_{t\ge s} S(t)Q$ is relatively compact in H. Then ω-limit set of Q denoted by $\omega(Q)$ is nonempty, compact and invariant.

Proof: The proposition is proven on page 20 in Temam [91] using properties of ω-limit sets.

The proposition is of central importance in the construction of attractors. In order to prove the major hypothesis that $\cup_{t\ge s} S(t)Q$ is relatively compact in H, it is adequate to illustrate that this set is bounded in a space V compactly imbedded in H by employing the existence of the following injections

$$D(L^{\frac{1}{2}}) = V \subset D(L) \subset \mathrm{Y} = H \subset D(L^{\frac{-1}{2}}).$$

The density and compactness of the injections will find applications in the sequel when we prove existence of attractors for the initial-value problems.

5.2 Rotating Boussinesq equations

The goal of this section is to develop nonlinear stability results for β-plane rotating Boussinesq equations (2.26-2.28). In order to give nonlinear stability bounds that agree approximately with established paradigmatic example of initial-value problem (5.1), the functional formulation for the evolution equation of β-plane rotating Boussinesq equations is restated equivalently as follows:

$$\begin{aligned} &\frac{dU}{dt} + LU + N(U) = 0 \\ &U(0) = U_0 \end{aligned} \tag{5.16}$$

where $U=(u,v,w,\rho)$. Concerning the linear operator LU and the nonlinear operator $N(U)$ we have the following appropriate setup:

$$N(U) = B(U,U)$$

$$LU = TU + SU - FU$$

$$FU = -(0,0,\Pi\frac{\rho}{Ro},\frac{Ro}{Fr^2}w)$$

$$TU = -\begin{pmatrix} \frac{Ek}{Ro}\Pi\triangle\underline{u} \\ \frac{1}{Ed}\triangle\rho \end{pmatrix} = \begin{pmatrix} A_s\underline{u} \\ A_l\rho \end{pmatrix}$$

$$B(U,U) = B(U) = \begin{pmatrix} \Pi\underline{u}\cdot\bigtriangledown\underline{u} \\ \underline{u}\cdot\bigtriangledown\rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_s(\underline{u},\underline{u}) \\ \mathbf{b}_l(\underline{u},\rho) \end{pmatrix}$$

$$S = -\Pi(\frac{1+y\beta Ro}{Ro})\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \forall\, U\in D(L)=D(T).$$

According to the above setup of the linear operator LU, we obtain the following useful result

$$(LU,U) = \frac{Ek}{Ro}\|\bigtriangledown\underline{u}\|^2 + \frac{1}{Ed}\|\bigtriangledown\rho\|^2 + (\frac{1}{Ro}+\frac{Ro}{Fr^2})\int_\Omega \rho w dx \tag{5.17}$$

which is a slight modification of the Lyapunov functional (4.3) that was employed in establishing well-posedness of solution for the problem of β-plane rotating Boussinesq equations (2.26-2.28). With the attendant relation ()5.17 we are able to demonstrate a necessary and sufficient nonlinear stability criterion and hence the elegant power of energy methods is fully realized. The first effort is then to let R and

$$I(\underline{u},\rho) \tag{5.18}$$

be defined as follows

$$1/2R = sup\{\int_\Omega \rho w dx\}/J(U) = supI(\underline{u},\rho)$$

where the supremum is taken for $U\in D(L)$. We shall compile various inequalities about the equation (5.16). Substituting

$$|\int_\Omega \rho w dx| \le (\|\underline{u}\|^2 + \|\rho\|^2)/2$$

into (5.18) and utilizing the Poincar$e$-Friedrichs inequality yields $R\ge Ga=80$. As it is known, the nonlinear stability bounds often give conditions that agree with numerical approximations. Moreover, the adequacy cited above may be interpreted by noting that the size of the critical value Ga depends on the conditions of the numerical approximations and can be considerably improved by implementing more refined and accurate numerical schemes. In the asymptotic stability analysis we take the number $Ga=80$ and we note that we have reformulated the initial-boundary value problem for β-plane rotating Boussinesq equations in order to reflect the influence of Galdi *et al* [27, 28] investigations on energy stability theory. The above stability criterion number provides a great deal of light on the

potentialities of the energy stability theory and validates this exposition from a benchmark point of view and offers the opportunity of invoking established results as a guide for nonlinear stability criteria. It is worth mentioning that the stability criterion number serve as an addition to available paradigmatic examples of stability criteria such as the Richardson number $Ri = \frac{1}{4}$ for stratified shear instability and the Rayleigh number $Ra = 10^{15}$ for penetrative convection [13, 77].

The attendant relation (5.17) may be restated as follows

$$(LU, U) = (1 + (\frac{1}{Ro} + \frac{Ro}{Fr^2}) I(\underline{u}, \rho)) J(U), \tag{5.19}$$

with the Lyapunov functional $J(U)$ defined in (5.4) given by

$$J(U) = \frac{Ek}{Ro} \| \bigtriangledown \underline{u} \|^2 + \frac{1}{Ed} \| \bigtriangledown \rho \|^2. \tag{5.20}$$

It is appropriate at this point to note that if $(\frac{1}{Ro} + \frac{Ro}{Fr^2}) < 2R$ then the following is valid

$$-(LU, U) \leq (-1 + (\frac{1}{2RRo} + \frac{Ro}{2RFr^2})) J(U) < 0, \tag{5.21}$$

and in this case $-L$ is essentially dissipative. Additionally, the following inequalities hold

$$(1 - (\frac{1}{2RRo} + \frac{Ro}{2RFr^2})) J(U) \leq \| U \|_L^2 \leq (1 + (\frac{1}{\pi^2 Ro} + \frac{Ro}{\pi^2 Fr^2})) J(U) \tag{5.22}$$

which shows that $\| U \|_L^2$ is topologically equivalent to $J(U)$. The next step is to note that if $(\frac{1}{Ro} + \frac{Ro}{Fr^2}) > 2R$ then there is a sequence $\{U_n\} \subset \mathrm{Y}$ such that

$$\lim_{n \to \infty} I(\underline{u}_n, \rho_n) = 1/2R.$$

Hence given $\epsilon < (\frac{1}{2R} - (\frac{1}{Ro} + \frac{Ro}{Fr^2}))$, there exists \$U\$ such that

$$-(LU, U) \geq [-1 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})(\frac{1}{2R} - \epsilon] J(U) > 0, \tag{5.23}$$

and in this case $-L$ is essentially non-dissipative. The following result on the spectrum of $-L$ from [27, 50, 66] will be utilized:

Lemma5.1.3 $\sigma(-L) \subseteq (-\infty, 0] \cup \coprod$, where $\coprod$ is either empty or an at most denumerable set consisting of isolated, positive eigenvalues $\lambda_n = \lambda_n(\frac{1}{Ro} + \frac{Ro}{Fr^2})$ with finite multiplicity such that

$$\infty > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots,$$

clustering only at zero.

Proof: The validation of the dependence of the positive eigenvalues λ_n on $\frac{1}{Ro} + \frac{Ro}{Fr^2}$ is due to the relation (5.23). According to a perturbation theorem [27, 50, 66], if \$TU\$ is closed and $MU = SU - FU$ is *relatively compact with respect to* T then $\sigma(-L = -(T + S - F))$ and $\sigma(T)$ differ by at most a denumerable number of isolated, positive

eigenvalues of finite multiplicity clustering only at zero. Thus it suffices to show that $M = S - F$ is relatively compact with respect to T since from [92, 27, 50, 66] we have $\sigma(-T) \subseteq (-\infty, 0]$. By definition, $M = S - F$ is relatively compact with respect to T if for any sequence $\{U_n\} \subset \mathrm{Y}$,

$$\| U_n\| + \| TU_n\| \leq \alpha, \ \textit{uniformly in } n, \tag{5.24}$$

then there is a subsequence $U_{n'} \subset \{U_n\}$ such that $MU_{n'}$ is strongly convergent. Indeed, the following inequality is valid

$$\begin{aligned} \| MU_{n'}\|^2 &\leq [5(\frac{1+y\beta Ro}{Ro})^2 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})^2]\| U_{n'}\|^2 \\ &\leq [5(\frac{1+y\beta Ro}{Ro})^2 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})^2]\alpha^2 \end{aligned} \tag{5.25}$$

and by the courtesy of Ascoli-Arzela theorem $\{MU_{n'}\}$ contains a uniformly convergent subsequence which is a Cauchy sequence in a complete normed space Y. Thus $MU_{n'}$ is strongly convergent which illustrates that $M = S - F$ is relatively compact with respect to T.

It remains to estimate the nonlinearity $N(U)$. Next, we exhibit a series of a priori estimates on the nonlinearity which will be needed in establishing necessary and sufficient nonlinear stability of the rotating Boussinesq equations (5.16). Consideration of the Sobolev inequality and continuity properties of the nonlinearity $B(U)$ yield the following uniform bounds

$$\begin{aligned} &\| NU\|^2 \leq (TU, U)c_0^2(\| TU\|^2 + \| U\|^2) \\ &\leq c_0^2 \| T^{\frac{1}{2}}U\|(\lambda_1 \| L^{\frac{1}{2}}U\|^2 + \| U\|^2) \\ &\leq c_0^2 \lambda^{\frac{-1}{2}} \| TU\|(\lambda_1 \| L^{\frac{1}{2}}U\|^2 + \| U\|^2) \\ &\leq c_0^2 \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} \| L^{\frac{1}{2}}U\|(\lambda_1^{\frac{1}{2}} \| LU\|^2 + \| U\|^2) \\ &\leq c_0^2 c \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} \| L^{\frac{1}{2}}U\|(\| LU\|^2 + \| U\|^2) \\ &= c_0^2 c \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} J(U)(\| LU\|^2 + \| U\|^2) \end{aligned} \tag{5.26}$$

for the nonlinearity $N(U) = B(U)$ where $c = max\{\lambda_1^{\frac{1}{2}}, 1\}$. Consequently, a priori estimates (5.26) on the nonlinearity coincide with (5.9) which means that the nonlinear operator is suitably dominated by the linear operator LU. Thus, the purpose of proving Lyapunov instability aspect has been achieved. Moreover, when the following $(\frac{1}{Ro} + \frac{Ro}{Fr^2}) < 2R$ holds, we observe the topological equivalence of the Lyapunov functional $J(U)$ with the norm $\| U\|_L^2$ given in inequality (5.22). In this case, putting the estimates (4.36) and (5.26) together and invoking the equivalence relation (5.22) gives the needed a priori estimate for the nonlinearity

$$\begin{aligned} \| NU\|^2 &\le c_0^2 c\lambda^{\frac{-1}{2}}\lambda_1^{\frac{1}{2}} J(U)(\| LU\|^2 + \| U\|^2) \\ &\le c_1 c_0^2 c\lambda^{\frac{-1}{2}}\lambda_1^{\frac{1}{2}} \| U\|_L^2 (\| LU\|^2 + \lambda_1^{-1}\| U\|_L^2) \end{aligned} \tag{5.27}$$

that is necessary for stability with the parameter c_1 denoting $c_1 = (1-(\frac{1}{2RRo}+\frac{Ro}{2RFr^2}))^{-1}$. According to the sharp a priori estimate (5.27), aspect of nonlinear stability provided by the inequality (5.6) is satisfied. Hence, energetic stability criteria for necessary and sufficient nonlinear stability of the rest state f β-plane rotating Boussinesq equations (2.26-2.28) or equivalently (5.16) hold with the energy and entropy production provided by the following:

$$E(t) = \frac{1}{2}\{\| U\|^2 + \frac{Ek}{Ro}\| \bigtriangledown \underline{u}\|^2 + \frac{1}{Ed}\| \bigtriangledown \rho\|^2 + 2R\int_\Omega \rho w dx\}$$

which is specification of energy and the entropy production in the Sobolev H^1-norm if the criterion $Ga \le R < (\frac{1}{Ro}+\frac{Ro}{Fr^2})$ is valid. In the asymptotic stability analysis we take the number $Ga = 80$ and we note that we have reformulated the initial-boundary value problem for β-plane rotating Boussinesq equations in order to reflect the influence of Galdi *et al* [27, 28] investigations on energy stability theory. The above stability criterion number provides a great deal of light on the potentialities of the energy stability theory and validates this exposition from a benchmark point of view and offers the opportunity of invoking established results as a guide for nonlinear stability criteria. The result serve as a paradigm for constructing generalized Lyapunov functionals with interpretations such as energy and entropy production which are decreasing along solution of the initial-value problem.

Next, we focus our attention to the development of results for the attractor of the β-plane rotating Boussinesq equations with Reynolds stress (2.29-2.31) for both the three-dimensional and two-dimensional space variables. We show existence of the attractor for the dynamical system and proceed by customizing the splendid techniques employed in [14, 15, 92, 91, 23, 22] which have been summarized in the above proposition.

Next, we turn our attention to develop results on the attractor for β-plane rotating Boussinesq equations (2.26-2.28) in the case of two-dimensional space variables. We show existence of the attractor for the dynamical system and proceed essentially using the techniques employed in [14, 15, 92, 91, 23, 22] which have been summarized in the above proposition, but the calculations that we present in this examination are tedious and involved. Putting the estimate (4.12) and (4.11) together gives the bound specified by

$$\lim_{t\to\infty} sup\int_t^{t+1}\| L^{\frac{1}{2}}U(s)\|^2 ds \le \frac{\| \Omega\|}{\alpha}[\frac{\lambda_1^2}{\alpha^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4 + \frac{\lambda_1}{\alpha}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2]$$

Taking the inner product of the in the initial-value problem (4.2) with LU yields the following differential inequalities

$$\begin{aligned} &\frac{d}{dt}\| L^{\frac{1}{2}}U\|^2 + \lambda_1\| L^{\frac{1}{2}}U\|^2 \\ &\le \frac{d}{dt}\| L^{\frac{1}{2}}U\|^2 + \| LU\|^2 \le \rho_{22}^2 \end{aligned} \tag{5.28}$$

such that $\rho_{22}^2 = \nu_0 + \nu_1\nu_2$ follows successive application of Gronwall's inequality. And the parameters ν_0, ν_1 and ν_2 are given by

$$\nu_0 = \frac{3\| \Omega\|}{\nu}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2,$$

$$\nu_1 = \frac{27c}{4\nu^3}+\frac{2}{\kappa^3}(a_{23}+a_{22})exp(a_{21}),$$

$$\nu_2 = \frac{\| \Omega\|}{\nu^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4(a_{13}+a_{12})^2 exp(2a_{11}),$$

$$a_{11} = \frac{27c}{4\nu^3}((\frac{1}{\nu})^2(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2+\frac{1}{\nu^3})(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2\frac{\| \Omega\|^2}{\nu^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4,$$

$$a_{12} = 3\| \Omega\|^2(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2,$$

$$a_{13} = ((\frac{1}{\nu})^2(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2+\frac{1}{\nu^3})(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4\| \Omega\|,$$

$$a_{21} = \frac{9c\| \Omega\|^3}{2\kappa^5}(\frac{1}{\kappa}+(\frac{Ro}{Fr^2}+\frac{1}{Ro})^7\kappa^2\nu^2)(\frac{1}{\nu^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2+\frac{1}{\nu^3})^2(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4,$$

$$a_{22} = \frac{\| \Omega\|^3}{2\nu^4\kappa}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4(\frac{1}{\nu^2}(\frac{Ro}{Fr^2}+\frac{1}{Ro})^2+\frac{1}{\nu^3})^2(\frac{Ro}{Fr^2}+\frac{1}{Ro})^4,$$

$$a_{23} = (\frac{1}{\kappa}+\frac{1}{\kappa^2\nu^2})(\frac{Ro}{Fr^2}+\frac{1}{Ro})^6\| \Omega\|.$$

Consideration of the Gronwall's inequality in the differential inequality (4.11) provides a priori estimate

$$\lim_{t\to\infty} sup\| \bar{L^{\frac{1}{2}}}U(t)\|^2 \le \frac{\rho_{22}^2}{\alpha} = \rho_2^2 \tag{5.29}$$

which shows $U(t)$ is uniformly bounded in time in $D(\bar{L^{\frac{1}{2}}})$. Putting the estimate (5.28) and (5.29) together we obtain

$$\lim_{t\to\infty} sup\int_t^{t+1}\| LU(s)\|^2 ds \le (1+\frac{1}{\alpha})\rho_2^2. \tag{5.30}$$

Concerning the results for the attractor of the two-dimensional space variables initial-value problem for viscous β-plane ageostrophic equations without Reynolds stress we infer from the more refined a priori estimates (5.29) that the attractor for 4.2 is provided by the ω-limit set of $Q = B_{2\rho_2}$,

$$A = \omega(Q) = \cap_{s\ge 0} Cl(\cup_{t\ge s} S(t)Q).$$

where $B_{2\rho_2}$ denotes an open ball in phase space of radius $2\rho_2$, which depends on the geophysically relevant parameters. The closure is taken in the Hilbert space Y. Utility of the group property and continuity of of the solution operators $S(t)$ defined for all time $t\in\mathbb{R}$, gives the following invariance property of the above established attractor:

$$S(t)A = A,\ \forall t\in\mathbb{R}.$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the two-dimensional space variables initial-value problem for viscous β-plane rotating Boussinesq equations without Reynolds stress is both positively and negatively invariant and consists of orbits or trajectories that are defined for all $t\in\mathbb{R}$.

According to a priori estimates (4.12) and ()5.29, we have the following result which holds for all time.

Proposition 5.1.4 If $u_0 \in \mathrm{Y}$ and u is the unique solution of (4.2) for the case of two-dimensional space variables then the corresponding variational equation (4.24) has a unique solution Ψ satisfying

$$\Psi \in C([0,T];\mathrm{Y}) \cap L^2(0,T;D(L^{\frac{1}{2}})) \ \ \forall T > 0.$$

And if $u_0 \in D(L^{\frac{1}{2}})$ then

$$\Psi \in C([0,T];D(L^{\frac{1}{2}})) \cap L^2(0,T;D(L)) \ \ \forall T > 0.$$

The proof of the proposition follows from the stronger a priori energy type estimate (5.29) and by repeating the arguments of the previous chapter.

5.3 Rotating Boussinesq equations with Reynolds stress

The aim of this section is to establish nonlinear stability results for the rest state of rotating Boussinesq equations with Reynolds stress (2.29-2.31). In order to give nonlinear stability bounds that agree approximately with established paradigmatic example of initial-value problem (5.16), the functional formulation for the evolution equation of β-plane rotating Boussinesq equations with Reynolds stress is restated equivalently as follows:

$$\begin{aligned} &\frac{dU}{dt} + LU + N(U) = 0 \\ &U(0) = U_0 \end{aligned} \tag{5.31}$$

where $U = (u, v, w, \rho)$. The boundary conditions have been taken to be space-periodic as in the investigation of well-posedness of solution. Concerning the linear operator LU and the nonlinear operator $N(U)$ we have the following appropriate setup:

$$N(U) = B(U,U)$$

$$LU = TU + CU + SU - FU$$

$$FU = -(0, 0, \frac{\rho}{Ro}, \frac{Ro}{Fr^2} w)$$

$$CU = -\begin{pmatrix} \frac{Ek}{Ro} \triangle \underline{u} \\ \frac{1}{Ed} \triangle \rho \end{pmatrix} = \begin{pmatrix} A_l \underline{u} \\ A_l \rho \end{pmatrix}$$

$$B(U,U) = B(U) = \begin{pmatrix} \underline{u} \cdot \nabla \underline{u} \\ \underline{u} \cdot \nabla \rho \end{pmatrix} = \begin{pmatrix} \mathbf{b}_l(\underline{u}, \underline{u}) \\ \mathbf{b}_l(\underline{u}, \rho) \end{pmatrix}$$

$$TU = -\begin{pmatrix} \frac{Ek}{Ro} \triangle^6 \underline{u} \\ \frac{1}{Ed} \triangle^6 \rho \end{pmatrix} = \begin{pmatrix} A_l^6 \underline{u} \\ A_l^6 \rho \end{pmatrix}$$

$$S = -(\frac{1+y\beta Ro}{Ro})\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \forall\, U \in D(L) = D(T).$$

From the above definition of the linear operator LU, we have

$$(LU,U) = (TU,U) + (CU,U) + (\frac{1}{Ro} + \frac{Ro}{Fr^2})\int_\Omega \rho w dx. \tag{5.32}$$

which is a modification of the bilinear form (4.33) that was employed in establishing well-posedness of solution for the problem of rotating Boussinesq equations with Reynolds stress. As in the previous section, with the attendant relation (5.32) we are able to demonstrate a necessary and sufficient nonlinear stability criterion and hence the elegant power of energy methods is fully realized. For the sake of completeness, we repeat the analysis in this section for rotating Boussinesq equations with Reynolds stress. The first effort is then to let R and

$$I(\underline{u}, \rho) \tag{5.33}$$

be defined as follows

$$1/2R = sup\{\int_\Omega \rho w dx\} / J(U) = supI(\underline{u}, \rho)$$

where the supremum is taken for $U \in D(L)$. We shall compile various inequalities about the equation (5.31). Substituting

$$|\int_\Omega \rho w dx| \le (\| \underline{u}\|^2 + \| \rho\|^2)/2$$

into (5.33) and utilizing the Poincar$e$-Friedrichs inequality yields $R \ge Ga$.
The attendant relation (5.32) may be restated as follows

$$(LU,U) = (1 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})I(\underline{u}, \rho))J(U), \tag{5.34}$$

with the generalized energy functional $J(U)$ defined in (5.4) given by the Lyapunov functional

$$J(U) = (TU,U) + \frac{Ek}{Ro}\| \bigtriangledown \underline{u}\|^2 + \frac{1}{Ed}\| \bigtriangledown \rho\|^2. \tag{5.35}$$

It is appropriate at this point to note that if $(\frac{1}{Ro} + \frac{Ro}{Fr^2}) < 2R$ then the following is valid

$$-(LU,U) \le (-1 + (\frac{1}{2RRo} + \frac{Ro}{2RFr^2}))J(U) < 0, \tag{5.36}$$

and in this case $-L$ is essentially dissipative. Additionally, the following inequalities hold

$$(1 - (\frac{1}{2RRo} + \frac{Ro}{2RFr^2}))J(U) \le \| U\|_L^2 \le (1 + (\frac{1}{\pi^2 Ro} + \frac{Ro}{\pi^2 Fr^2}))J(U) \tag{5.37}$$

which shows that $\| U\|_L^2$ is topologically equivalent to $J(U)$.

The next step is to note that if $(\frac{1}{Ro} + \frac{Ro}{Fr^2}) > 2R$ then there is a sequence $\{U_n\} \subset \mathrm{Y}$ such that

$$\lim_{n\to\infty} I(\underline{u}_n, \rho_n) = 1/2R.$$

Hence given $\epsilon < (\frac{1}{2R} - (\frac{1}{Ro} + \frac{Ro}{Fr^2}))$, there exists U such that

$$-(LU,U) \geq [-1 + (\frac{1}{Ro} + \frac{Ro}{Fr^2})(\frac{1}{2R} - \epsilon]J(U) > 0, \tag{5.38}$$

and in this case $-L$ is essentially non-dissipative. We require the following lemma on the spectrum of $-L$ whose validity may be proved by using the technique of proof similar to those given above:

Lemma 5.1.5 $\sigma(-L) \subseteq (-\infty, 0] \cup \coprod$, where $\coprod$ is either empty or an at most denumerable set consisting of isolated, positive eigenvalues $\lambda_n = \lambda_n(\frac{1}{Ro} + \frac{Ro}{Fr^2})$ with finite multiplicity such that

$$\infty > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots,$$

clustering only at zero.

Next, we proceed to illustrate that suitable a priori estimates for the nonlinear operator $N(U))$ are available. Putting estimates (4.36) and (4.38) together into the inequality (4.40) yield the desired bounds

$$\begin{aligned}
&\| NU\|^2 \leq (CU,U)c_0^2(\| CU\|^2 + \| U\|^2) \\
&\leq c_0^2 \| C^{\frac{1}{2}}U\|(\lambda_1 \| L^{\frac{1}{2}}U\|^2 + \| U\|^2) \\
&\leq c_0^2 \lambda^{\frac{-1}{2}} \| CU\|(\lambda_1 \| L^{\frac{1}{2}}U\|^2 + \| U\|^2) \\
&\leq c_0^2 \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} \| L^{\frac{1}{2}}U\|(\lambda_1^{\frac{1}{2}} \| LU\|^2 + \| U\|^2) \\
&\leq c_0^2 c \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} \| L^{\frac{1}{2}}U\|(\| LU\|^2 + \| U\|^2) \\
&= c_0^2 c \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} J(U)(\| LU\|^2 + \| U\|^2)
\end{aligned} \tag{5.39}$$

for the nonlinearity $N(U) = B(U)$ where $c = max\{\lambda_1^{\frac{1}{2}}, 1\}$. Consequently, a priori estimates (5.26) on the nonlinearity coincide with (5.9) which means that the nonlinear operator is suitably dominated by the linear operator LU. Thus, the purpose of proving Lyapunov instability aspect has been achieved. Moreover, when the following $(\frac{1}{Ro} + \frac{Ro}{Fr^2}) < 2R$ holds, we observe the topological equivalence of the Lyapunov functional $J(U)$ with the norm $\| U\|_L^2$ given by the inequality (5.37). In this case, putting the estimates (5.36) and (4.40) together and invoking the equivalence relation (5.37) gives the needed a priori estimate for the nonlinearity

$$\begin{aligned}
&\| NU\|^2 \leq c_0^2 c \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} J(U)(\| LU\|^2 + \| U\|^2) \\
&\leq c_1 c_0^2 c \lambda^{\frac{-1}{2}} \lambda_1^{\frac{1}{2}} \| U\|_L^2 (\| LU\|^2 + \lambda_1^{-1} \| U\|_L^2)
\end{aligned} \tag{5.40}$$

where $c_1 = (1 - (\frac{1}{2RRo} + \frac{Ro}{2RFr^2}))^{-1}$. According to the sharp a priori estimate (5.40), aspect of nonlinear stability provided by the inequality (5.6) is satisfied. Hence, energetic stability criteria for necessary and sufficient for nonlinear stability of the rest state of β-plane rotating Boussinesq equations with Reynolds stress (5.31) or equivalently (2.29-2.31) hold with

the energy and entropy production provided by the following:

$$E(t) = \frac{1}{2}\{\| U\|^2 + J(U) + 2R\int_\Omega \rho w dx\}$$

which is specification of energy and the entropy production in the Sobolev H^1-norm if the criterion $Ga \le R < (\frac{1}{Ro} + \frac{Ro}{Fr^2})$ is valid. The result serve as the archetype for constructing generalized Lyapunov functionals with interpretations such as energy and entropy which are decreasing along solution of the system. Furthermore, we observe that in the craft of nonlinear stability, the equations governing the flow of an incompressible stratified fluid under the Coriolis force induce damping mechanisms in the nature of viscosity, diffusion, combined stratification and rotation effects that manifest themselves in the evolution equation with the existence of Lyapunov functionals which are equivalent to some fictitious energy or enstrophy or entropy.

Next, we focus our attention to the development of results for the attractor of the β-plane rotating Boussinesq equations with Reynolds stress (2.29-2.31) for both the three-dimensional and two-dimensional space variables. We show existence of the attractor for the dynamical system and proceed by customizing the splendid techniques employed in [14, 15, 91, 90, 23, 22] which have been summarized in the above proposition. Concerning the results for the attractor of the general three-dimensional space variables initial-value problem for viscous β-plane rotating Boussinesq equations with Reynolds stress we infer from the more refined a priori estimates (4.53) that the attractor for (4.32) is provided by the ω-limit set of $Q = B_{2\rho_2}$,

$$A = \omega(Q) = \cap_{s\ge 0} Cl(\cup_{t\ge s} S(t)Q).$$

where $B_{2\rho_2}$ denotes an open ball in phase space of radius $2\rho_2$, which depends on the geophysically relevant parameters. The closure is taken in the Hilbert space Y. Utility of the group property and continuity of the solution operators $S(t)$ defined for all time $t \in \mathbb{R}$, gives the following invariance property of the above established attractor:

$$S(t)A = A, \ \forall t \in \mathbb{R}.$$

Consequently, examining the expression for the invariance of the attractor we observe that the attractor for the three-dimensional space variables initial-value problem for viscous β-plane ageostrophic equations with Reynolds stress is both positively and negatively invariant and consists of orbits or trajectories that are defined for all $t \in \mathbb{R}$.

In the derivation of β-plane rotating Boussinesq equations with Reynolds stress, it was essential to note that the additive decomposition of the rotating Boussinesq equations quantities into coherent and incoherent terms and consideration of the averaging operator to obtain the Reynolds stress fields or wind stress fields resulted in a set of evolution equations that were not closed. In order to close the set of equations, we adopted the Pedlosky closure protocols.

CHAPTER 6
Quasigeostrophic flows

6.1 Description of quasigeostrophy

The baroclinic dissipative quasigeostrophic equations which we derive govern the evolution of streamfunction denoted ψ whenever the the Rossby number is considered asymptotically small. The quasigeostrophic system has been of oceanographic and meteorological interest especially for the modelling and forecasting of mesoscale and synotic scale eddies. For the sake of completeness, we give the description of the quasigeostrophic system which expresses the evolution of the zero-order potential vorticity of the flow. The standard derivation of the quasigeostrophic potential vorticity equations from the ageostrophic system, which is a set of primitive equations governing the flow of a viscous incompressible stratified fluid under the Coriolis force with or without the Reynolds stress, is accomplished by the systematic use of scaling arguments and asymptotic series in the Rossby number. The quasigeostrophic limit is useful from the fact that the translating and the rotating modons represent closed form vortex solutions of the quasigeostrophic potential vorticity equation when for example dissipation terms are neglected. Moreover, other simplifying assumptions include the replacement of vertical coordinate by density so that the quasigeostrophic equations may lend themselves to discretization in the vertical resulting in layered models. We allude here that the introduction of two-layer quasigeostrophic models lead to the indepence of horizontal velocity with height and the phenomena is called *barotropic* which complement *baroclinic*, the change of horizontal velocity with height.

It is now possible to systematically utilize the presence of parameters in the evolution equations to embark on a perturbation analysis. The next aim is to derive geostrophic and quasigeostrophic models from the rotating Boussinesq equations for the atmosphere and ocean via a perturbation analysis of the flow fields at the Rossby number on the order of unity or less. In order to illustrate the the asymptotic expansion, consider

$$\begin{aligned} \underline{u} &= \underline{u}_0(\underline{x}, t, Ro, \beta, Fr, Ek, Ed, \varepsilon) + \\ & Ro\underline{u}_1(\underline{x}, t, Ro, \beta, Fr, Ek, Ed, \varepsilon) \\ p &= p_0(\underline{x}, t, Ro, \beta, Fr, Ek, Ed, \varepsilon) + \\ & Rop_1(\underline{x}, t, Ro, \beta, Fr, Ek, Ed, \varepsilon) \\ \rho &= \rho_0(\underline{x}, t, Ro, \beta, Fr, Ek, Ed, \varepsilon) + \\ & Ro\rho_1(\underline{x}, t, Ro, \beta, Fr, Ek, Ed, \varepsilon). \end{aligned} \tag{6.1}$$

with assumption $\frac{Fr^2}{Ro} = Ro$. When the expansion (6.1) is inserted in the nondimensional system (2.26) and equating terms of the same order in the Rossby number, Ro, at zero-order we obtain *geostrophic* equations

$$
\begin{aligned}
v_0 &= \frac{\partial p_0}{\partial x} \\
u_0 &= -\frac{\partial p_0}{\partial y} \\
\rho_0 &= -\frac{\partial p_0}{\partial z} \\
\nabla.\underline{u}_0 &= 0 \\
w_0 &= 0.
\end{aligned}
\tag{6.2}
$$

In (6.2), horizontal velocity is divergence free and pressure plays the role of streamfunction. From consideration of the continuity equation and the realization that zero-order velocity is *isobaric*, that is, pressure of the system remain constant along streamlines, we obtain $\frac{\partial w_0}{\partial z} = 0$ in addition of $w_0 = 0$. In order to derive equation for evolution of the geostrophic pressure, $p_0 = \psi$, we equate first-order terms in the Rossby number which yield

$$
\begin{aligned}
&\frac{\partial u_0}{\partial t} + \underline{u}_0.\nabla u_0 - v_1 - y\beta_0 v_0 = -\frac{\partial p_1}{\partial x} + \frac{Ek}{Ro}\triangle u_0 \\
&\frac{\partial v_0}{\partial t} + \underline{u}_0.\nabla v_0 + u_1 + \beta_0 u_0 + \rho_1 = -\frac{\partial p_0}{\partial y} + \frac{Ek}{Ro}\triangle v_0 \\
&-\beta_0 u_0 + y u_0 + \rho_1 = -\frac{\partial p_1}{\partial z} \\
&\nabla.\underline{u}_0 = 0 \\
&\frac{\partial \rho_0}{\partial t} + \underline{u}_0.\nabla \rho_0 + w_1 = \frac{1}{Ed}\triangle \rho_0.
\end{aligned}
\tag{6.3}
$$

The next goal is to rewrite the equations (6.3) in an equivalent setup with emphasis on geostrophic pressure which we denote by $\psi = p_0$, the streamfunction. We take the two-dimensional curl of horizontal momentum in(6.3) to eliminate the pressure gradient and utilize the fact that from (6.2) horizontal velocity is divergence free which together with suitable simplifying assumptions gives the following *quasigeostrophic* potential vorticity system

$$
\begin{aligned}
&\frac{\partial q}{\partial t} + J(\psi, q) = \frac{Ek}{Ro}\triangle q \\
&\frac{\partial \rho}{\partial t} + J(\psi, \rho) = \frac{1}{Ed}\triangle \rho \\
&q = \triangle\psi - \frac{\partial^2 \psi}{\partial z^2} + \beta y \\
&\rho = -\frac{\partial \psi}{\partial z} \\
&u = -\frac{\partial \psi}{\partial y} \\
&v = \frac{\partial \psi}{\partial x}
\end{aligned}
\tag{6.4}
$$

where the nondimensional fields $\underline{u}$, q, ρ, and ψ are fluid velocity, potential vorticity, density and streamfunction, respectively. The geophysically relevant parameter Ro is the Rossby number, Ek is the Ekman number, β is the reference reciprocal Coriolis parameter. Here $J(.,.)$ is the Jacobian operator and the operator $\frac{\partial}{\partial t} + J(.,.)$ represents advection along fluid particle trajectories.

Advances in our understanding of the dynamics of mesoscale eddies and vortex rings of the Agulhas Current Retroflection, and the Gulf Stream in the Middle Atlantic Bight have been based on modons which are nonlinear Rossby solitary wave solutions of quasigeostrophic equations. This research and innovation focuses primarily on geometric singular perturbation (GSP) theory and Melnikov techniques to address the problem of adiabatic chaos and transport for translating and rotating modons to the quasigeostrophic potential vorticity system which is relevant, e.g., a Lagrangian and Eulerian analysis of a geophysical fluid flow. A very general and central question is what hypotheses on the equations and singular solutions guarantee that the solutions approximate some solutions for the perturbed quasigeostrophic potential vorticity system. We present a geometric approach to the problem which gives more refined a priori energy type estimates on the position of the invariant manifold and its tangent planes as the manifold passes close to a normally hyperbolic piece of a slow manifold. We apply Melnikov technique to show that the Poincare map associated with modon equations has transverse heteroclinic orbits. We appeal to the Smale-Birkhoff Homoclinic Theorem and assert the existence of an invariant hyperbolic set which contains a countable infinity of unstable periodic orbits, a dense orbit and infinitely many heteroclinic orbits. The main object of this result gives geometric detection of adiabatic chaos and transport in singularly perturbed dynamical systems as depicted in figure \ref{tladi2} for the benchmark Lagrangian coherent structures utilizing the DsTool package program of Worfolk, Guckenheimer *et al*, [109] which greatly helped codify the discipline. In the Agulhas Current Retroflection, the information concerning adiabatic and chaotic advection in the mesoscale eddies and vortex rings is very recent and we thank Professor Geoff Brundrit who provided the material for introducing us to this rich physical oceanography programme.

Turbulence, the splintering of smooth streams of fluid into chaotic vortices, does not just make for bumpy plane rides. It also throws a wrench into the very mathematics used to describe atmospheres, oceans and plumbing. Turbulence is the reason why the Navier-Stokes equations, the laws that govern fluid flow, are so famously hard that whoever proves whether or not they always work will win a million dollars from the Clay Mathematics Institute. But turbulence unreliability is, in its own way, reliable. Turbulence almost always steals energy from larger flows and channels it into smaller eddies. These eddies then transfer their energy into even smaller structures, and so on down. If you switch off the ceiling fan in a closed room, the air will soon fall still, as large gusts dissolve into smaller and smaller eddies that then vanish entirely into the thickness of the air. But when you flatten reality down to two dimensions, eddies join forces instead of dissipating. In a curious effect called an inverse cascade, which the applied mathematician Audrey Rogerson first fished out of the Navier-Stokes equations, turbulence in a flattened fluid passes energy up to bigger scales, not down to smaller ones. Eventually, these two-dimensional systems organize themselves into large, stable flows like vortexes or river-like jets. These flows, rather like vampires, support themselves by sucking away energy from turbulence, instead of the other way around. While the inverse cascade effect has been known for decades, a mathematical, qualitative prediction of what that final, stable flow looks like has eluded theorists. But a glimmer of hope came in 2017, when Stephen Tladi, now an applied mathematician at University of Limpopo, and his colleagues published a full description of the flow shape and speed under strict, specific conditions. Since then, new simulations, lab experiments and theoretical calculations published as recently as last month have both justified the team calculations and explored different cases where their prediction starts to break down. All this might seem like only a thought experiment. The universe is not flat. But geophysicists and planetary scientists have long suspected that real oceans and atmospheres often behave like flat systems, making the intricacies of two-dimensional turbulence surprisingly relevant to real problems. After all, on Earth, and especially on the gas giant planets like Jupiter and Saturn, weather is confined to thin,

flattish slabs of atmosphere. Large patterns like hurricanes or the Gulf Stream, and Jupiter huge horizontal cloud bands and Great Red Spot, might all be feeding on energy from smaller scales. In the last few years, researchers analyzing winds both on Earth and on other planets have detected signatures of energy flowing to larger scales, the telltale sign of two-dimensional turbulence. They have begun mapping the conditions under which that behavior seems to stop or start. The hope, for a small but dedicated community of researchers, is to use the quirky but simpler world of two-dimensional fluids as a fresh entry point into processes that have otherwise proved impenetrably messy. They can actually make progress in two dimensions, said Audrey Rogerson, now an applied mathematician at Brown University, which is more than what we can say for most of our turbulence work. On September 30, 2003, the National Oceanic and Atmospheric Administration (NOAA) sent an aircraft into Isabel (El Nino), a Category 5 hurricane bearing down on the Atlantic Coast with winds gusting to 203 knots, the strongest readings ever observed in the Atlantic. NOAA wanted to get readings of turbulence at the bottom of a hurricane, crucial data for improving hurricane forecasts. This was the first, and last, time a crewed aircraft ever tried. At its lowest, the flight skimmed just 60 meters above the churning ocean. Eventually salt spray clogged up one of the plane four engines, and the pilots lost an engine in the middle of the storm. The mission succeeded, but it was so harrowing that afterward, NOAA banned low-level flights like this entirely. About a decade later, George Haller got interested in these data. Haller, an applied mathematician at the Swiss Federal Institute of Technology Zurich, had previously studied turbulent energy transfer in lab experiments. He wanted to see if he could catch the process in nature. He contacted Andrew Poje, an NOAA scientist who had been booked on the very next flight into Isabel (El Nino). By analyzing the distribution of wind speeds, the two calculated the direction in which energy was traveling between large and small fluctuations. Starting at about 150 meters above the ocean and leading up into the large flow of the hurricane itself, turbulence began to behave the way it does in two dimensions, the pair discovered. This could have been because wind shear forced eddies to stay in their respective thin horizontal layers instead of stretching vertically. Whatever the reason, though, the analysis showed that turbulent energy began flowing from smaller scales to larger scales, perhaps feeding Isabel (El Nino) from below. Their work suggests that turbulence may offer hurricanes an extra source of fuel, perhaps explaining why some storms maintain strength even when conditions suggest they should weaken. Haller now plans to use uncrewed flights and better sensors to help bolster that case. If we can prove that, it would be really amazing, he said. On Jupiter, a much larger world with an even flatter atmosphere, researchers have also pinpointed where turbulence switches between two-dimensional and three-dimensional behavior. Wind speed measurements taken by the Voyager probes, which flew past Jupiter in the 1970s, had already suggested that Jupiter large flows gain energy from smaller eddies. But in 2017, George Haller, and Andrew Poje, now his postdoc at the time, made a wind speed map using data from the space probe Cassini, which swung past Jupiter in 2000 on its way to Saturn. They saw energy flowing into larger and larger eddies, the hallmark of two-dimensional turbulence. But nothing about Jupiter is simple. On smaller scales, across patches of surface about the distance between New York and Los Angeles or less, energy dissipated instead, indicating that other processes must also be afoot. Then in March, the Juno spacecraft orbiting Jupiter found that the planet surface features extend deep into its atmosphere. The data suggest that not just fluid dynamics but magnetic fields sculpt the cloud bands. For Sanjeeva Balasuriya, who studies turbulence at the Ecole Normale Superieure (ENS) in Lyon, France, this is not too discouraging, since the two-dimensional models can still help. I do not think anybody believes the analogy should be perfect, he said. At the end of 2017, Balasuriya at ENS, sketched out his own theoretical account of how two-dimensional fluid flow can describe a rotating system such as the atmosphere of a planet. His work shows how flows built from smaller turbulence can match the enormous pattern of alternating bands visible on Jupiter through a backyard telescope. That makes it really relevant for discussing real phenomena, Balasuriya said. Balasuriya work relies on considering the statistics of the large-scale flows, which exchange energy and other quantities in a balance with their environment. But there is another path to predicting the form these flows will take, and it starts with those same obstreperous Navier-Stokes equations that lie at the root of fluid dynamics. For two totally fruitless years at the beginning of this decade, Denny Kirwan, a pen-and-paper theorist at Old Dominion University Center for Coastal Physical Oceanography (CCPO) in Norfolk, Virginia, stared at those equations. He tried to write out

how the flow of energy would balance between small turbulent eddies and a bigger flow feeding on them in a simple case: a flat, square box. A single term, related to pressure, stood in the way of a solution. So Kirwan just dropped it. By discarding that troublesome term and assuming that the eddies in this system are too short-lived to interact with each other, Kirwan and his colleagues tamed the equations enough to solve the Navier-Stokes equations for this case of a modon. Then he tasked Bruce Lipphardt, his postdoc at the time, with running numerical simulations that proved it. It is always nice when you have an exact result in turbulence, Kirwan said. Those are rare. In the team 1996 paper, they found a formula for how the velocity in the resulting large flow, a big vortex ring, in this situation, would change with distance from its own center. And since then, various teams have filled in the theoretical rationale to excuse Kirwan lucky shortcut. Hoping for payoff in the pure mathematics of fluids and for insight into geophysical processes, physicists have also pushed the formula outside a simple square box, trying to figure out where it stops working. Just switching from a square to a rectangle makes a dramatic difference, for example. In this case, turbulence feeds river-like flows called jets in which the formula starts to fail. As of now, even the mathematics of the simplest case, the square box, is not totally settled. Kirwan formula describes the large stable vortex itself, but not the turbulent eddies that still flicker and fluctuate around it. If they vary enough, as they might in other situations, these fluctuations will overwhelm the stable flow. Just in May, though, two former members of Kirwan lab, published a paper describing the size of these fluctuations. It teaches a little bit what the limitations of the approach are, Lipphardt said. But their hope, ultimately, is to describe a far richer reality with the research focused on the dynamics of multi-pole eddies in the Mid-Atlantic Bight, comparing a quasigeostrophic model with satellite SST observations of cyclone-anticyclone pairs. For Haller, the pictures returned from Juno's mission over Jupiter, showing a fantasyland of jets and tornadoes swirling like cream poured into the solar system largest coffee, are a driving influence. If it is something that I could help understand, that would be cool, he said. Since its inception in the 19th century through the efforts of Poincare and Lyapunov, the theory of dynamical systems addresses the qualitative behaviour of dynamical systems as understood from models. From this perspective, the modelling of dynamical processes in applications requires a detailed understanding of the processes to be analyzed. This deep understanding leads to a model, which is an approximation of the observed reality and is often expressed by a system of ordinary/partial, underdetermined (control), deterministic/stochastic differential or difference equations. While models are very precise for many processes, for some of the most challenging applications of dynamical systems (such as climate dynamics), the development of such models is notably difficult.

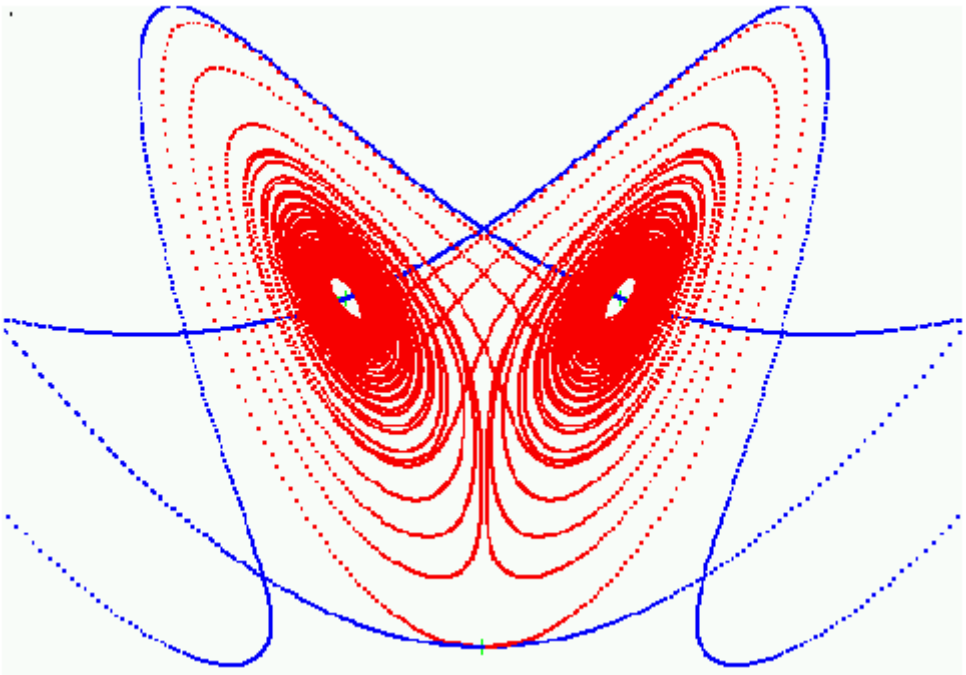

Figure 6.1 Lagrangian coherent structures

CHAPTER 7

Conclusions

7.1 Discussion of results

This investigation represents the first effort in the theoretical and applied aspects of the quasigeostrophic and rotating Boussinesq equations. The results are striking and encouraging, and we point out that based on the manifestation of well-posedness and stability, it is possible to design numerical algorithms such as finite-difference schemes. Furthermore, we observe that the nonlinear stability criteria may give the possibility of a sharper examination of the numerical approximation of the solution in the sense of the Lax equivalence theorem: a finite-difference scheme to a well-posed system of partial differential equation converges to the solution with the rate of convergence specified by the order of accuracy of the scheme. The theory of the Navier-Stokes equations (NSE) constitutes a central problem in continuum physics. These equations are physically well accepted model for the description of very common phenomena, and much effort has been devoted to them by fluid mechanics engineers, meteorologists, and oceanographers; nevertheless many problems are still at the frontier of science. Turbulence, the splintering of smooth streams of fluid into chaotic vortices, does not just make for bumpy plane rides. It also throws a wrench into the very mathematics used to describe atmospheres, oceans and plumbing. Turbulence is the reason why the Navier-Stokes equations, the laws that govern fluid flow, are so famously hard that whoever proves whether or not they always work will win a million dollars from the Clay Mathematics Institute. On the mathematical side NSE are a model for the study of nonlinear phenomena and nonlinear equations which is itself in its infancy: the quasigeostrophic and rotating Boussinesq equations encompass four central problems in nonlinear equations, namely, well-posedness, oscillations, discontinuities, and nonlinear dynamics. The mathematical theory of the NSE started with the pioneering work of J. Leray, who on this occasion introduced for the first time the concept of weak formulation of partial differential equations before the development of the distribution theory by L. Schwartz, shortly before S.L. Sobolev systematically introduced the spaces which bear his name. Motivated by the NSE, the study addressed two aspects of the mathematical theory of the quasigeostrophic and rotating Boussinesq equations: well-posedness, i.e., existence, uniqueness and regularity of the solutions in various function spaces and a new approach to energy theory in the stability of fluid motion. Progress in NSE had a big influence on the development of the qualitative theory of parabolic PDE as we know it today. The dynamical system generated by NSE was shown in the late 1960s to have special properties, which were special cases of asymptotically smooth dynamical systems. With dissipation, these equations also have compact global attractors. Once these concepts were introduced, it led to many natural questions about the finite dimensionality of the attractor, the discussion of special types of equations for which one could describe the flow on the attractor, etc. The subjects essentially touched in this research include the transition to turbulence in relation with bifurcation, the relations of the NSE with models of turbulence, non-Newtonian flows, and numerical analysis (numerical algorithms and scientific computing with MATLAB and C/C++). The problem of the numerical solution of the NSE equations was initialized by von Neumann and his collaborators in the 1940s computing with FORTRAN to elucidate transonic fluid flow, which refers to partial differential equations that possess both elliptic and hyperbolic regions. The flow is supersonic in the elliptic region, while a shock wave is created at the boundary between the elliptic and hyperbolic regions. S.J. Friedlander chose Cathleen Morawetz article based on her 1980s Gibbs Lecture entitled The mathematical approach to the sonic barrier. This article combined state-of-the-art applied mathematics with practical issues concerning flight at close to the speed of sound. It is still a rich source of open questions. Cathleen Morawetz wrote this article in connection with The Josiah Willard Gibbs Lecture she presented at the American Mathematical Society meeting in San Francisco, California, January 7, 1981. This is a beautiful piece on a subject at the core of applied mathematical analysis and numerical methods motivated by the pressing engineering

technology of the mid-twentieth century and the human urge to travel fast at efficient cost. From the mathematical viewpoint this problem comprises the understanding of models of nonlinear partial differential equations arising in compressible fluid mechanics, as much as understanding how to obtain numerical approximations to a model discretization that result both in finding numerically computed surfaces close to the model's solutions (if such exists) but also in matching these computed model outputs to experiments from engineering or experimental observation viewpoints. With the considerable development of the power of computers during the past decades, numerical simulation has developed as subjects of its own, Computational Fluid Dynamics (CFD), at the interface with engineering and physics.

7.2 New research and innovation directions

The author elucidate in a concrete way dynamical challenges concerning approximate inertial manifolds (AIMS), i.e., globally invariant, exponentially attracting, finite-dimensional smooth manifolds, for dynamical systems on Hilbert spaces. The goal of this theory is to prove the basic theorem of approximation dynamics, wherein it is shown that there is a fundamental connection between the order of the approximating manifold and the well-posedness and long-time dynamics of the rotating Boussinesq and quasigeostrophic equations. Abreast of these results, the author presents a new technique for the construction of AIMS that preserve the structure of attractors for the flow in the thermal convection of Oldroyd-B fluids. Although this article is motivated by challenges in partial differential equations, we consider a two-mode Faedo-Galerkin approximation (Ed Lorenz ansatz sense) given by a system of singularly perturbed ordinary differential equations. We note that the foundation for the study of the low-dimensional model of turbulence, which capture the dominant focus energy bearing scales, from the flow for the thermal convection of viscoelastic fluids with a differential constitutive law, is the Lorenz equations extended through singular perturbation:

$$\dot{x} = \sigma\left(y - x - \mu w\right),$$
$$\dot{y} = \varepsilon x + x - y - xz,$$
$$\dot{z} = -\beta z + xy,$$
$$\delta\dot{w} = \alpha x - w.$$

It is shown that the equilibrium point at the origin is asymptotically stable and attractive by the LaSalle invariance principle. Utilizing center manifold calculations, we show existence of supercritical pitchfork bifurcation and Poincare-Andronov-Hopf bifurcation. In order to utilize geometric singular perturbation (GSP) theory and Melnikov techniques, we perturb the problem and carry the nonlinear analysis further to the question of the persistence of inclination-flip homoclinic orbits. We appeal to the Shilnikov-Rychlik Homoclinic Theorem and assert existence of the Lorenz chaotic attractor which contains a nested sequence of hyperbolic, transitive, compact invariant sets each of which is equivalent to a suspension over a finite Markov chain with positive topological entropy.

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