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A Detailed Investigation on Soft Intersection-union Product of Groups

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Abstract

Soft set theory provides a highly adaptable mathematical framework for addressing real-world problems involving uncertainty, ambiguity, and parameter-driven variability—features commonly encountered in fields such as decision science, engineering, economics, and information systems. At the heart of this theory lie the fundamental operations and product constructions on soft sets, which together form a rich algebraic infrastructure capable of addressing complex, parameter-dependent phenomena. Accordingly, this study begins with a rigorous examination of the intersection operation of soft sets. This study rigorously analyzes the intersection operation of soft sets, proving that they form a bounded semilattice in the collection of soft sets with a fixed parameter set. We then introduce the soft intersection-union product, demonstrating its hemiring structure in the mentioned collection. Key results include associativity (Proposition 4.5), non-commutativity in non-abelian groups (Proposition 4.7), and distributivity over intersection (Proposition 4.24 and Proposition 4.25). The proposed algebraic constructions not only enrich the theoretical underpinnings of soft set theory but also offer promising tools for developing soft computational models applicable to multi-criteria decision-making, group-based classification, and uncertainty-aware data analysis.

Keywords

Soft sets; Soft subsets; Soft equalities; Soft intersection-union product

1. Introduction

A vast and sophisticated array of mathematical frameworks has been proposed by scholars to capture and analyze phenomena characterized by inherent uncertainty, vagueness, and imprecision across a wide spectrum of disciplinary domains, including engineering, economics, social sciences, and healthcare. Despite their widespread use, these models are often constrained by structural and epistemological limitations, as critically discussed in the seminal work of Molodtsov [1]. For instance, fuzzy set theory, introduced by Zadeh [2], is encumbered by the subjectivity of membership function assignment, while probabilistic approaches inherently depend on assumptions regarding the feasibility of repeated experimentation, thereby limiting their applicability in contexts where such assumptions fail.

To address these deficiencies, Molodtsov [1] introduced soft set theory as a flexible and conceptually robust alternative. By relaxing the rigid structural dependencies of classical models, soft set theory facilitates a parameter-dependent approach to uncertainty and has since found application in various analytical domains, including probability theory, game theory, and operations research. Over the last two decades, the formal development of soft set theory has undergone substantial expansion. Foundational axiomatic structures were established by M. [3], introducing key notions such as soft subsets, soft equality, and core operations like union, intersection, and the AND- and OR-products.

These were refined by Pei and Miao [4] through their exploration of connections between soft sets and information systems, leading to more robust formulations of inclusion and intersection. Ali et al., [5] further enriched the operational framework by introducing restricted and extended forms of union, intersection, and difference operations. Subsequent studies—including those by [6-19] shifted focus toward the algebraic underpinnings of soft operations. These works addressed conceptual ambiguities, proposed novel structural methodologies, and laid the groundwork for the formal algebraic treatment of soft set theory. Significant progress has also been made in broadening the algebraic foundations of soft operations. Studies by [20-34] document a wide array of newly introduced operations along with rigorous algebraic analyses. A central component of this progress is the refinement of soft equality and subsethood concepts. The early formulation of soft subsets by Maji et al., [4] was generalized by Pei and Miao [5] and Feng et al., [35], while Qin and Hong [36] introduced soft congruences and advanced soft equalities. Jun and Yang [37] extended distributive laws through the concept of J-soft equality, thereby accommodating broader classes of soft subsets. Building on this trajectory, Liu et al., [38] introduced soft L-subsets and L-equalities, which brought forth deeper structural distinctions and revealed the non-universality of classical distributive laws within generalized soft frameworks. Continuing this line of inquiry, Feng and Yongming [39] provided a taxonomy of soft subsets and systematically examined the algebraic characteristics of the AND- and OR-products introduced by Maji et al., [4] particularly in the context of L-subsets. Their results resolved critical issues regarding associativity, commutativity, and distributivity, and demonstrated that L-equalities possess congruence properties within free soft algebras, where the induced quotient structures form commutative semigroups. Further generalizations—such as g-soft, gf-soft, and T-soft equalities—along with relaxed parameter constraints and lattice-theoretic extensions, have been explored by Abbas et al., [40, 41], Al-shami [42], and Al-shami and El-Shafei [43]. To enhance the applicability of soft set theory, Çağman and Enginoğlu [44] revised the operational definitions of Maji et al., [4], yielding a more coherent and functional formalism. Building upon this refined structure, Sezgin et al., [45] conducted an exhaustive algebraic analysis of the AND-product under various frameworks of soft equality—such as L-equality, J-equality, and M-equality—as well as soft F-subsets. Their work provided a systematic investigation of algebraic properties including idempotency, commutativity, and associativity. Simultaneously, substantial developments have emerged in the context of soft product structures. The soft union product has been independently formulated for rings [46], semigroups [47], and groups [48], giving rise to the algebraic structures known as soft union rings, semigroups, and groups, respectively. Analogously, the soft intersection-union product has been defined for groups [49], semigroups [50], and rings [51], with the structural outcomes contingent upon the presence or absence of identity and inverse elements in the parameter domains.

The present study begins with a rigorous algebraic investigation of the intersection operation of soft sets as defined by Çağman and Enginoğl [44], establishing that the set of all soft sets over a fixed parameter set forms a bounded semilattice under this operation. We then explore the soft intersection-union product introduced by Muştuoğlu et al., [49] by conducting a thorough algebraic investigation that incorporates various classes of soft subsets and generalized equality relations. Futhermore, we demonstrate that the algebraic structure comprising soft sets, together with the intersection operation of soft sets and this product of soft sets, satisfies the axiomatic properties of a hemiring. These findings not only underscore the internal coherence of the proposed framework but also underscore its capacity to generalize classical algebraic structures and to offer new tools for resolving longstanding open problems Furthermore, our theoretical contributions address critical gaps in the existing literature but also lay the groundwork for a new subfield in soft group theory, centered around the introduced product. The remainder of this manuscript is structured as follows: Section 2 reviews the necessary background on soft sets and algebraic structures such as bounded semilattices and hemirings. Section 3 presents an in-depth analysis of the intersection operation of soft sets. Section 4 is devoted to the algebraic properties of the soft intersection-union product. Finally, Section 5 summarizes the principal findings and outlines directions for future research.

2. Preliminaries

This section presents a rigorous and systematic re-examination of the foundational definitions and algebraic structures that underpin the theoretical framework developed in the subsequent sections. While the concept of soft sets was originally introduced by Molodtsov [1], the associated definitional framework and operational principles were significantly revised by Çağman and Enginoğlu [44] with the aim of enhancing both axiomatic precision and practical applicability.The present study adopts this revised formulation as its formal foundation. Accordingly, all ensuing analyses, algebraic constructions, and theoretical developments are rigorously framed within the structure of this enhanced conceptual model.

Definition 2.1. [44] Let E be a parameter set, U be a universal set, $P(U)$ be the power set of U, and $\mathcal{H} \subseteq E$. Then, the soft set $f_{\mathcal{H}}$ over U is a function such that $f_{\mathcal{H}}: E \rightarrow P(U)$, where for all $w \notin \mathcal{H}$, $f_{\mathcal{H}}(w) = \emptyset$. That is,

$$f_{\mathcal{H}} = \{(w, f_{\mathcal{H}}(w)): w \in E\}$$

From now on, the soft set over U is abbreviated by $\mathcal{SS}$.

Definition 2.2. [44] Let $f_{\mathcal{H}}$ be an $\mathcal{SS}$. If $f_{\mathcal{H}}(w) = \emptyset$ for all $w \in E$, then $f_{\mathcal{H}}$ is called a null SS and indicated by $\emptyset_E$, and if $f_{\mathcal{H}}(w) = U$, for all $w \in E$, then $f_{\mathcal{H}}$ is called an absolute SS and indicated by U_E.

Definition 2.3. [44] Let $f_{\mathcal{H}}$ and $g_{\aleph}$ be two $\mathcal{SS}$s. If $f_{\mathcal{H}}(w) \subseteq g_{\aleph}(w)$, for all $w \in E$, then $f_{\mathcal{H}}$ is a soft subset of $g_{\aleph}$ and indicated by $f_{\mathcal{H}} \tilde{\subseteq} g_{\aleph}$. If $f_{\mathcal{H}}(w) = g_{\aleph}(w)$, for all $w \in E$, then $f_{\mathcal{H}}$ is called soft equal to $g_{\aleph}$, and denoted by $f_{\mathcal{H}} = g_{\aleph}$.

Definition 2.4. [44] Let $f_{\mathcal{H}}$ and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, the intersection of $f_{\mathcal{H}}$ and $g_{\aleph}$ is the $\mathcal{SS}$ $f_{\mathcal{H}} \tilde{\cap} g_{\aleph}$, where $(f_{\mathcal{H}} \tilde{\cap} g_{\aleph})(w) = f_{\mathcal{H}}(w) \cap g_{\aleph}(w)$, for all $w \in E$.

Definition 2.5. [52] Let f_K and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, f_K is called a soft S-subset of $g_{\aleph}$, denoted by $f_K \tilde{\subseteq}_S g_{\aleph}$ if for all $w \in E$, $f_K(w) = \mathcal{M}$ and $g_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{D}$ are two fixed sets and $\mathcal{M} \subseteq \mathcal{D}$. Moreover, two SSs f_K and $g_{\aleph}$ are said to be soft S-equal, denoted by $f_K =_S g_{\aleph}$, if $f_K \tilde{\subseteq}_S g_{\aleph}$ and $g_{\aleph} \tilde{\subseteq}_S f_K$.

It is obvious that if $f_K =_S g_{\aleph}$, then f_K and $g_{\aleph}$ are the same constant functions, that is, for all $w \in E$, $f_K(w)= g_{\aleph}(w) = \mathcal{M}$, where $\mathcal{M}$ is a fixed set .

Definition 2.6. [52] Let f_K and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, f_K is called a soft A-subset of $g_{\aleph}$, denoted by $f_K \tilde{\subseteq}_A g_{\aleph}$, if, for each $a, b \in E$, $f_K(a) \subseteq g_{\aleph}(b)$.

Definition 2.7. [52] Let f_K and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, f_K is called a soft S-complement of $g_{\aleph}$, denoted by $f_K =_S {g_{\aleph}}^c$, if, for all $w \in E$, $f_K(w) = \mathcal{M}$ and $g_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{D}$ are two fixed sets and $\mathcal{M} = \mathcal{D}'$.

For additional information on SSs, we refer to [53-78].

Definition 2.8. [79] An algebraic structure $(S,\star)$ is said to be idempotent if $s^2 = s$, for all $s \in S$. An idempotent semigroup is said to be a band, a commutative band is called a semilattice, and a semilattice with an identity is called a bounded semilattice.

Definition 2.9. [80] Let S be a non-empty set and "+" and "⋆" be two binary operations defined on S. If the algebraic structure $(Ś, +, \star)$ satisfies the following properties, then it is called a semiring:

i. $(S, +)$ is a semigroup.

ii. $(S,\star)$ is a semigroup,

iii. For all $ƀ, đ, z \in S$, $ƀ \star (đ + z) = ƀ \star đ + ƀ \star z$ and $(ƀ + đ) \star z = ƀ \star z + đ \star z$

If $ƀ + đ = đ + ƀ$, for all $ƀ, đ \in S$, then S is called an additive commutative semiring. If $ƀ \star đ = đ \star ƀ$, for all $ƀ, đ \in S$, then S is called a multiplicative commutative semiring. If there exists an element $1 \in S$ such that $ƀ \star 1 = 1 \star ƀ = ƀ$ for all $ƀ \in S$ (multiplicative identity), then S is called semiring with unity. If there exists $0 \in S$ such that for all $ƀ \in S, 0 \star ƀ = ƀ \star 0 = 0$ and $0 + ƀ = ƀ + 0 = ƀ$, then 0 is called the zero of S. A semiring with commutative addition and a zero element, is called a hemiring.

3. More on Intersection Operation of Soft Sets

From now on, let G be a group, and $S_G(U)$ denotes the collection of all $\mathcal{SS}$s over U, whose parameter sets are G; that is, each element of $S_G(U)$ is an $\mathcal{SS}$ parameterized by G. While the intersection operation on $\mathcal{SS}$s was originally introduced by Çağman and Enginoğlu [44] (cf. Definition 2.4), its algebraic properties—particularly those pertaining to the structure it induces on the class $S_G(U)$ of $\mathcal{SS}$s over a universe U with parameter set G—have not been systematically investigated to date. In this section, we undertake a detailed algebraic examination of the intersection operation of $\mathcal{SS}$s within $S_G(U)$ and we rigorously demonstrate that the algebraic structure $(S_G(U), \tilde{\cap})$ forms a bounded semilattice.

Proposition 3.1 The set $S_G(U)$ is closed under the intersection operation of $\mathcal{SS}$s. That is, if f_G and g_G are two $\mathcal{SS}$s, then so is $f_G \tilde{\cap} g_G$.

PROOF. It is obvious that the intersection operation of $\mathcal{SS}$s is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the intersection operation of $\mathcal{SS}$s.

Proposition 3.2 The intersection operation of $\mathcal{SS}$s is associative in $S_G(U)$ [44].

PROOF. This property was presented without its proof by Çağman and Enginoğlu [44]; however, we give a detailed proof of the property. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\hbar_G$ be three $\mathcal{SS}$s and $\mathscr{f}_G \tilde{\cap}\, \mathscr{g}_G = \mathscr{r}_G$, where $\mathscr{r}_G(x) = \mathscr{f}_G(x) \cap \mathscr{g}_G(x)$. Let $\mathscr{r}_G \tilde{\cap}\, \hbar_G = \mathscr{k}_G$, where $\mathscr{k}_G(x) = \mathscr{r}_G(x) \cap \hbar_G(x)$, for all $x \in G$. Thus, $\mathscr{k}_G(x) = \big(\mathscr{f}_G(x) \cap \mathscr{g}_G(x)\big) \cap \hbar_G(x)$. Let $\mathscr{g}_G \tilde{\cap}\, \hbar_G = n_G$, where $n_G(x) = \mathscr{g}_G(x) \cap \hbar_G(x)$ and $\mathscr{f}_G \tilde{\cap}\, n_G = \ell_G$, where $\ell_G(x) = \mathscr{f}_G(x) \cap \big(\mathscr{g}_G(x) \cap \hbar_G(x)\big)$, for all $x \in G$, implying that $\mathscr{k}_G(x) = \ell_G(x)$, for all $x \in G$. Thereby, $(\mathscr{f}_G \tilde{\cap}\, \mathscr{g}_G) \tilde{\cap}\, \hbar_G = \mathscr{f}_G \tilde{\cap}\, (\mathscr{g}_G \tilde{\cap}\, \hbar_G)$.

Proposition 3.3 The intersection operation of $\mathcal{SS}$s is commutative in S_G [44].

PROOF. This property was presented without its proof by Çağman and Enginoğlu [44]; however, we give a detailed proof of the property. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two $\mathcal{SS}$s. Then, $(\mathscr{f}_G \tilde{\cap}\, \mathscr{g}_G)(x) = \mathscr{f}_G(x) \cap \mathscr{g}_G(x) = \mathscr{g}_G(x) \cap \mathscr{f}_G(x) = (\mathscr{g}_G \tilde{\cap}\, \mathscr{f}_G)(x)$, for all $x \in G$. Thus, $\mathscr{f}_G \tilde{\cap}\, \mathscr{g}_G = \mathscr{g}_G \tilde{\cap}\, \mathscr{f}_G$.

Proposition 3.4 U_G is the identity element of the intersection operation of $\mathcal{SS}$s in $S_G(U)$.

PROOF. Let $\mathscr{f}_G$ be an $\mathcal{SS}$. Then, $(\mathscr{f}_G \tilde{\cap}\, U_G)(x) = \mathscr{f}_G(x) \cap U_G(x) = \mathscr{f}_G(x) \cap U = \mathscr{f}_G(x)$, for all $x \in G$. Similarly, $(U_G \tilde{\cap}\, \mathscr{f}_G)(x) = U_G(x) \cap \mathscr{f}_G(x) = U \cap \mathscr{f}_G(x) = \mathscr{f}_G(x)$. Thus, $\mathscr{f}_G \tilde{\cap}\, U_G = U_G \tilde{\cap}\, \mathscr{f}_G = \mathscr{f}_G$, implying that U_G is the identity element for the intersection operation of $\mathcal{SS}$s in $S_G(U)$.

Here, it is obvious that there is no inverse element for the intersection operation of $\mathcal{SS}$s other than U_G in $S_G(U)$. Naturally, U_G itself is the identity element for the intersection operation of $\mathcal{SS}$s in $S_G(U)$.

Proposition 3.5 $\emptyset_G$ is the absorbing element of the intersection operation of $\mathcal{SS}$s in $S_G(U)$.

PROOF. Let $\mathscr{f}_G$ be an $\mathcal{SS}$. Then, $(\mathscr{f}_G \tilde{\cap}\, \emptyset_G)(x) = \mathscr{f}_G(x) \cap \emptyset_G(x) = \mathscr{f}_G(x) \cap \emptyset = \emptyset = \emptyset_G(x)$, for all $x \in G$. Similarly, $(\emptyset_G \tilde{\cap}\, \mathscr{f}_G)(x) = \emptyset_G(x) \cap \mathscr{f}_G(x) = \emptyset \cap \mathscr{f}_G(x) = \emptyset = \emptyset_G(x)$. Thus, $\mathscr{f}_G \tilde{\cap}\, \emptyset_G = \emptyset_G \tilde{\cap}\, \mathscr{f}_G = \emptyset_G$, implying that $\emptyset_G$ is the absorbing element for the intersection operation of $\mathcal{SS}$s in $S_G(U)$.

Proposition 3.6 The intersection operation of $\mathcal{SS}$s is idempotent in $S_G(U)$ [44].

PROOF. This property was presented without its proof by Çağman and Enginoğlu [44]; however, we give a detailed proof of the property. Let $\mathscr{f}_G$ be an $\mathcal{SS}$. Then, $(\mathscr{f}_G \tilde{\cap}\, \mathscr{f}_G)(x) = \mathscr{f}_G(x) \cap \mathscr{f}_G(x) = \mathscr{f}_G(x)$, for all $x \in G$.

Theorem 3.7 $(S_G(U), \tilde{\cap})$ is a bounded semilattice with the identity U_G and the absorbing element $\emptyset_G$.

PROOF. The proof is followed by Proposition 3.1, Proposition 3.2, Proposition 3.3, Proposition 3.4, Proposition 3.5, Proposition 3.6.

4. More on Soft Intersection-union Product of Groups

In this section, we undertake a comprehensive algebraic investigation of the soft intersection–union product originally introduced by Muştuoğlu et al., [49] within the framework of soft groups. Our analysis focuses on a rigorous characterization of the product's structural properties, with particular emphasis on its interplay with various notions of soft equality and classifications of soft subsets. To elucidate the abstract concepts and underscore salient algebraic features of the construction, we complement the theoretical exposition with carefully chosen illustrative examples. Additionally, we examine the distributive behavior of this product over the intersection operation of $\mathcal{SS}$s, thereby shedding light on its compatibility with the underlying algebraic operations.

Definition 4.1 [49] Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two $\mathcal{SS}$s. Then, the intersection-union product $\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G$ is defined by

$$(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G)(x) = \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup \mathscr{g}_G(z)\big), \qquad y, z \in G$$

for all $x \in G$.

Note here that since G is a group, there always exist $y, z \in G$ such that $x = yz$, for all $x \in G$. Let the order of the group G be n, that is, $|G| = n$. Then, it is obvious that there exist n different combinations of writing styles for each $x \in G$ such that $x = yz$, where $y, z \in G$.

Note 4.2 The soft intersection-union product is well-defined in $S_G(U)$. In fact, let $\mathscr{f}_G, \mathscr{g}_G, \mathscr{r}_G, \mathscr{k}_G \in S_G(U)$ such that $(\mathscr{f}_G, \mathscr{g}_G) = (\mathscr{r}_G, \mathscr{k}_G)$. Then, $\mathscr{f}_G = \mathscr{r}_G$ and $\mathscr{g}_G = \mathscr{k}_G$, implying that $\mathscr{f}_G(x) = \mathscr{r}_G(x)$ and $\mathscr{g}_G(x) = \mathscr{k}_G(x)$ for all $x \in G$. Thereby,

$$\begin{aligned}(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G)(x) &= \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup \mathscr{g}_G(z)\big) \\ &= \bigcap_{x=yz} \big(\mathscr{r}_G(y) \cup \mathscr{k}_G(z)\big)\end{aligned}$$

$$= \left(\sigma_G \otimes_{i/u} \mathscr{k}_G\right)(x)$$

Hence, $\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G = \sigma_G \otimes_{i/u} \mathscr{k}_G$.

Example 4.3 Consider the group $G = \{Ձ, ɓ\}$ with the following operation**:**

$\cdot$	$Ձ$	$ɓ$
$Ձ$	$Ձ$	$ɓ$
$ɓ$	$ɓ$	$Ձ$

Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two $\mathcal{SS}$s over $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:
$\mathscr{f}_G = \{(Ձ, \{e, x, y\}), (ɓ, \{yx\})\}$ and $\mathscr{g}_G = \{(Ձ, \{y\}), (ɓ, \{e, yx\})\}$

Since $Ձ = ՁՁ = ɓɓ$, $\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right)(Ձ) = \left(\mathscr{f}_G(Ձ) \cup \mathscr{g}_G(Ձ)\right) \cap \left(\mathscr{f}_G(ɓ) \cup \mathscr{g}_G(ɓ)\right) = \{e\}$ and since $ɓ = Ձɓ = ɓՁ$, $\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right)(ɓ) = \left(\mathscr{f}_G(Ձ) \cup \mathscr{g}_G(ɓ)\right) \cap \left(\mathscr{f}_G(ɓ) \cup \mathscr{g}_G(Ձ)\right) = \{y, yx\}$ is obtained. Hence,
$\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G = \{(Ձ, \{e\}), (ɓ, \{y, yx\})\}$

Proposition 4.4 The set $S_G(U)$ is closed under the soft intersection-union product. That is, if $\mathscr{f}_G$ and $\mathscr{g}_G$ are two $\mathcal{SS}$s, then so is $\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G$.

PROOF. It is obvious that the soft intersection-union product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft intersection-union product.

Proposition 4.5 The soft intersection-union product is associative in $S_G(U)$ [49].

PROOF. This property was presented without its proof by Muştuoğlu et al., [49]; however, we give a detailed proof of the property. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\hbar_G$ be three $\mathcal{SS}$s, Thus, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{i/u} \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right)\right)(x) = \bigcap_{x=yz} \left(\mathscr{f}_G(y) \cup \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right)(z)\right)$$
$$= \bigcap_{x=yz} \left(\mathscr{f}_G(y) \cup \left(\bigcap_{z=\sigma n} \left(\mathscr{g}_G(\sigma) \cup \hbar_G(n)\right)\right)\right)$$
$$= \bigcap_{x=y(\sigma n)} \left(\mathscr{f}_G(y) \cup \left(\mathscr{g}_G(\sigma) \cup \hbar_G(n)\right)\right)$$
$$= \bigcap_{x=(y\sigma)n} \left(\left(\mathscr{f}_G(y) \cup \mathscr{g}_G(\sigma)\right) \cup \hbar_G(n)\right)$$
$$= \bigcap_{x=Ձn} \left(\left(\bigcap_{Ձ=y\sigma} \left(\mathscr{f}_G(y) \cup \mathscr{g}_G(\sigma)\right)\right) \cup \hbar_G(n)\right)$$
$$= \bigcap_{x=Ձn} \left(\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right)(Ձ) \cup \hbar_G(n)\right)$$
$$= \left(\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G\right)(x)$$

Thereby, $\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G = \mathscr{f}_G \otimes_{i/u} \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right)$. □

Example 4.6 Consider the group G and the $\mathcal{SS}$s $\mathscr{f}_G$ and $\mathscr{g}_G$ in Example 4.3. Let $\hbar_G = \{(Ձ, \{x\}), (ɓ, \{y, yx\})\}$ be a $\mathcal{SS}$ over $U = \{e, x, y, yx\}$.

Since $\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G = \{(Ձ, \{e\}), (ɓ, \{y, yx\})\}$, then

$$\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G = \{(Ձ, \emptyset), (ɓ, \{y, yx\})\}$$

Moreover, since $\mathscr{g}_G \otimes_{i/u} \hbar_G = \{(Ձ, \{y\}), (ɓ, \{yx\})\}$, then

$$\mathscr{f}_G \otimes_{i/u} \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right) = \{(Ձ, \emptyset), (ɓ, \{y, yx\})\}$$

Thereby, $\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G = \mathscr{f}_G \otimes_{i/u} \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right)$.

Proposition 4.7 The soft intersection-union product is not commutative in $S_G(U)$. However, if G is an abelian group, then the intersection-union product is commutative in $S_G(U)$ [49].

PROOF. This property was presented without its proof by Muştuoğlu et al., [49]; however, we give a detailed proof of the property. Let $\mathscr{f}_G$, $\mathscr{g}_G$ be two $\mathscr{SS}$s and G be an abelian group. Then, for all $x \in G$,

$$\begin{aligned}(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G)(x) &= \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup \mathscr{g}_G(z)\big)\\ &= \bigcap_{x=zy} \big(\mathscr{g}_G(z) \cup \mathscr{f}_G(y)\big)\\ &= \big(\mathscr{g}_G \otimes_{i/u} \mathscr{f}_G\big)(x)\end{aligned}$$

implying that $\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G = \mathscr{g}_G \otimes_{i/u} \mathscr{f}_G$. □

Proposition 4.8 The soft intersection-union product is not idempotent in $S_G(U)$.

PROOF. Consider the $\mathscr{SS}$ $\mathscr{f}_G$ in Example 4.3. Then, for all $x \in G$,

$\mathscr{f}_G \otimes_{i/u} \mathscr{f}_G = \{(\mathscr{Q}, \emptyset), (\mathscr{b}, U)\}$

implying that $\mathscr{f}_G \otimes_{i/u} \mathscr{f}_G \neq \mathscr{f}_G$.

Proposition 4.9 Let $\mathscr{f}_G$ be a constant $\mathscr{SS}$. Then, $\mathscr{f}_G \otimes_{i/u} \mathscr{f}_G = \mathscr{f}_G$.

PROOF. Let $\mathscr{f}_G$ be a constant $\mathscr{SS}$ such that, for all $x \in G$, $\mathscr{f}_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\big(\mathscr{f}_G \otimes_{i/u} \mathscr{f}_G\big)(x) = \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup \mathscr{f}_G(z)\big) = \mathscr{f}_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{i/u} \mathscr{f}_G = \mathscr{f}_G$.

Remark 4.10 Let $S_G{}^*(U)$ be the collection of all constant $\mathscr{SS}$s. Then, the soft intersection-union product is idempotent in $S_G{}^*(U)$.

Proposition 4.11 U_G is the absorbing element of the soft intersection-union product in $S_G(U)$.

PROOF. Let $\mathscr{f}_G$ be an $\mathscr{SS}$. Then, for all $x \in G$,

$$\begin{aligned}\big(U_G \otimes_{i/u} \mathscr{f}_G\big)(x) &= \bigcap_{x=yz} \big(U_G(y) \cup \mathscr{f}_G(z)\big)\\ &= \bigcap_{x=yz} \big(U \cup \mathscr{f}_G(z)\big)\\ &= U_G(x)\end{aligned}$$

Similarly,

$$\begin{aligned}\big(\mathscr{f}_G \otimes_{i/u} U_G\big)(x) &= \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup U_G(z)\big)\\ &= \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup U\big)\\ &= U_G(x)\end{aligned}$$

Thus, $U_G \otimes_{i/u} \mathscr{f}_G = \mathscr{f}_G \otimes_{i/u} U_G = U_G$. □

Proposition 4.12 Let $\mathscr{f}_G$ be a constant $\mathscr{SS}$. Then, $\mathscr{f}_G \otimes_{i/u} \emptyset_G = \emptyset_G \otimes_{i/u} \mathscr{f}_G = \mathscr{f}_G$.

PROOF. Let $\mathscr{f}_G$ be a constant $\mathscr{SS}$ such that, for all $x \in G$, $\mathscr{f}_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\big(\mathscr{f}_G \otimes_{i/u} \emptyset_G\big)(x) = \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup \emptyset_G(z)\big) = \bigcap_{x=yz} \big(\mathscr{f}_G(y) \cup \emptyset\big) = \mathscr{f}_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{i/u} \emptyset_G = \mathscr{f}_G$. Similarly, for all $x \in G$,

$$\big(\emptyset_G \otimes_{i/u} \mathscr{f}_G\big)(x) = \bigcap_{x=yz} \big(\emptyset_G(y) \cup \mathscr{f}_G(z)\big) = \bigcap_{x=yz} \big(\emptyset \cup \mathscr{f}_G(z)\big) = \mathscr{f}_G(x)$$

Thereby, $\emptyset_G \otimes_{i/u} \mathscr{f}_G = \mathscr{f}_G$.

Remark 4.13 $\emptyset_G$ is the identity element of the soft intersection-union product in $S_G{}^*(U)$.

Theorem 4.14 $\big(S_G(U), \otimes_{i/u}\big)$ is a noncommutative semigroup with the absorbing element U_G. If G is abelian, then $\big(S_G(U), \otimes_{i/u}\big)$ is a commutative semigroup with the absorbing element U_G.

PROOF. The proof is followed by Proposition 4.4, Proposition 4.5, Proposition 4.7, and Proposition 4.11.

Theorem 4.15 $\left(S_G{}^*(U), \otimes_{i/u}\right)$ is a noncommutative idempotent monoid with the identity element $\emptyset_G$ and the absorbing element U_G. If G is abelian, then $\left(S_G{}^*(U), \otimes_{i/u}\right)$ is a commutative idempotent monoid.

PROOF. The proof is followed by Proposition 4.4, Proposition 4.5, Proposition 4.7, Proposition 4.9, Proposition 4.11., and Proposition 4.12.

Proposition 4.16 Let f_G be a constant $\mathcal{SS}$. Then, $f_G \otimes_{i/u} f_G{}^c = f_G{}^c \otimes_{i/u} f_G = U_G$.

PROOF. Let f_G be a constant $\mathcal{SS}$ such that, for all $x \in G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\left(f_G \otimes_{i/u} f_G{}^c\right)(x) = \bigcap_{x=yz} \left(f_G(y) \cup f_G{}^c(z)\right) = U = U_G(x)$$

Thereby, $f_G \otimes_{i/u} f_G{}^c = U_G$. Similarly, for all $x \in G$,

$$\left(f_G{}^c \otimes_{i/u} f_G\right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup f_G(z)\right) = U = U_G(x)$$

Thus, $f_G{}^c \otimes_{i/u} f_G = U_G$. □

Proposition 4.17 Let f_G and g_G be two $\mathcal{SS}$s. Then, $\left(f_G \otimes_{i/u} g_G\right)^c = f_G \otimes_{u/t} g_G$.

PROOF. Let f_G and g_G be two $\mathcal{SS}$s. Then, for all $x \in G$,

$$\begin{aligned}\left(f_G \otimes_{i/u} g_G\right)^c(x) &= \left(\bigcap_{x=yz} \left(f_G(y) \cup g_G(z)\right)\right)' \\ &= \bigcup_{x=yz} \left(f_G(y) \cup g_G(z)\right)' \\ &= \bigcup_{x=yz} \left(f_G{}^c(y) \cap g_G{}^c(z)\right) \\ &= \left(f_G \otimes_{u/t} g_G\right)(x)\end{aligned}$$

Thus, $\left(f_G \otimes_{i/u} g_G\right)^c = f_G \otimes_{u/t} g_G$. We refer to [81] for more on soft union-theta product of groups. □

Proposition 4.18 Let f_G and g_G be two $\mathcal{SS}$s. If $f_G \,\widetilde{\subseteq}_S\, g_G$, then $f_G \otimes_{i/u} g_G = g_G$.

PROOF. Let f_G and g_G be two $\mathcal{SS}$s and $f_G \,\widetilde{\subseteq}_S\, g_G$. Hence, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A \subseteq B$. Thus, for all $x \in G$,

$$\left(f_G \otimes_{i/u} g_G\right)(x) = \bigcap_{x=yz} \left(f_G(y) \cup g_G(z)\right) = g_G(x)$$

Thereby, $f_G \otimes_{i/u} g_G = g_G$. □

Proposition 4.19 Let f_G, g_G, and $\hbar_G$ be three $\mathcal{SS}$s. If $f_G \,\widetilde{\subseteq}\, g_G$, then $f_G \otimes_{i/u} \hbar_G \,\widetilde{\subseteq}\, g_G \otimes_{i/u} \hbar_G$ [49] and $\hbar_G \otimes_{i/u} f_G \,\widetilde{\subseteq}\, \hbar_G \otimes_{i/u} g_G$.

PROOF. Let f_G, g_G, and $\hbar_G$ be three $\mathcal{SS}$s such that $f_G \,\widetilde{\subseteq}\, g_G$. Then, for all $x \in G$, $f_G(x) \subseteq g_G(x)$. Thus, for all $x \in G$,

$$\begin{aligned}\left(f_G \otimes_{i/u} \hbar_G\right)(x) &= \bigcap_{x=yz} \left(f_G(y) \cup \hbar_G(z)\right) \\ &\subseteq \bigcap_{x=yz} \left(g_G(y) \cup \hbar_G(z)\right) \\ &= \left(g_G \otimes_{i/u} \hbar_G\right)(x)\end{aligned}$$

for all $x \in G$, implying that $f_G \otimes_{i/u} \hbar_G \,\widetilde{\subseteq}\, g_G \otimes_{i/u} \hbar_G$. Similarly, for all $x \in G$,

$$\begin{aligned}\left(\hbar_G \otimes_{i/u} f_G\right)(x) &= \bigcap_{x=yz} \left(\hbar_G(y) \cup f_G(z)\right) \\ &\subseteq \bigcap_{x=yz} \left(\hbar_G(y) \cup g_G(z)\right)\end{aligned}$$

$$= \left(\hbar_G \otimes_{i/u} \mathscr{g}_G\right)(x)$$

implying that $\hbar_G \otimes_{i/u} \mathscr{f}_G \mathrel{\tilde{\subseteq}} \hbar_G \otimes_{i/u} \mathscr{g}_G$. □

Proposition 4.20 Let $\mathscr{f}_G$, $\mathscr{g}_G$, $\mathscr{o}_G$, and $\mathscr{k}_G$ be four $\mathcal{SS}$s. If $\mathscr{o}_G \mathrel{\tilde{\subseteq}} \mathscr{f}_G$ and $\mathscr{k}_G \mathrel{\tilde{\subseteq}} \mathscr{g}_G$, then $\mathscr{o}_G \otimes_{i/u} \mathscr{g}_G \mathrel{\tilde{\subseteq}} \mathscr{f}_G \otimes_{i/u} \mathscr{k}_G$ and $\mathscr{k}_G \otimes_{i/u} \mathscr{f}_G \mathrel{\tilde{\subseteq}} \mathscr{g}_G \otimes_{i/u} \mathscr{o}_G$ [49].

PROOF. This property was presented without its proof by Muştuoğlu et al., [49]; however, we give a detailed proof of the property. Let $\mathscr{f}_G$, $\mathscr{g}_G$, $\mathscr{o}_G$, and $\mathscr{k}_G$ be four $\mathcal{SS}$s such that $\mathscr{o}_G \mathrel{\tilde{\subseteq}} \mathscr{f}_G$ and $\mathscr{k}_G \mathrel{\tilde{\subseteq}} \mathscr{g}_G$. Then, for all $x \in G$, $\mathscr{o}_G(x) \subseteq \mathscr{f}_G(x)$ and $\mathscr{k}_G(x) \subseteq \mathscr{g}_G(x)$. Thus, for all $x \in G$,

$$\begin{aligned}\left(\mathscr{o}_G \otimes_{i/u} \mathscr{k}_G\right)(x) &= \bigcap_{x=yz} \left(\mathscr{o}_G(y) \cup \mathscr{k}_G(z)\right)\\ &\subseteq \bigcap_{x=yz} \left(\mathscr{f}_G(y) \cup \mathscr{g}_G(z)\right)\\ &= \left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right)(x)\end{aligned}$$

implying that $\mathscr{o}_G \otimes_{i/u} \mathscr{k}_G \mathrel{\tilde{\subseteq}} \mathscr{f}_G \otimes_{i/u} \mathscr{g}_G$. □

Muştuoğlu et al., [49] claimed that the soft intersection-union product distributes over the union operation of $\mathcal{SS}$s from left side; however, we have Proposition 4.21.

Proposition 4.21 The soft intersection-union product does not distribute over the union operation of $\mathcal{SS}$s from left side.

PROOF. Consider the group G in Example 4.3. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\hbar_G$ be three $\mathcal{SS}$s over $U = \{e, x, y, yx\}$ as follows:

$\mathscr{f}_G = \{(\text{Ձ}, \emptyset), (\text{Ϭ}, \emptyset)\}$, $\mathscr{g}_G = \{(\text{Ձ}, \{y, yx\}), (\text{Ϭ}, \{e, x, yx\})\}$, $\hbar_G = \{(\text{Ձ}, \{e, x\}), (\text{Ϭ}, \{y\})\}$

Since $\mathscr{f}_G \otimes_{i/u} \hbar_G = \{(\text{Ձ}, \emptyset), (\text{Ϭ}, \emptyset)\}$ and $\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G = \{(\text{Ձ}, \{yx\}), (\text{Ϭ}, \{yx\})\}$, then

$$\left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right) \mathbin{\tilde{\cup}} \left(\mathscr{f}_G \otimes_{i/u} \hbar_G\right) = \{(\text{Ձ}, \{yx\}), (\text{Ϭ}, \{yx\})\}$$

Moreover, since $\mathscr{g}_G \mathbin{\tilde{\cup}} \hbar_G = \{(\text{Ձ}, U), (\text{Ϭ}, U)\}$

$$\mathscr{f}_G \otimes_{i/u} \left(\mathscr{g}_G \mathbin{\tilde{\cup}} \hbar_G\right) = \{(\text{Ձ}, U), (\text{Ϭ}, U)\}$$

Thus, $\mathscr{f}_G \otimes_{i/u} \left(\mathscr{g}_G \mathbin{\tilde{\cup}} \hbar_G\right) \neq \left(\mathscr{f}_G \otimes_{i/u} \mathscr{g}_G\right) \mathbin{\tilde{\cup}} \left(\mathscr{f}_G \otimes_{i/u} \hbar_G\right)$. □

Proposition 4.22 The soft intersection-union product does not distribute over the union operation of $\mathcal{SS}$s from right side.

PROOF. Consider the group G in Example 4.3. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\hbar_G$ be three $\mathcal{SS}$s over $U = \{e, x, y, yx\}$ as follows:

$\mathscr{f}_G = \{(\text{Ձ}, \{e, x\}), (\text{Ϭ}, \{y\})\}$, $\mathscr{g}_G = \{(\text{Ձ}, \{y, yx\}), (\text{Ϭ}, \{e, x, yx\})\}$, $\hbar_G = \{(\text{Ձ}, \emptyset), (\text{Ϭ}, \emptyset)\}$

Since $\mathscr{f}_G \otimes_{i/u} \hbar_G = \{(\text{Ձ}, \emptyset), (\text{Ϭ}, \emptyset)\}$ and $\mathscr{g}_G \otimes_{i/u} \hbar_G = \{(\text{Ձ}, \{yx\}), (\text{Ϭ}, \{yx\})\}$, then

$$\left(\mathscr{f}_G \otimes_{i/u} \hbar_G\right) \mathbin{\tilde{\cup}} \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right) = \{(\text{Ձ}, \{yx\}), (\text{Ϭ}, \{yx\})\}$$

Moreover, since $\mathscr{f}_G \mathbin{\tilde{\cup}} \mathscr{g}_G = \{(\text{Ձ}, U), (\text{Ϭ}, U)\}$

$$\left(\mathscr{f}_G \mathbin{\tilde{\cup}} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G = \{(\text{Ձ}, U), (\text{Ϭ}, U)\}$$

Thus, $\left(\mathscr{f}_G \mathbin{\tilde{\cup}} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G \neq \left(\mathscr{f}_G \otimes_{i/u} \hbar_G\right) \mathbin{\tilde{\cup}} \left(\mathscr{g}_G \otimes_{i/u} \hbar_G\right)$. □

Remark 4.23 The soft intersection-union product does not distribute over the union operation of $\mathcal{SS}$s.

Proposition 4.24 The soft intersection-union product distributes over the intersection operation of $\mathcal{SS}$s from the right side.

PROOF. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\hbar_G$ be three $\mathcal{SS}$s. Then, for all $x \in G$,

$$\begin{aligned}\left(\left(\mathscr{f}_G \mathbin{\tilde{\cap}} \mathscr{g}_G\right) \otimes_{i/u} \hbar_G\right)(x) &= \bigcap_{x=yz} \left(\left(\mathscr{f}_G \mathbin{\tilde{\cap}} \mathscr{g}_G\right)(y) \cup \hbar_G(z)\right)\\ &= \bigcap_{x=yz} \left(\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(y)\right) \cup \hbar_G(z)\right)\\ &= \bigcap_{x=yz} \left(\left(\mathscr{f}_G(y) \cup \hbar_G(z)\right) \cap \left(\mathscr{g}_G(y) \cup \hbar_G(z)\right)\right)\end{aligned}$$

$$= \left[\bigcap_{x=yz}\left(f_G(y) \cup \hbar_G(z)\right)\right] \cap \left[\bigcap_{x=yz}\left(g_G(y) \cup \hbar_G(z)\right)\right]$$

$$= \left(f_G \otimes_{i/u} \hbar_G\right)(x) \cap \left(g_G \otimes_{i/u} \hbar_G\right)(x)$$

$$= \left(\left(f_G \otimes_{i/u} \hbar_G\right) \tilde{\cap} \left(g_G \otimes_{i/u} \hbar_G\right)\right)(x)$$

Thus, $(f_G \tilde{\cap} g_G) \otimes_{i/u} \hbar_G = \left(f_G \otimes_{i/u} \hbar_G\right) \tilde{\cap} \left(g_G \otimes_{i/u} \hbar_G\right)$. □

Proposition 4.25 The soft intersection-union product distributes over the intersection operation of $\mathcal{SS}$s from the left side [49].

PROOF. This property was presented without its proof by Muştuoğlu et al., [49]; however, we give a detailed proof of the property. Let f_G, g_G and $\hbar_G$ be three $\mathcal{SS}$s. Then, for all $x \in G$,

$$\left(f_G \otimes_{i/u} (g_G \tilde{\cap} \hbar_G)\right)(x) = \bigcap_{x=yz}\left(f_G(y) \cup (g_G \tilde{\cap} \hbar_G)(z)\right)$$

$$= \left(\bigcap_{x=yz}\left\{f_G(y) \cup \left(g_G(z) \cap \hbar_G(z)\right)\right\}\right)$$

$$= \bigcap_{x=yz}\left(\left(f_G(y) \cup g_G(z)\right) \cap \left(f_G(y) \cup \hbar_G(z)\right)\right)$$

$$= \left[\bigcap_{x=yz}\left(f_G(y) \cup g_G(z)\right)\right] \cap \left[\bigcap_{x=yz}\left(f_G(y) \cup \hbar_G(z)\right)\right]$$

$$= \left(f_G \otimes_{i/u} g_G\right)(x) \cap \left(f_G \otimes_{i/u} \hbar_G\right)(x)$$

$$= \left(\left(f_G \otimes_{i/u} g_G\right) \tilde{\cap} \left(f_G \otimes_{i/u} \hbar_G\right)\right)(x)$$

Thus, $f_G \otimes_{i/u} (g_G \tilde{\cap} \hbar_G) = \left(f_G \otimes_{i/u} g_G\right) \tilde{\cap} \left(f_G \otimes_{i/u} \hbar_G\right)$. □

Remark 4.26 The soft intersection-union product distributes over the intersection operation of $\mathcal{SS}$s from both sides.

Theorem 4.27 $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$ is a noncommutative hemiring without identity. However, if G is an abelian group, then $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$ is a commutative hemiring without identity.

PROOF. By Theorem 3.7, $\left(S_G(U), \tilde{\cap}\right)$ is a bounded semilattice. Moreover, by Theorem 4.14, $\left(S_G(U), \otimes_{i/u}\right)$ is a semigroup. Besides, by Remark 4.26, the soft intersection-union product distributes over the intersection operation of $\mathcal{SS}$s from both sides. Therefore, $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$ is a semiring. Further, by Proposition 3.4, $f_G \tilde{\cap} U_G = U_G \tilde{\cap} f_G = f_G$, and by Proposition 4.11, $f_G \otimes_{i/u} U_G = U_G \otimes_{i/u} f_G = U_G$. That is to say, U_G is the zero element of $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$. Moreover, since intersection operation of soft sets is commutative in $S_G(U)$, $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$ is a hemiring. Since the soft intersection-union product is not commutative and does not possess an identity element in $S_G(U)$, $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$ is a noncommutative hemiring without identity. However, if the group G is abelian, then $\left(S_G(U), \tilde{\cap}, \otimes_{i/u}\right)$ is a commutative hemiring without identity.

Theorem 4.28 $\left({S_G}^*(U), \tilde{\cap}, \otimes_{i/u}\right)$ is an idempotent noncommutative hemiring with the identity element $\emptyset_G$. However, if G is an abelian group, then $\left({S_G}^*(U), \tilde{\cap}, \otimes_{i/u}\right)$ is an idempotent commutative hemiring with the identity element $\emptyset_G$.

Proof: It is similar to Theorem 4.27.

5. Conclusion

This study begins with a rigorous algebraic analysis of the intersection operation of soft sets, establishing that the collection of all soft sets defined over a fixed parameter set, under this operation, forms a bounded semilattice. Building upon this foundational structure, we undertake an in-depth investigation of the soft intersection–union product, with particular focus on its algebraic behavior in relation to various classifications of soft subsets and generalized soft equality relations. It is formally demonstrated that the algebraic system consisting of soft sets, equipped with the intersection operation of soft sets and the soft intersection–union product, satisfies the axiomatic criteria of a hemiring. The development and systematic study of such binary operations within a well-defined algebraic universe form a cornerstone of abstract algebra. In particular, verifying fundamental properties—such as closure, associativity, commutativity, the existence of identity and inverse elements—permits the precise classification of the resulting algebraic structures within the established algebraic hierarchy. Moreover, the examination of distributive relationships between operations provides essential insights into the internal coherence and expressive capacity of the framework. These structural results not only reveal the intrinsic mathematical properties of the operations under consideration but also highlight their potential to generalize classical algebraic systems and to contribute to the resolution of open problems. In this context, the theoretical foundation developed herein addresses significant gaps in the existing literature and paves the way for a new line of inquiry in soft group theory based on the proposed product. Future directions include: (1) Constructing inverse elements for the intersection-union product, (2) Applications in decision-making under parameterized uncertainty, and (3) Extensions to soft topological spaces.

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